Original Article

Bulletin of Pure and Applied Sciences.
Vol. 38E (Math & Stat.), No.1, 2019. P.135-154

Available online at
www.bpasjournals.com

Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2019.00012.2

TRANSLATIONS AND MULTILICATIONS OF JUN'S CUBIC SETS IN CUBIC SUBALGEBRAS AND CUBIC FILTERS OF *CI*-ALGEBRAS

S.R. Barbhuiya[1,*], A.K. Dutta[2]

Author Affiliation:
[1]Department of Mathematics, Srikishan Sarda College, Hailakandi, Assam 788151, India.
E-mail: saidurbarbhuiya@gmail.com
[2]Department of Mathematics, D.H.S.K. College, Dibrugarh, Assam 786001, India.
E-mail: akdutta28@gmail.com

***Corresponding Author:**
S.R. Barbhuiya, Associate Professor and Head, Department of Mathematics, Srikishan Sarda College, Hailakandi, Assam 788151, India.

E-mail: saidurbarbhuiya@gmail.com

Received on 05.08.2018 Revised on 27.01.2019 Accepted on 27.03.2019

Abstract: In this paper, we introduce the concept of translation and multiplication of cubic subalgebras and cubic filters of *CI*-algebras and investigate some of their basic properties. The notion of cubic extensions of cubic subalgebras and cubic filters is also introduced and several related properties are investigated.

Keywords: *CI*-algebra , Interval-valued fuzzy set , Cubic set , Cubic subalgebra, Cubic filters , Translation, Multiplication, Extension.

2010 AMS Mathematics Subject Classification: 06F35, 03E72, 03G25, 08A72.
.

1. INTRODUCTION

In real physical world, we often encounter a class of objects which do not have precisely defined criteria of membership, for instance, the "collection of young men" in the set of all men, "collection of beautiful girls" in the set of all girls, "collection of costly vehicles" in the set of all vehicles etc. Instead, their boundaries seem vague and the transition from member to nonmember appears gradualy rather than abrupt. Such situations in our real life which are characterized by vagueness or imprecision can not be answered just in two valued logic (binary) yes or no. To eliminate the sharp boundary dividing members of the class from non members, Lotfi A. Zadeh ([34]) a professor of electrical engineering at the University of California in Berkeley and Dieter Klaua ([19, 20]) a German mathematician, introduced the notion of a fuzzy set to describe vagueness mathematically. Rosenfeld ([30]) first fuzzified the algebraic concept of 'Group' into fuzzy subgroup and opened up a new insight in the field of pure mathematics. Since then, a host of mathematicians have been engrossed in extending the concepts and results of abstract algebra to boarder framework of fuzzy setting. After the introduction of BCK-algebra by Imai and Iseki ([9])in 1966, Xi([32]) applied the concept of fuzzy set to BCK-algebra and introduced fuzzy subalgebra and fuzzy ideals in BCK-algebra etc. Since then, a huge volume of literature has been produced on the theory of BCK/BCI-algebras. There exist several generalizations of BCK/BCI-algebras, as such BCH-algebras[8], dual BCK-algebras [22], d-algebras[29], etc. In [25] B.L.Meng introduced the notion of a CI-algebra as a generation of a BE-algebra and dual BCK/BCI/BCH-algebras.

In [35] Zadeh made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i. e. a fuzzy set with interval valued membership function). In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval $[0,1]$ are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Atanassov and Gargov [3] introduced the concept of interval-valued intuitionistic fuzzy set. In [4] different operators over interval-valued intuitionistic fuzzy sets are defined. Saeid [31] first introduced the concept of interval valued fuzzy subgroup as a generalization of Rosenfeld's fuzzy subgroup. In [7, 27, 12, 31] different authors applied interval-valued fuzzy sets in various algebraic structures.
Combining the fuzzy set and an interval-valued fuzzy set, Jun et al. [14] introduced a new type of set called a cubic set, and investigated several properties. They also applied the cubic set theory to BCK/BCI-algebras (see [15]). Subsequently the theory of cubic sets attracted several mathematicians. Jun et al. [16, 17, 18] furhter studied the theory of cubic sets in different algebraic structures. Yaqoob et al. [33] investigated some properties of cubic KU-ideals of KU-algebras ([33]). Ahn et al ([1]) introduced cubic subalgebras and cubic filters in CI-algebras and investigated several related properties. Lee et al. ([23]) and Jun([13]) discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras and ideals in BCK/BCI-algebras. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. Motivated by these works, in this paper we introduce the concept of fuzzy translation and fuzzy multiplication in cubic fuzzy subalgebras and filters of *CI*-algebras.

