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K-odd sequential harmonious labeling of some graphs *

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Abstract A labeling of a graph is the assigning of natural values to the vertices of the graph in some way that induces edge labels according to certain pattern. Interest in graph labeling began in 1967 when A. Rosa (On certain valuations of the vertices of a graph, Theory of Graphs, (International Symposium, Rome, July 1966), (1967), Gordon and Breach, New York and Dunod Pairs, 349–355) published his paper on graph labeling. In this paper, we prove that Ladder, Pyramid and the degree splitting of Star and Bistar graphs are k- odd sequential harmonious graphs.

Key words k-odd sequential harmonious labeling, Ladder graph, Pyramid graph, Star graph, Bistar graph.

2010 Mathematics Subject Classification 05C78, 05C99.

1 Introduction

Graph labeling is one of the fascinating areas of graph theory with its wide range of applications. Most graph labeling methods trace their origin to the introductory work of Rosa [5] in 1967 in this field. In [4] Muthuramakrishnan defined a labeling called the k- odd sequential harmonious labeling. A graph $G(V, E)$ is said to be a k- odd sequential graph if f is an injection from V to $\{k-1,\ k,\ k+1, \ldots, k+2q-1\}$ such that the induced mapping f^+ from E to $\{\ 2k-1, 2k+1,\ 2k+3, \ldots 2k+2q-3\}$ defined by

$$f^+(uv) = \begin{cases} f(u) + f(v) + 1, & \text{if } \ f(u) + f(v) \text{ is even,} \\ f(u) + f(v), & \text{if } f(u) + f(v) \text{ is odd,} \end{cases}$$

is distinct. A graph G is called a k- *odd sequential graph* if it admits k- odd sequential harmonious labeling. In [6] some of the graphs are proved to be degree splitting graphs on graceful, felicitous and elegant labeling by Selvaraju et al. In [2,3] some of the graphs that are shown to be k-odd sequential harmonious are: paths, cycles, triangular snakes, double quadrilateral, bistar, grids, etc. Further results on k- odd sequential harmonious labeling can be found in the dynamic survey of Gallian [1].

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Graph labeling is an active area of research in graph theory which is mainly evolved through its rigorous applications in coding theory, communication networks, optimal circuits, layouts and graph decomposition problems etc.
In this paper, it is proven that the graphs *Ladder, Pyramid* and the *degree splitting of Star and Bistar graphs* admits k-odd sequential harmonious labeling.

2 Preliminary definitions

We give below some preliminary definitions which will be used in this paper by us.

Definition 2.1. ***$k-$odd sequential graph:*** A graph $G\,(V,E)$ is said to be a k- odd sequential harmonious if f is an injection from V to $\{k-1,k,k+1,\ldots,k+2q-1\}$ such that the induced mapping f^+ from E to $\{2k-1,2k+1,2k+3,\ldots,2k+2q-3\}$ defined by $f^+(u,v)= f\,(u)+f\,(u)+1$ if $f\,(u)+f\,(v)$ is even and $f^+(u,v)= f\,(u)+f\,(u)$ if $f\,(u)+f\,(u)$ is odd are distinct. A graph G is called a k- odd sequential graph if it admits a k- odd sequential harmonious labeling.

Definition 2.2. ***Ladder graph:*** The graph $P_n \times P_2$ is called the ladder graph L_n.

Definition 2.3. ***Pyramid graph:*** A pyramid graph is obtained by arranging the vertices into a finite number of lines with i vertices in the i^{th} line and in every line the j^{th} vertex in that line is joined to the j^{th} vertex and the $(j+1)^{\text{th}}$ vertex of the next line. A pyramid graph that has $\frac{n(n+1)}{2}$ vertices is denoted by J_n.

Definition 2.4. ***Star graph:*** The star graph S_n of order n is a tree on n nodes with one node having a vertex of degree $n-1$ and other $n-1$ nodes having a vertex of degree 1.

