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Some cordial labelings on square graph of a comb *

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Abstract In this paper, we prove the existence of 3− total cordial edge magic labeling, 3− total sum cordial labeling and total product cordial labeling for the square graph of a comb.

Key words 3− total cordial edge magic labeling, 3− total sum cordial labeling, square graph of a comb.

2010 Mathematics Subject Classification Primary: 05C78, 05C90, 05C99.

1 Introduction

Graph theory has various applications in the field of computer programming and networking, marketing and communications, business administration and so on. Some major research topics in graph theory are Graph coloring, Spanning trees, Planar graphs, Networks and Graph labeling.Graph labeling is an interesting area of research in graph theory. The concept of graph labeling was introduced by Alexander Rosa [8] on certain valuations of the vertices of a graph in the year 1967 . A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. If the domain of mapping is the set of vertices (edges) then the labeling is called a vertex (an edge labeling). J.A. Gallian [5] has given a detailed survey on different kinds of graph labeling.
Sedlacek [9] has introduced the concept of A-magic graph. A graph is called an *A-magic graph* if the edges have distinct non-negative labels which satisfy the condition that the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Chou and Lee [3] investigated the concept of Z_3-magic graphs. A graph G admits Z_3-magic labeling, if there exists a function f from E to $\{1, 2\}$ such that the sum of the labels on the edges incident at each vertex $v \in V$ is a constant. Jayapriya and Thirusangu [6] introduced and proved the existence of $0-$Edge magic labeling for some class of graphs.
The concept of cordial labeling was introduced by I. Cahit [2]. It is proved there that every tree is cordial; K_n is cordial iff $n \leq 3$; $K_{m,n}$ is cordial for all $m$and n. A function from V to $\{0, 1\}$ is said to

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be a cordial labeling if each edge uv receives the label $|f(u) - f(v)|$ such that the number of vertices labeled with '0' and the number of vertices labeled with '1' differ by at most one, and the number of edges labeled with '0' and the number of edges labeled with '1' differ by at most one.
Moreover, it admits total cordial labeling if the number of vertices and edges labeled with '0' and the number of vertices and edges labeled with '1' differ by at most one.
In 2018 Esakkiammal et al. [4] introduced the concept of square graph of a comb . They have proved that the square graph of a comb admits square sum labeling, cube sum labelling and cube difference labeling.
Motivated by this study, in this paper we prove the existence of $3-$ total cordial edge magic labeling, $3-$ total sum cordial labeling and the total product cordial labeling for the square graph of a comb.

2 Preliminaries

Definition 2.1. Consider a function $f : V \cup E \rightarrow \{0, 1, 2\}$, assign a label to the vertices and the edges such that $|f(u) + f(v) + f(uv)|\,(mod3)$ is constant for all edges uv of a graph G. The number of vertices and edges labeled 'i ' and the number of vertices and edges labeled 'j ' differ by at most one for $i \neq j$ and $i, j \in \{0, 1, 2\}$. Then f is called a $3-$ total cordial edge magic labeling of G.

Definition 2.2. Consider a function $f : V \rightarrow \{0, 1, 2\}$. For each edge uv, assign the label $|f(u) + f(v)|$ $(mod3)$ satisfying the condition that the number of vertices and edges labeled 'i ' and the number of vertices and edges labeled 'j ' differ by at most one for $i \neq j$ and $i, j \in \{0, 1, 2\}$. Then f is called the $3-$total sum cordial labeling of G.

Definition 2.3. Consider a function $f : V \rightarrow \{0, 1\}$. For each edge uv, assign the label $[f(u) * f(v)]$ satisfying the condition that the total number of vertices and edges labeled '0' and the number of vertices and edges labeled '1' differ by atmost one . Then f is called the total product cordial labeling of G.

3 Main Results

In this section, we prove the existence of $3-$ total cordial edge magic labeling, $3-$ total sum cordial labeling and total product cordial labeling for the square graph of a comb by presenting algorithms.

Definition 3.1. Let P_n be a path graph with n vertices and $n-1$ edges. The comb graph is defined as $P_n \cup K_1$. The comb is a graph is obtained by joining a pendant edge to each vertices of the path. It is denoted by $(\text{comb}\, n)$. The comb graph has $2n$ vertices and $2n-1$ edges.

Definition 3.2. The square of a graph $G = (V, E)$ is the graph $G^2 = \left(V, E^2\right)$ such that $(u, w) \in E^2$ iff for some $v \in V$ both $(u, v) \in E$ and $(v, w) \in E$; i.e., there is a path of exactly two edges. The square of a comb graph is a square comb graph which is denoted by $(\text{comb}\, n)^2$. The square comb graph has $2n$ vertices and $5(n-1)$ edges.