2. PRELIMINARIES

In this section, we recall some concepts related to *CI*-algebra, interval-valued fuzzy sets and cubic fuzzy sets.
Definition 2.1: ([12, 13]) An algebraic system $(X,*,1)$ of type (2,0) is called a CI-algebra if it satisfies the following axioms:
(i) $x*x=1$
(ii) $1*x=x$
(iii) $x*(y*z)=y*(x*z)\ \forall\ x,y,z\in X$.
A *CI*-algebra *X* satisfying the condition $x*1=1$, is called a *BE*-algebra. In any *CI*-algebra *X* one can define a binary relation $'\leq'$ by $x\leq y$ if and only if $x*y=1$.
In any *CI*-algebra *X* the following properties hold $\forall\ x,y\in X$:
(i) $y*((y*x)*x)=1$,
(ii) $(x*1)*(y*1)=(x*y)*1$,
(iii) if $1\leq x$, then $x=1$, for all $x,y\in X$.

A non-empty subset *S* of a *CI*-algebra *X* is called a subalgebra of *X* if $x*y\in S$, for all $x,y\in S$. A nonempty subset *I* of a *CI*-algebra *X* is called an ideal (see, [13]) of *X* if (I_1) $0\in I$ (I_2) $x*y\in I$ and $y\in I\Rightarrow x\in I$ for all $x,y\in X$.

Definition 2.2: A fuzzy subset μ of a *CI*-algebra *X* is called a subalgebra of *X* if $\mu(x*y)\geq min\{\mu(x),\mu(y)\}$ for all $x,y\in S$. A nonempty subset F of a *CI*-algebra X is called a filter of X if
(i) $1\in F$,
(ii) $x\in F, x*y\in F\Rightarrow y\in F, \forall\ x,y\in X$.
A filter *F* of *X* is said to be closed if $x\in F$ implies $x*1\in F$.

Definition 2.3: ([13]) A fuzzy set μ in X is called a fuzzy filter (FF) of X if it satisfies the following conditions:
(i) $\mu(1)\geq\mu(x)$,
(ii) $\mu(y)\geq\min\{\mu(x*y),\mu(x)\}\ \forall\ x,y\in X$.

The notion of interval-valued fuzzy set was introduced by Zadeh ([35]). To consider the notion of interval-valued fuzzy sets, we need the following definitions. An interval number on $[0,1]$, denoted by $\tilde{a}$, is defined as the closed sub interval of $[0,1]$, where $\tilde{a}=[\underline{a},\overline{a}]$,satisfying $0\le \underline{a}\le \overline{a}\le 1$. Let $D[0,1]$ denote the set of all such interval numbers on $[0,1]$ and also denote the interval numbers $[0,0]$ and $[1,1]$ by $\tilde{0}$ and $\tilde{1}$ respectively. Let $\tilde{a}_1=[\underline{a}_1,\overline{a}_1]$ and $\tilde{a}_2=[\underline{a}_2,\overline{a}_2]\in D[0,1]$. Define on D[0, 1] the relations $\le,=,<,+,.$ by

1. $\tilde{a}_1\le\tilde{a}_2 \Leftrightarrow \underline{a}_1\le\underline{a}_2$ and $\overline{a}_1\le\overline{a}_2$
2. $\tilde{a}_1=\tilde{a}_2 \Leftrightarrow \underline{a}_1=\underline{a}_2$ and $\overline{a}_1=\overline{a}_2$
3. $\tilde{a}_1<\tilde{a}_2 \Leftrightarrow \underline{a}_1<\underline{a}_2$ and $\overline{a}_1<\overline{a}_2$
4. $\tilde{a}_1+\tilde{a}_2 \Leftrightarrow [\underline{a}_1+\underline{a}_2,\overline{a}_1+\overline{a}_2]$
5. $\tilde{a}_1.\tilde{a}_2 \Leftrightarrow [min(\underline{a}_1\underline{a}_2,\underline{a}_1\overline{a}_2,\overline{a}_1\underline{a}_2,\overline{a}_1\overline{a}_2),max(\underline{a}_1\underline{a}_2,\underline{a}_1\overline{a}_2,\overline{a}_1\underline{a}_2,\overline{a}_1\overline{a}_2)]=[\underline{a}_1\underline{a}_2,\overline{a}_1\overline{a}_2]$
6. $k\tilde{a}=[k\underline{a},k\overline{a}]$ where $0\le k\le 1$.