Definition 2.5. ***Bistar graph:*** The bistar $(B_{n,n})$ is a graph obtained by joining the centre (apex) vertices of two copies of $K_{1,n}$ by an edge. The vertex set of $B_{n,n}$ is $V\,(B_{n,n}) = \{u,v,u_i,v_i/1\le i\le n\}$, where u,v are apex vertices and u_i,v_i are pendent vertices. The edge set of $B_{n,n}$ is $E\,(B_{n,n}) = \{uv,uu_i,vv_i/1\le i\le n$, so $|V\,(B_{n,n})| = 2n+2$ and $|E\,(B_{n,n})| = 2n+1$.

Definition 2.6. ***Degree splitting:*** Let $G = (V,E)$ be a graph with $V = S_1\cup S_2\cup S_3\cup\cdots\cup S_i\cup T$ where S_i is a set of vertices having atleast two vertices of the same degree and $T = V\backslash\cup S_i$. The degree splitting graph of G denoted by $DS\,(G)$ is obtained from G by adding vertices $w_1,w_2,\ldots,w_i$ and joining them to each vertex of S_i for $1\le i\le t$.

3 Results

Now we prove a number of results and also give illustrations of these results.

Theorem 3.1. *The graph ladder L_n is a $k-$ odd sequential harmonious graph.*

Proof. Let $L_n = P_n\times P_2$ be the ladder graph. Let V,E denotes the vertex set and edge set of the ladder graph respectively. Let the vertices of the ladder graph be u_1,u_2,$u_3,\ldots,u_n$,v_1,v_2,$v_3,\ldots,v_n$. Let p,q denotes the number of vertices and number of edges of the ladder graph (see Fig. 1).
Then $p = |V| = 2n$ and $q = |E| = 3n-2$. The vertex and edge labels of the ladder are follows:
We define $f : V\,(G)\to\{k-1,k,k+1,\ldots,k+2q-1\}$ as below:

$$f\,(u_1) = k-1$$

$$f\,(u_i) = 3i+k-4 \quad \text{for } 1\le i\le n$$
$$f\,(v_1) = k$$
$$f\,(v_i) = 3i+k-3 \quad \text{for } 1\le i\le n$$

The corresponding edge labels are

$$f : E\,(G)\to\{2k-1,2k+1,2k+3,\ldots,2k+2q-3\}$$

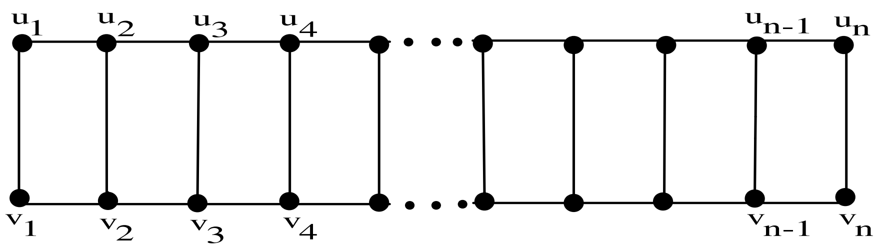

Fig. 1: The Ladder graph.

Let us define the edge sets as

$$\begin{aligned} A &= \{\ f^+(u_iu_{i+1})\ ;;1 \le i \le n\} \\ &= \{6i + 2k - 5\ ;;1 \le i \le n \\ B &= \{\ f^+(v_iv_{i+1})\ ;;1 \le i \le n\} \\ &= \{6i + 2k - 3\ ;;1 \le i \le n \\ C &= \{\ f^+(u_iv_i)\ ;;1 \le i \le n\} \\ &= \{6i + 2k - 7\}\,. \end{aligned}$$

It is clear that all the vertex labels are distinct. Also, it is easily verified that the edge label sets are mutually disjoint. That is $A \cap B \cap C = \emptyset$ and $A \cup B \cup C = \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$. Therefore, the edge induced labels of L_n have q disjoint values. Hence the graph L_n is a $k-$ odd sequential harmonious graph.