Definition 3.3. (Structure of the square graph of a comb). The square graph of a comb graph has a vertex set $V = \{v_i \cup u_i : 1 \leq i \leq n\}$ and the edge set is

$$E = \{v_i u_i : 1 \leqslant i \leqslant n\} \cup \{v_i v_{i+2} : 1 \leqslant i \leqslant n-2\} \cup \{v_i v_{i+1} \cup v_i u_{i+1} \cup u_i v_{i+1} : 1 \leqslant i \leqslant n-1\}.$$

Algorithm 3.4. Procedure: $3-$ Total cordial edge magic labeling of square graph of a comb
Input: Square graph of a comb
// assignment of labels to the vertices with $0, 1, 2$//
If $n \equiv 2\,(mod3)$
For $i = 1, \ldots, n$ do

$$\{u_i \text{ and } v_i \quad \leftarrow \quad \begin{cases} 0 & \text{if} \quad i \equiv 0\,(mod3) \\ 1 & \text{if} \quad i \equiv 1\,(mod3) \\ 2 & \text{if} \quad i \equiv 2(mod3) \end{cases}$$

}
else

For $i = 1, \ldots, n$ do

$$\{v_i \quad \leftarrow \begin{cases} 0 & \text{if} \quad i \equiv 1\,(mod3) \\ 1 & \text{if} \quad i \equiv 2\,(mod3) \\ 2 & \text{if} \quad i \equiv 0\,(mod3) \end{cases}$$

$$u_i \quad \leftarrow \begin{cases} 0 & \text{if} \quad i \equiv 0\,(mod3) \\ 1 & \text{if} \quad i \equiv 1\,(mod3) \\ 2 & \text{if} \quad i \equiv 2(mod3) \end{cases}$$

}
End procedure.
Output: 3− total cordial edge magic labeling of a graph with a magic number 0.

Theorem 3.5. *The square graph of a comb* $(\text{comb}\, n)^2, n \geq 3$ *admits* 3− *total cordial edge magic labeling.*

Proof. The vertices in the graph is labeled with labels from the set Z_3 as given in the Algorithm 3.4. The edges in the graph are labeled satisfying the function $|f(u) + f(v) + f(uv)|\,(mod3) = 0$ (a constant) for all edges uv. Thus the edge labels are as follows:

1. For $1 \leq i \leq n, f(v_i u_i) = 1$,
2. For $1 \leq i \leq n-2$,

$$f(v_i v_{i+2}) = \begin{cases} 1 & \text{if} \quad i \equiv 1\,(mod3) \\ 2 & \text{if} \quad i \equiv 2\,(mod3) \\ 0 & \text{if} \quad i \equiv 0\,(mod3) \end{cases}$$

3. For $1 \leq i \leq n-1$

$$f(v_i v_{i+1}) = \begin{cases} 2 & \text{if} \quad i \equiv 1\,(mod3) \\ 0 & \text{if} \quad i \equiv 2\,(mod3) \\ 1 & \text{if} \quad i \equiv 0\,(mod3) \end{cases}$$

$$f(v_i u_{i+1}) = 2, f(u_i v_{i+1}) = 0$$

The vertices and edges receive the label $0, 1$ and 2 as follows in Table 1 :

Table 1:

Cases	Vertices and Edge labels		
	0	1	2
$n \in \{3m : m \in \mathbb{N}\}$	$\frac{7n-6}{3}$	$\frac{7n-3}{3}$	$\frac{7n-6}{3}$
$n \in \{3m+1 : m \in \mathbb{N}\}$	$\frac{7n-7}{3}$	$\frac{7n-4}{3}$	$\frac{7n-4}{3}$
$n \in \{3m+2 : m \in \mathbb{N}\}$	$\frac{7n-5}{3}$	$\frac{7n-5}{3}$	$\frac{7n-5}{3}$

From the above table, it is clear that the number of vertices and edges labeled together by $0, 1$ and 2 differ by at most one for all the cases.
Hence the square graph of a comb $(\text{comb}\ n)^2, n \geq 3$ admits 3− total cordial edge magic labeling with a magic constant 0.

□

Example 3.6. 3− Total cordial edge magic labeling of the square comb graph $(\text{comb}\ 7)^2$ is shown in Fig. (1).

Algorithm 3.7. Procedure: 3− **Total sum cordial labeling of square graph of a comb**
Input: Square graph of a comb
// assignment of labels to the vertices with $0, 1, 2$//
If $n \equiv 1\,(mod3)$

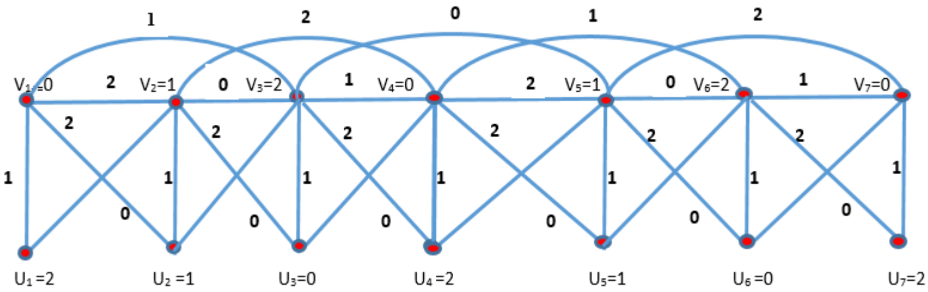

Fig. 1: 3$-$ Total cordial edge magic labeling of the square comb graph (comb 7)^2.