Now consider two intervals $\tilde{a}_1=[\underline{a}_1,\overline{a}_1],\tilde{a}_2=[\underline{a}_2,\overline{a}_2]\in D[0,1]$ then we define refine minimum $rmin$ as $rmin(\tilde{a}_1,\tilde{a}_2)=[min(\underline{a}_1,\underline{a}_2),min(\overline{a}_1,\overline{a}_2)]$ and refine maximum $rmax$ as $rmax(\tilde{a}_1,\tilde{a}_2)=[max(\underline{a}_1,\underline{a}_2),max(\overline{a}_1,\overline{a}_2)]$ generally if $\tilde{a}_i=[\underline{a}_i,\overline{a}_i],\tilde{b}_i=[\underline{b}_i,\overline{b}_i]\in D[0,1]$ for $i=1,2,3,\ldots$, then we define $rmax(\tilde{a}_i,\tilde{b}_i)=[max(\underline{a}_i,\underline{b}_i), max(\overline{a}_i,\overline{b}_i)]$ and $rmin(\tilde{a}_i,\tilde{b}_i)=[min(\underline{a}_i,\underline{b}_i), min(\overline{a}_i,\overline{b}_i)]$ and $rinf_i(\tilde{a}_i)=[\wedge_i\underline{a}_i,\wedge_i\overline{a}_i]$ and $rsup_i(\tilde{a}_i)=[\vee_i\underline{a}_i,\vee_i\overline{a}_i]$.

$(D[0,1],\le)$ is a complete lattice with $\wedge=rmin,\vee=rmax,\tilde{0}=[0,0]$ and $\tilde{1}=[1,1]$ being the least and the greatest element respectively.

Definition 2.4: An interval-valued fuzzy set (IVFS) defined on a non empty set *X* is an object having the form $\tilde{\mu}=\{x,[\underline{\mu}(x),\overline{\mu}(x)]\},\forall x\in X$ where $\underline{\mu}$ and $\overline{\mu}$ are two fuzzy sets in *X* such that $\underline{\mu}(x)\le\overline{\mu}(x)$ for all $x\in X$. Let $\tilde{\mu}(x)=[\underline{\mu}(x),\overline{\mu}(x)],\forall x\in X$. Then $\tilde{\mu}(x)\in D[0\,1],\forall x\in X$.

If $\tilde{\mu}$ and $\tilde{\nu}$ be two interval-valued fuzzy sets in *X*, then we define

- $\tilde{\mu}\subset\tilde{\nu} \Leftrightarrow$ for all $x\in X,\underline{\mu}(x)\le\underline{\nu}(x)$ and $\overline{\mu}(x)\le\overline{\nu}(x)$.
- $\tilde{\mu}=\tilde{\nu} \Leftrightarrow$ for all $x\in X,\underline{\mu}(x)=\underline{\nu}(x)$ and $\overline{\mu}(x)=\overline{\nu}(x)$.
- $(\tilde{\mu}\tilde{\cup}\tilde{\nu})(x)=\hat{\mu}(x)\vee\hat{\nu}(x)=[max\{\underline{\mu}(x),\underline{\nu}(x)\},max\{\overline{\mu}(x),\overline{\nu}(x)\}]$.
- $(\tilde{\mu}\tilde{\cap}\tilde{\nu})(x)=\hat{\mu}(x)\wedge\hat{\nu}(x)=[min\{\underline{\mu}(x),\underline{\nu}(x)\},min\{\overline{\mu}(x),\overline{\nu}(x)\}]$.
- $(\tilde{\mu}\times\tilde{\nu})(x,y)=\hat{\mu}(x)\wedge\hat{\nu}(y)=[min\{\underline{\mu}(x),\underline{\nu}(y)\},min\{\overline{\mu}(x),\overline{\nu}(y)\}]$.
- $\tilde{\mu}^c(x)=[1-\overline{\mu}(x),1-\underline{\mu}(x)]$.