□

Illustration 3.2. An illustration of the k-odd sequential harmonious labeling of the ladder graph for $k = 3$ is shown in Fig. 2 below:

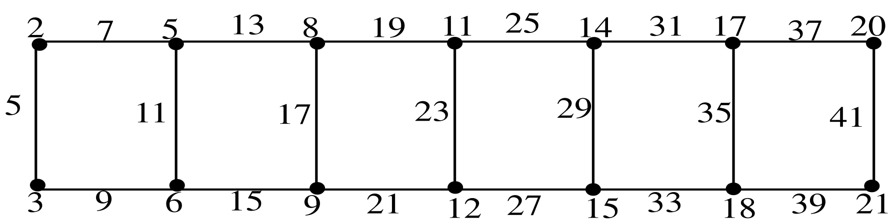

Fig. 2: The k-odd sequential harmonious labeling of the ladder graph for $k = 3$.

Theorem 3.3. *The pyramid graph J_n is a $k-$ odd sequential graph.*

Proof. Let J_n be the pyramid graph (see Fig. 3). Let V, E denotes the vertex set and edge set of the pyramid graph respectively. Then,

$$V = \{v_{ij}/\ 1 \le i \le n\ ;;1 \le j \le i\},$$

$$E = \{v_{ij}v_{i+1,j}, v_{ij}v_{i+1,j+1}/1 \le i \le n-1, \quad 1 \le j \le i\}.$$

Let p, q denotes the number of vertices and number of edges of the pyramid graph. Then $p = |V(J_n)| = \frac{(n^2+n)}{2}$ and $q = |E(J_n)| = (n^2 - n)$. The vertex and edge labels of the pyramid graph are as follows:

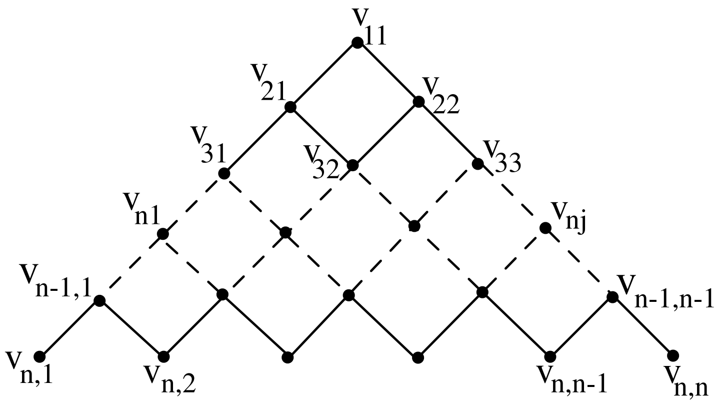

Fig. 3: The pyramid graph J_n.

We define $f : V(G) \to \{k-1, k, k+1, \ldots, k+2q-1\}$ as below:

$$f(v_{ij}) = k - 1 + (i-1)^2 + 2(j-1) \text{ for } 1 \le i \le n \text{ ; } 1 \le j \le i.$$

The corresponding edge labels are $f : E(G) \to \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$. Let us define the edge sets by

$$\begin{aligned} A &= \left\{ f^+(v_{ij}v_{i+1,j}) \;/ 1 \le i \le n-1 \;; 1 \le j \le i \right\} \\ &= \{\; 2i(i-1) + 4j + 2k - 5 \;; 1 \le j \le i \;\} \\ B &= \left\{ f^+(v_{ij}v_{i+1,j+1}) \;/ 1 \le i \le n \;; 1 \le j \le i \right\} \\ &= \;\{2i(i-1) + 4j + 2k - 3 \;; 1 \le j \le i\}\,. \end{aligned}$$

It is clear that all the vertex labels are unique. Also, it is easily verified that the edge label sets are independent. That is $A \cap B = \emptyset$ and $A \cup B = \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$. Therefore, the edge induced labels of J_n have q disjoint values. Hence the graph J_n is a $k-$ odd sequential graph.