For $i = 1, \ldots, n$ do

$$\{\ v_i \leftarrow \begin{cases} 0 & \text{if} \quad i \equiv 1\ (mod3) \\ 1 & \text{if} \quad i \equiv 2\ (mod3) \\ 2 & \text{if} \quad i \equiv 0\ (mod3) \end{cases}$$

$$u_i \leftarrow \begin{cases} 0 & \text{if} \quad i \equiv 3\ (mod4) \\ 1 & \text{if} \quad i \equiv 0\ (mod2) \\ 2 & \text{if} \quad i \equiv 1 (mod4) \end{cases}$$

$\}$

else,

For $i = 1, \ldots, n$ do

$$\{\ v_i \leftarrow \begin{cases} 0 & \text{if} \quad i \equiv 0\ (mod3) \\ 1 & \text{if} \quad i \equiv 2\ (mod3) \\ 2 & \text{if} \quad i \equiv 1\ (mod3) \end{cases}$$

$$u_i \leftarrow \begin{cases} 0 & \text{if} \quad i \equiv 0\ (mod3) \\ 1 & \text{if} \quad i \equiv 1\ (mod3) \\ 2 & \text{if} \quad i \equiv 2 (mod3) \end{cases}$$

$\}$

End procedure.
Output: 3$-$ total sum cordial labeling of a graph.

Theorem 3.8. *The square graph of a comb* $(\text{comb } n)^2, n \geq 3$ *admits* 3$-$ *total sum cordial labeling.*

Proof. The vertices in the graph are labeled with labels from the set Z_3as given in the Algorithm 3.7. The edges in the graph are labeled satisfying the condition that , for each edge uv, assign the label $[f(u) + f(v)]\,(mod3)$. Thus the edge labels are as follows:

1. For $1 \leq i \leq n$, $f(v_i u_i) = 2$
2. For $1 \leq i \leq n-2$

$$f(v_i v_{i+2}) = \begin{cases} 2 & \text{if} \quad i \equiv 1\ (mod3) \\ 0 & \text{if} \quad i \equiv 2\ (mod3) \\ 1 & \text{if} \quad i \equiv 0\ (mod3) \end{cases}$$

3. For $1 \leq i \leq n-1$

$$f(v_i v_{i+i}) = \begin{cases} 0 & \text{if} \quad i \equiv 1\ (mod3) \\ 1 & \text{if} \quad i \equiv 2\ (mod3) \\ 2 & \text{if} \quad i \equiv 0\ (mod3) \end{cases}$$

$$f(v_i u_{i+1}) = 0, f(u_i v_{i+1}) = 0$$

Table 2:

Cases	Vertices and Edge labels		
	0	1	2
$n \in \{3m : m \in \mathbb{N}\}$	$\frac{7n-6}{3}$	$\frac{7n-6}{3}$	$\frac{7n-3}{3}$
$n \in \{3m+1 : m \in \mathbb{N}\}$	$\frac{7n-4}{3}$	$\frac{7n-4}{3}$	$\frac{7n-5}{3}$
$n \in \{3m+2 : m \in \mathbb{N}\}$	$\frac{7n-5}{3}$	$\frac{7n-5}{3}$	$\frac{7n-5}{3}$

The vertices and edges receive the labels $0, 1$ and 2 as follows in Table 2.
From the above table, it is clear that the number of vertices and edges labeled together by $0, 1$ and 2 differ by at most one for all the cases. Hence the square graph of a comb $(\text{comb } n)^2, n \geq 3$ admits $3-$ total sum cordial labeling.

□

Example 3.9. 3 - Total sum cordial labeling of the square comb graph $(\text{comb } 6)^2$.

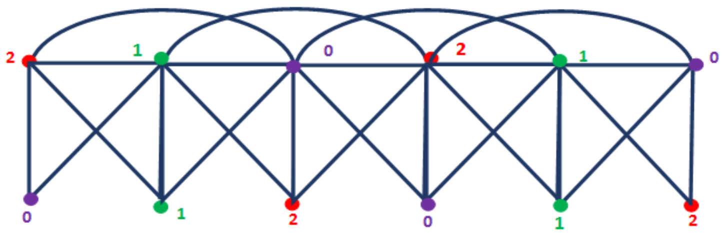

Fig. 2: 3 - Total sum cordial labeling of the square comb graph $(\text{comb } 6)^2$.