Definition 2.5: Let $\tilde{\mu}$ be an interval-valued fuzzy set in *X*. Then for every $[0,0]\prec\tilde{t}\le[1,1]$, the crisp set $\tilde{\mu}_t=\{x\in X\mid\tilde{\mu}(x)\ge\tilde{t}\}$ is called the level subset of $\tilde{\mu}$.

Definition 2.6: An interval-valued fuzzy set $\tilde{\mu}$ in *CI*-algebra *X* is called an interval-valued fuzzy subalgebra of *X* if $\tilde{\mu}(x*y)\ge rmin\{\tilde{\mu}(x),\tilde{\mu}(y)\}$, for all $x,y\in X$.

Definition 2.7: An interval-valued fuzzy set $\tilde{\mu}$ in *CI*-algebra *X* is called an interval-valued fuzzy filter of *X* if

(i) $\tilde{\mu}(1)\ge\tilde{\mu}(x)$,

(ii) $\tilde{\mu}(y)\ge rmin\{\tilde{\mu}(x*y),\tilde{\mu}(x)\}$ for all $x,y\in X$.

Combining the notion of interval-valued fuzzy set and the fuzzy set Jun et al. introduced the concept of cubic set in [14] which is defined as follows:

Definition 2.8: Let X be a nonempty set. A cubic set A in X is a structure

$$A=\{<x,\tilde{\mu}_A(x),\nu_A(x)>\mid x\in X\},$$

which is briefly denoted by $A=(\tilde{\mu}_A,\nu_A)$ where $\tilde{\mu}_A=[\underline{\mu}_A,\overline{\mu}_A]$ is an IVFS in X and ν_A is a fuzzy set in X and it is written as $A=<\tilde{\mu}_A,\nu_A>$.

For two cubic sets $A=<\tilde{\mu}_A,\nu_A>$ and $B=<\tilde{\mu}_B,\nu_B>$ in X, we define

- $A\subseteq B\Leftrightarrow\tilde{\mu}_A\leq\tilde{\mu}_B$ and $\nu_A\geq\nu_B$
- $A=B\Leftrightarrow$ for all $x\in X$, $\tilde{\mu}_A(x)=\tilde{\mu}_B(x)$ and $\nu_A(x)=\nu_B(x)$
- $A\cap B=\{<x\mid(\tilde{\mu}_A\tilde{\cap}\tilde{\mu}_B)(x),(\nu_A\cup\nu_B)(x)>\mid x\in X\}$

where $(\tilde{\mu}_A\tilde{\cap}\tilde{\mu}_B)(x)=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_B(x)\}$ and $(\nu_A\cup\nu_B)(x)=max\{\nu_A(x),\nu_B(x)\}$

- $A\cup B=\{<x\mid(\tilde{\mu}_A\tilde{\cup}\tilde{\mu}_B)(x),(\nu_A\cap\nu_B)(x)>\mid x\in X\}$

where $(\tilde{\mu}_A\tilde{\cup}\tilde{\mu}_B)(x)=rmax\{\tilde{\mu}_A(x),\tilde{\mu}_B(x)\}$ and $(\nu_A\cap\nu_B)(x)=min\{\nu_A(x),\nu_B(x)\}$.

Definition 2.9: ([1]) Let $A=(\tilde{\mu}_A,\nu_A)$ be cubic set in X, where X is a *CI*-algebra, then the set A is cubic *CI* -subalgebra over the binary operator $*$ if it satisfies the following conditions:

$$(i)\ \tilde{\mu}_A(x*y)\geq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\},$$

$$(ii)\ \nu_A(x*y)\leq max\{\nu_A(x),\nu_A(y)\}\ \forall x,y\in X.$$

Example 2.1: Consider *CI*-algebra $X=\{1,a,b,c\}$ with the following Cayley table.

*	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X by

$$\tilde{\mu}_A=\begin{pmatrix}1 & a & b & c\\ [0.5,0.8] & [0.3,0.5] & [0.4,0.7] & [0.2,0.4]\end{pmatrix}$$

and
$$\nu_A=\begin{pmatrix}1 & a & b & c\\ 0.1 & 0.2 & 0.4 & 0.8\end{pmatrix}.$$

Then by routine calculations it can be easily verified that $A=(\tilde{\mu}_A,\nu_A)$ is a cubic subalgebra of X.