□

Illustration 3.4. An illustration of the Theorem 3.3 for the case $k = 3$ is depicted in Fig. 4 below:

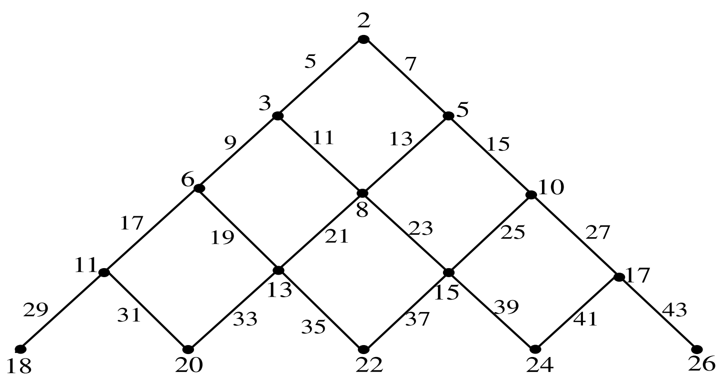

Fig. 4: The k-odd sequential harmonious labeling of the pyramid graph for $k = 3$.

Theorem 3.5. *The degree splitting of a star graph admits $k-$ odd sequential harmonious labeling.*

Proof. Let $DS(K_{1,n})$ be the degree splitting of star graph as shown in Fig. 5. The degree splitting of the star graph $(K_{1,n})$, denoted by $DS(K_{1,n})$, is obtained from $(K_{1,n})$ by adding vertices u_1,u_2 and joining them to each vertex of v_1 and each vertex of v_{ij}. Let V, E denotes the vertices and edges of the $DS(K_{1,n})$. Let p, q denote the number of vertices and the number of edges of the degree splitting star graph. Let v_1 be the apex of degree splitting of the star graph. Consider $K_{1,n}$ with $V(DS(K_{1,n})) = \{v_1, v_{ij}, u_1, u_2\}$ where, v_{ij} $(1 \leq j \leq n)$ are pendant vertices. Then $|V(DS(K_{1,n}))| = n+3$ and $|E(DS(K_{1,n}))| = 2n+1$. The vertex and edge labels of the $DS(K_{1,n})$ are as follows:
We define $f : V(DS(K_{1,n})) \to \{k-1, k, k+1, \ldots, k+2q-1\}$ as under:

$$f(v_1) = k-1$$

$$f(v_{1j}) = 2j+k-2, \text{ for } 1 \leq j \leq n$$
$$f(u_1) = 2n+k-1$$
$$f(u_2) = 2q+k-2$$

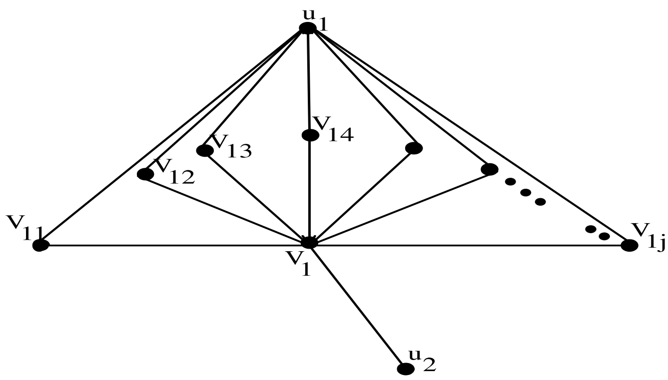

Fig. 5: The degree splitting of the Star graph $(K_{1,n})$.

The corresponding edge labels are $f : E(G) \to \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$. Let us define the edge sets by

$$A = \{f^+(v_1 v_{1j}) ; 1 \leq j \leq n\}$$
$$= \{2k+2j-3 ; 1 \leq j \leq n\}.$$
$$B = \left\{f^+(v_{1j} u_1) ; 1 \leq j \leq n\right\}$$
$$= \{2j+2n+2k-3 ; 1 \leq j \leq n\}.$$
$$C = \{f^+(v_1 u_2)\}$$
$$= \{2q+2k-3\}.$$

It is clear that all the vertex labels are different. Also, it is easily verified that the edge label sets are mutually disjoint. That is $A \cap B \cap C = \emptyset$ and $A \cup B \cup C = \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$. Therefore, the edge induced labels of $DS(K_{1,n})$ have q disjoint values. Hence the graph $DS(K_{1,n})$ admits $k-$ odd sequential harmonious labeling.