Algorithm 3.10. Procedure: Total product cordial labeling of square graph of a comb
Input: Square graph of a comb
// assignment of labels to the vertices with 0 or 1 //
If $n \equiv 1 \, (mod2)$
For $i = 1$ to n do

$$\{ \; v_i \quad \leftarrow \quad \begin{cases} 1 & \text{if} \quad i \equiv 1 \, (mod2) \\ 0 & \text{if} \quad i \equiv 0 \, (mod2) \end{cases}$$

$$u_i \leftarrow 1$$

$$\}$$

$$u_n \leftarrow 0$$

else
If $n \equiv 0 \, (mod2)$

$$k \leftarrow \frac{n}{2} - 1$$

For $i = 1, \ldots, n$ do

$$\{ \; v_i \leftarrow \begin{cases} 1 & \text{if} \quad i \leq \frac{n}{2}) \\ 0 & \text{otherwise} \end{cases}$$

$$u_i \leftarrow \quad \begin{cases} 1 & \text{if} \quad i \leq n-k \\ 0 & \text{otherwise} \end{cases}$$

$$\}$$

End procedure.
Output: Total product cordial graph.

Theorem 3.11. *The square graph of a comb,* $(\text{comb } n)^2, n \geq 3$ *admits total product cordial labeling.*

Proof. The vertices in the graph are labeled with labels from the set Z_2 as given in the Algorithm 3.10. The edges in the graph are labeled satisfying the condition that, for each edge uv, assign the label $[f(u) * f(v)]$ satisfying cordial labeling condition in vertices. Thus the edge labels $n \equiv 1\,(mod2)$ are as follows:

1. For $1 \leq i \leq n-1$
$$f(v_i u_i) = \begin{cases} 1 & \text{if} \quad i \equiv 1\,(mod2) \\ 0 & \text{if} \quad i \equiv 0\,(mod2) \end{cases}$$
$$\text{and} \qquad f(v_n u_n) = 0$$

2. For $1 \leq i \leq n-2$
$$f(v_i v_{i+2}) = \begin{cases} 1 & \text{if} \quad i \equiv 1\,(mod2) \\ 0 & \text{if} \quad i \equiv 0\,(mod2) \end{cases}$$

3. For $1 \leq i \leq n-1$
$$f(v_i v_{i+1}) = 0$$
$$f(v_i u_{i+1}) = \begin{cases} 1 & \text{if} \quad i \equiv 1\,(mod2) \\ 0 & \text{if} \quad i \equiv 0\,(mod2) \end{cases}$$
$$f(u_i v_{i+1}) = \begin{cases} 0 & \text{if} \quad i \equiv 1\,(mod2) \\ 1 & \text{if} \quad i \equiv 0\,(mod2) \end{cases}$$

The vertices and edges receive the label 0 and 1 as follows in Table 3:

Table 3:

Cases	0	1
$n \in \{2m+1 : m \in \mathbb{N}\}$	$\frac{7n-5}{2}$	$\frac{7n-5}{2}$
$n \in \{2m+2 : m \in \mathbb{N}\}$	$\frac{7n-6}{2}$	$\frac{7n-4}{2}$

From the above table, it is clear that the number of vertices and edges labeled together by 0 and 1 differ at most by one for all the cases.
Hence the square graph of a comb, $(\text{comb } n)^2, n \geq 3$ admits total product cordial labeling . $\square$

Example 3.12. Total product cordial labeling of the square comb graph.
Case 1 : $n \equiv 1\,(mod2)$ $(\text{comb } 5)^2$

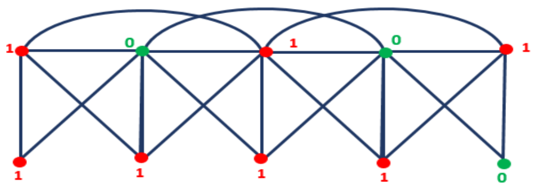

Fig. 3: Total product cordial labeling of the square comb graph.

Case 2 : $n \equiv 0\,(mod2)$ $(\text{comb } 4)^2$

4 Conclusion

In this paper, we proved the existence of 3− total cordial edge magic labeling, 3− total sum cordial labeling and total product cordial labeling for the square graph of a comb.

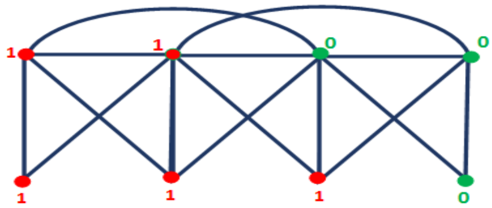

Fig. 4: Total product cordial labeling of the square comb graph.

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