Definition 2.10: ([1]) Let $A=(\tilde{\mu}_A,\nu_A)$ be cubic set in X, where X is a *CI* -algebra, then the set A is called cubic filter of *CI*-algebra X over the binary operator `$*$' if it satisfies the following conditions:

$$(i)\tilde{\mu}_A(1)\geq\tilde{\mu}_A(x),$$

$$(ii)\nu_A(1)\leq\nu_A(x),$$

$$(iii)\tilde{\mu}_A(y)\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\},$$

$$(iv)\nu_A(y)\leq max\{\nu_A(x*y),\nu_A(x)\}\quad\forall\quad x,y\in X.$$

Example 2.2: Consider the *CI* -algebra X and define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X by

$$\tilde{\mu}_A=\begin{pmatrix}1 & a & b & c\\ [0.7,0.8] & [0.4,0.7] & [0.3,0.6] & [0.1,0.5]\end{pmatrix}$$

and $$\nu_A = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.3 & 0.5 & 0.8 \end{pmatrix}.$$

Then it is easy to verify that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic filter of *X*.

3.TRANSLATION AND MULTIPLICATION OF CUBIC SUBALGEBRAS AND CUBIC FILTERS OF *CI* – ALGEBRAS

In what follows, let *X* denotes a *CI*-algebra, and for any cubic set $A = (\tilde{\mu}_A, \nu_A)$ of *X*, let $\tau = inf\{\nu_A(x) \mid x \in X\}$ and $\tilde{T} = (\underline{T}, \overline{T})$ $(\underline{T} \le \overline{T}.)$ where $\overline{T} = 1 - sup\{\overline{\mu_A}(x) \mid x \in X.\}$

Definition 3.1: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of *X* and $0 \le \overline{\alpha} \le \overline{T}$ where $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in D[0, \overline{T}]$ and $\beta \in [0, \tau]$. An object of the form $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ is called an cubic $(\tilde{\alpha}, \beta)$-translation of *A* if it satisfies $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha}$ and $(\nu_A)^T_{\beta}(x) = \nu_A(x) - \beta$ for all $x \in X$.

Example 3.1: Consider *CI*-algebra *X* as in Example 2.1. Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in *X* by

$$\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.5, 0.8] & [0.3, 0.5] & [0.4, 0.7] & [0.2, 0.4] \end{pmatrix}$$

and $$\nu_A = \begin{pmatrix} 1 & a & b & c \\ 0.15 & 0.2 & 0.4 & 0.8 \end{pmatrix}.$$

Here $\overline{T} = 1 - sup\{\overline{\mu}_A(x) \mid x \in X\} = 1 - 0.8 = 0.2$ and $\tau = inf\{\nu_A(x) \mid x \in X\} = 0.15$. Let $\tilde{\alpha} = [0.1, 0.18] \in D[0, \overline{T}]$ and $\beta = 0.1 \in [0, \tau]$. Then the $(\tilde{\alpha}, \beta)$ translation $((\tilde{\mu}_A)^T_{[0.1,\,0.18]}, (\nu_A)^T_{0.1})$ of cubic set $A = (\tilde{\mu}_A, \nu_A)$ is given by

$$(\tilde{\mu}_A)^T_{[0.1,\,0.18]} = \begin{pmatrix} 1 & a & b & c \\ [0.6, 0.98] & [0.4, 0.68] & [0.5, 0.88] & [0.3, 0.58] \end{pmatrix}$$

and $$(\nu_A)^T_{0.1} = \begin{pmatrix} 1 & a & b & c \\ 0.05 & 0.1 & 0.3 & 0.7 \end{pmatrix}.$$

Definition 3.2: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic fuzzy subset of *X* and $\gamma \in (0,1]$. An object having the form $A^M_{\gamma} = ((\tilde{\mu}_A)^M_{\gamma}, (\nu_A)^M_{\gamma})$ is called a cubic γ-multiplication of *A* if $(\tilde{\mu}_A)^M_{\gamma}(x) = \tilde{\mu}_A.\gamma$ and $(\nu_A)^M_{\gamma} = \nu_A.\gamma$, for all $x \in X$.