□

Illustration 3.6. We provide an illustration for the Theorem 3.5 in Fig. 6 below for the case $n = 7$.

Theorem 3.7. *The degree splitting of the bistar graph admits $k-$ odd sequential harmonious labeling.*

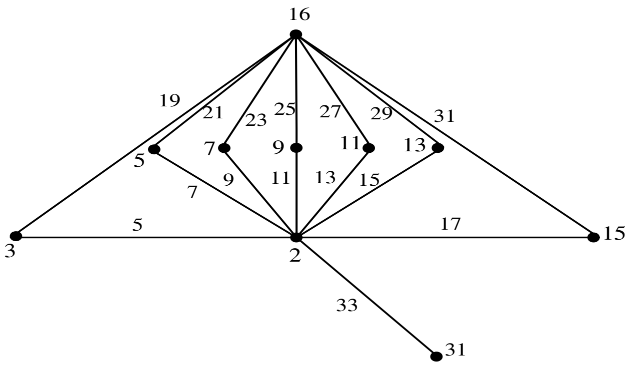

Fig. 6: The k-odd sequential harmonious labeling of $DS\left(K_{1,7}\right)$.

Proof. Let $DS(B_{n,n})$ be the degree splitting of the bistar graph (see Fig. 7). The degree splitting of the bistar graph $(B_{n,n})$, denoted by $DS(B_{n,n})$, is obtained from $(B_{n,n})$ by adding vertices u_1,u_2 and joining them to each vertex of v_{1j} and v_{2j}. Let (V, E) denotes the vertices and edges of the $DS(K_{1,n})$. Let p, q denote the number of vertices and number of edges of the degree splitting of the bistar graph. Consider $B_{n,n}$ with $V(B_{n,n}) = \{v_1,v_2,\ v_{1j},v_{2j}\}$ where, v_{1j} , v_{2j} are pendant vertices with $1 \leq j \leq n$. Here $V(DS(B_{n,n})) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2,\ v_{1j},v_{2j}, 1 \leq j \leq n\}$ and $V_2 = \{u_1,u_2\}$. Now, $|V(DS(B_{n,n}))| = 2n+4, |E(DS(B_{n,n}))| = 4n+3$ and the vertex and edge labels of the $DS(B_{n,n})$ are as follows:
We define $f : V(DS(B_{n,n})) \to \{k-1, k, k+1, \ldots, k+2q-1\}$ as under:

$$f\left(u_1\right) = k-1$$

$$f\left(v_{1j}\right) = 2j+k-2\ ;\ 1 \leq j \leq n$$
$$f\left(v_{2j}\right) = 2n+2j+k-3\ ;\ 1 \leq j \leq n$$
$$f\left(v_1\right) = 4n+k-2$$
$$f\left(v_2\right) = 4n+k$$
$$f\left(u_2\right) = q+k-1$$

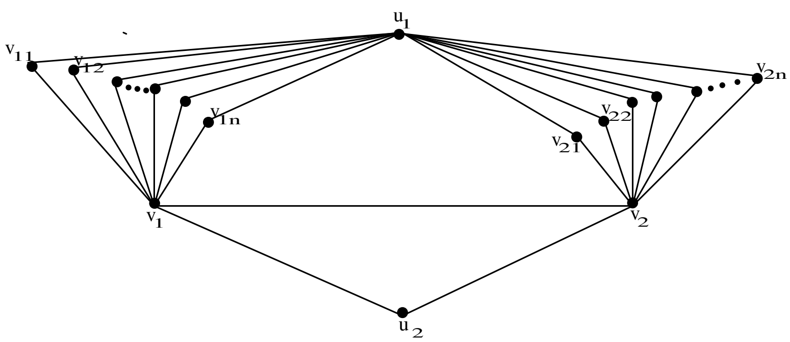

Fig. 7: The degree splitting of the Bistar graph $(B_{n,n})$.