Example 3.2: Consider a cubic fuzzy set $A = (\tilde{\mu}_A, \nu_A)$ in *X* as in Example 2.1. Let $\gamma = 0.6 \in (0,1]$. Then the γ-multiplication of $((\tilde{\mu}_A)^M_{0.6}, (\nu_A)^M_{0.6})$ cubic set $A = (\tilde{\mu}_A, \nu_A)$ is given by

$$(\tilde{\mu}_A)^M_{0.6} = \begin{pmatrix} 1 & a & b & c \\ [0.3, 0.48] & [0.18, 0.3] & [0.24, 0.42] & [0.12, 0.24] \end{pmatrix}$$

and $$(\nu_A)^M_{0.6} = \begin{pmatrix} 1 & a & b & c \\ 0.06 & 0.12 & 0.24 & 0.48 \end{pmatrix}.$$

Definition 3.2: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic fuzzy subset of *X* and $0 \le \overline{\alpha} \le \overline{T}$ where $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in D[0, \overline{T}]$, $\beta \in [0, \tau]$ and $\gamma \in (0,1]$. A mapping $A^{MT}_{(\tilde{\alpha},\beta;\gamma)} = ((\tilde{\mu}_A)^{MT}_{\tilde{\alpha};\gamma}, (\nu_A)^{MT}_{\beta;\gamma}) : X \to [D[0,1],[0,1]]$ is said to be a cubic magnified $(\tilde{\alpha}, \beta; \gamma)$ translation of $A = (\tilde{\mu}_A, \nu_A)$ if it satisfies $(\tilde{\mu}_A)^{MT}_{\tilde{\alpha};\gamma}(x) = \gamma\tilde{\mu}_A(x) + \tilde{\alpha}$ and $(\nu_A)^{MT}_{\beta;\gamma}(x) = \gamma\nu_A(x) - \gamma\beta$ for all $x \in X$.

Example 3.3: Consider a cubic fuzzy set $A=(\tilde{\mu}_A,\nu_A)$ in X as in Example 2.1. For this cubic set $\overline{T}=1-sup\{\overline{\mu}_A(x)\mid x\in X\}=1-0.8=0.2$ and $\tau=inf\{\nu_A(x)\mid x\in X\}=0.1$. Let $\tilde{\alpha}=[0.1,\ 0.2]\in D[0,\overline{T}]$, $\beta=0.1\in[0,\tau]$ and $\gamma=0.5\in(0,1]$. Then the cubic magnified $(\tilde{\alpha},\beta;\gamma)$ translation $A^{MT}_{([0.1,0.2],0.1;0.5)}$ of cubic set $A=(\tilde{\mu}_A,\nu_A)$ is given by

$$(\tilde{\mu}_A)^{MT}_{([0.1,\,0.2];0.5)}=\begin{pmatrix} 0 & a & b & c \\ [0.35,0.6] & [0.25,0.45] & [0.3,0.55] & [0.2,0.40] \end{pmatrix}$$

and
$$(\nu_A)^{MT}_{(0.1;0.5)}=\begin{pmatrix} 0 & a & b & c \\ 0.0 & 0.05 & 0.15 & 0.35 \end{pmatrix}.$$

4. TRANSLATIONS OF CUBIC SUBALGEBRAS

Theorem 4.1: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of *X* and let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. Then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic subalgebra of *X*.

Proof. Here $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ and let $x,y\in X$. Then

$$\begin{aligned}(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)&=\tilde{\mu}_A(x*y)+\tilde{\alpha}\\&\geq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}+\tilde{\alpha}\\&=rmin\{\tilde{\mu}_A(x)+\tilde{\alpha},\tilde{\mu}_A(y)+\tilde{\alpha}\}\\&=rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\\(\nu_A)^T_{\beta}(x*y)&=\nu_A(x*y)-\beta\\&\leq max\{\nu_A(x),\nu_A(y)\}-\beta\\&=max\{\nu_A(x)-\beta,\nu_A(y)-\beta\}\\&=max\{(\nu_A)^T_{\beta}(x),(\nu_A)^T_{\beta}(y)\}\quad\forall\quad x,y\in X.\end{aligned}$$

Therefore the cubic fuzzy $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic subalgebra of *X*.

Theorem 4.2: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subset of *X* such that the cubic $(\hat{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic subalgebra of *X* for some $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. Then $A=(\hat{\mu}_A,\nu_A)$ is a cubic subalgebra of X.

Proof. Here $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ is a cubic subalgebra of X and let $x,y\in X$, we have

$$\begin{aligned}\tilde{\mu}_A(x*y)+\tilde{\alpha}&=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)\\&\geq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\\&=rmin\{\tilde{\mu}_A(x)+\tilde{\alpha},\tilde{\mu}_A(y)+\tilde{\alpha}\}\\&=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}+\tilde{\alpha}\\\nu_A(x*y)-\beta&=(\nu_A)^T_{\beta}(x*y)\\&\leq max\{(\nu_A)^T_{\beta}(x),(\nu_A)^T_{\beta}(y)\}\\&=max\{\nu_A(x)-\beta,\nu_A(y)-\beta\}\\&=max\{\nu_A(x),\nu_A(y)\}-\beta\quad\forall\quad x,y\in X.\end{aligned}$$

which implies that $\tilde{\mu}_A(x*y)\geq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(x)\}$ and $\nu_A(x*y)\leq max\{\nu_A(x),\nu_A(x)\}$. Hence $A=(\tilde{\mu}_A,\nu_A)$ is a cubic subalgebra of X.

Theorem 4.3: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subset of *X* and $0\le\overline{\alpha}\le\overline{T}$ where $\tilde{\alpha}=[\underline{\alpha},\overline{\alpha}]\in D[0,\overline{T}]$, $\beta\in[0,\tau]$ and $\gamma\in(0,1]$. Then $A=(\tilde{\mu}_A,\nu_A)$ is cubic subalgebra of *X* iff $A^{MT}_{(\tilde{\alpha},\beta;\gamma)}$ is cubic subalgebra of *X*.

Proof. It follows from Theorem 4.1 and Theorem 4.3.

Theorem 4.4: The intersection and the union of any two $(\hat{\alpha},\beta)$ -translations of a cubic subalgebra of *X* is also a cubic subalgebra of *X*.

Proof. Let $A^T_{(\tilde{\alpha},\beta)}$ and $A^T_{(\tilde{\gamma},\delta)}$ be two fuzzy translatons a cubic subalgebra $A=(\hat{\mu}_A,\nu_A)$ of *X*, where $\tilde{\alpha},\tilde{\gamma}\in D[0,\overline{T}],\beta,\delta\in[0,\tau]$. Assume that $\tilde{\alpha}\le\tilde{\gamma}$ and $\beta\ge\delta$, then by Theorem 4.1 $A^T_{(\tilde{\alpha},\beta)}$ and $A^T_{(\tilde{\gamma},\delta)}$ are cubic subalgebras of *X*. Now

$$
\begin{aligned}
((\tilde{\mu}_A)^T_{\tilde{\alpha}}\tilde{\cup}(\tilde{\mu}_A)^T_{\tilde{\gamma}})(x) &= rmax\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\gamma}}(x)\}\\
&= rmax\{\tilde{\mu}_A(x)+\tilde{\alpha},\tilde{\mu}_A(x)+\tilde{\gamma}\}\\
&= rmax\{[\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\alpha}},\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\alpha}}],[\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\gamma}},\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\gamma}}]\}\\
&= [max\{\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\alpha}},\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\gamma}}\},max\{\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\alpha}},\overline{\hat{\mu}_A}(x)+\overline{\tilde{\gamma}}\}]\\
&= [\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\gamma}},\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\gamma}}]\\
&= \tilde{\mu}_A(x)+\tilde{\gamma}=(\tilde{\mu}_A)^T_{\tilde{\gamma}}(x).
\end{aligned}
$$

$$
\begin{aligned}
((\nu_A)^T_{\beta}\cup(\nu_A)^T_{\delta})(x) &= min\{(\nu_A)^T_{\beta}(x),(\nu_A)^T_{\delta}(x)\}\\
&= min\{\nu_A(x)-\beta,\nu_A(x)-\delta\}\\
&= \nu_A(x)-\beta\\
&= (\nu_A)^T_{\beta}(x)
\end{aligned}
$$