The corresponding edge labels are $f : E\left(G\right) \to \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$.
Let us define the edge sets

$$A = \left\{f^{+}(v_1v_2)\right\}$$

$$= \begin{cases} 8n + 2k - 1, & \text{if } f(v_1) + f(v_2) \text{ is even,} \\ 8n + 2k - 2, & \text{if } f(v_1) + f(v_2) \text{ is odd,} \end{cases}$$

$$B = \{f^+(v_1 v_{1j})\ ; 1 \le j \le n\}$$

$$= \begin{cases} 2j + 4n + 2k - 3, & \text{if } f(v_1) + f(v_{1j}) \text{ is even,} \\ 2j + 4n + 2k - 4, & \text{if } f(v_1) + f(v_{1j}) \text{ is odd,} \end{cases}$$

$$C = \{f^+(v_2 v_{2j})\ ; 1 \le j \le n\}$$

$$= \begin{cases} 2j + 2n + 2k - 3, & \text{if } f(v_1) + f(v_{1j}) \text{ is even,} \\ 2j + 2n + 2k - 4 & \text{if } f(v_1) + f(v_{1j}) \text{ is odd,} \end{cases}$$

$$D = \{f^+(v_1 u_2)\}$$

$$= \begin{cases} 4n + 2k + q - 2, & \text{if } f(v_1) + f(u_2) \text{ is even,} \\ 4n + 2k + q - 3, & \text{if } f(v_1) + f(u_2) \text{ is odd,} \end{cases}$$

$$E = \{f^+(v_2 u_2)\}$$

$$= \begin{cases} 4n + 2k + q, & \text{if } f(v_2) + f(u_2) \text{ is even,} \\ 4n + 2k + q - 1, & \text{if } f(v_2) + f(u_2) \text{ is odd,} \end{cases}$$

$$F = \{f^+(v_2 v_{2j})\ ; 1 \le j \le n\}$$
$$= \{2j + 6n + 2k - 3\ ; 1 \le j \le n\},$$
$$G = \{f^+(u_1 v_{1j})\ ; 1 \le j \le n\}$$
$$= \{2k - 3 + 2j\ ; 1 \le j \le n\}.$$

It is clear that all the vertex labels are independent. Also, it is easily verified that the edge label sets are mutually disjoint. That is $A \cap B \cap C \cap D \cap E \cap F \cap G = \emptyset$ and $A \cup B \cup C \cup D \cup E \cup F \cup G = \{2k-1, 2k+1, 2k+3, \ldots, 2k+2q-3\}$. Therefore, the edge induced labels of $DS(B_{n,n})$ have q disjoint values. Hence the graph $DS(B_{n,n})$ admits $k-$ odd sequential harmonious labeling.

□

Illustration 3.8. We provide an illustration for the Theorem 3.7 in Fig. 8 for the case $n = 4$.

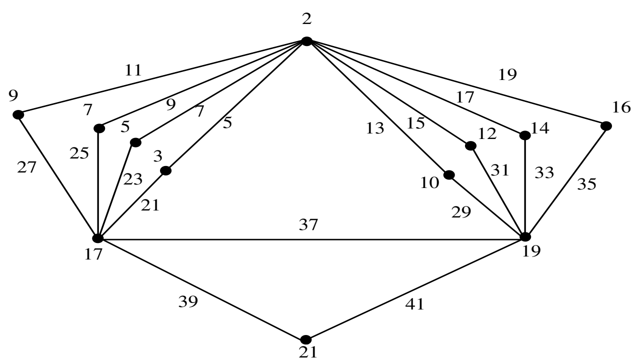

Fig. 8: k-odd sequential harmonious labeling of $DS(B_{4,4})$.

4 Conclusion

In this paper, we have proved that the ladder graph , the pyramid graph and the degree splitting of the star and the bistar graphs admit $k-$ odd sequential harmonious labeling.

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