Also

$$
\begin{aligned}
((\tilde{\mu}_A)^T_{\tilde{\alpha}}\tilde{\cap}(\tilde{\mu}_A)^T_{\tilde{\gamma}})(x) &= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\gamma}}(x)\}\\
&= rmin\{\tilde{\mu}_A(x)+\tilde{\alpha},\tilde{\mu}_A(x)+\tilde{\gamma}\}\\
&= rmin\{[\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\alpha}},\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\alpha}}],[\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\gamma}},\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\gamma}}]\}\\
&= [min\{\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\alpha}},\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\gamma}}\},min\{\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\alpha}},\overline{\hat{\mu}_A}(x)+\overline{\tilde{\gamma}}\}]\\
&= [\underline{\tilde{\mu}_A}(x)+\underline{\tilde{\alpha}},\overline{\tilde{\mu}_A}(x)+\overline{\tilde{\alpha}}]\\
&= \tilde{\mu}_A(x)+\tilde{\alpha}=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)
\end{aligned}
$$

$$
\begin{aligned}
((\nu_A)^T_{\beta}\cap(\nu_A)^T_{\delta})(x) &= max\{(\nu_A)^T_{\beta}(x),(\nu_A)^T_{\delta}(x)\}\\
&= max\{\nu_A(x)-\beta,\nu_A(x)-\delta\}\\
&= \nu_A(x)-\delta\\
&= (\nu_A)^T_{\delta}(x)
\end{aligned}
$$

Definition 4.1: For two cubic sets $A=(\hat{\mu}_A,\nu_A)$ and $B=(\hat{\mu}_B,\nu_B)$ of *X*. We define $A\subseteq B\Leftrightarrow\tilde{\mu}_A\le\tilde{\mu}_B,\nu_A\ge\nu_B$ for all $x\in X$. Then we say that *B* is an cubic extension of *A*.

Definition 4.2. Let $A=(\tilde{\mu}_A,\nu_A)$ and $B=(\tilde{\mu}_B,\nu_B)$ be two cubic subsets of *X*. Then, *B* is called a cubic *S*-extension (subalgebra-extension) of *A* if the following assertions are valid:
(i) *B* is a cubic extension of *A*.
(ii) If *A* is a cubic subalgebra of *X*, then *B* is a cubic subalgebra of *X*.

Theorem 4.5: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subset of *X* and $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. Then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic *S*-extension of *A*.

Proof: Here $A=(\tilde{\mu}_A,\nu_A)$ and $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$. Now, $\tilde{\mu}_A(x)+\tilde{\alpha}=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$, which implies $\tilde{\mu}_A(x)\le(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$. Again $\nu_A(x)-\beta=(\nu_A)^T_{\beta}(x)$, which implies $\nu_A(x)\ge(\nu_A)^T_{\beta}(x)$. Therefore $A\subseteq A^T_{(\tilde{\alpha},\beta)}$. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic extension of *A*. Since *A* is a cubic subalgebra of *X*, therefore $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic subalgebra of *X*. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic fuzzy *S*-extension of *A*.

Remark 4.1: The converse of above theorem is not true as is seen by the following example.

Example 4.1: Consider *CI*-algebra *X* as in Example 2.1. Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in *X* by

$$\tilde{\mu}_A=\begin{pmatrix} 1 & a & b & c \\ [0.5,0.9] & [0.3,0.5] & [0.4,0.7] & [0.2,0.4] \end{pmatrix}$$

and
$$\nu_A=\begin{pmatrix} 1 & a & b & c \\ 0.1 & 0.3 & 0.4 & 0.8 \end{pmatrix}$$

Then it is easy to verify that *A* is cubic fuzzy subalgebra of *X*. Let $B=(\hat{\mu}_B,\nu_B)$ be a cubic fuzzy subset of *X* defined by

$$\tilde{\mu}_B=\begin{pmatrix} 1 & a & b & c \\ [0.6,0.91] & [0.33,0.57] & [0.44,0.8] & [0.25,0.49] \end{pmatrix}$$

and
$$\nu_B=\begin{pmatrix} 1 & a & b & c \\ 0.05 & 0.23 & 0.34 & 0.78 \end{pmatrix}.$$

Then $B=(\hat{\mu}_B,\nu_B)$ is an cubic *S*-extension of *A*. But it is not the cubic $(\tilde{\alpha},\beta)$ -translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ of *A* for all $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$.

Theorem 4.6: The intersection of any two cubic *S*-extensions of a cubic subalgebra of *X* is also a cubic *S*-extension of *X*.

Proof. Let $S=(\tilde{\mu}_S,\nu_S)$ and $T=(\tilde{\mu}_T,\nu_T)$ be two cubic *S*-extensions of a cubic subalgebra $A=(\tilde{\mu}_A,\nu_A)$. Therefore, we have $A\subseteq S\Leftrightarrow\tilde{\mu}_A\le\tilde{\mu}_S,\nu_A\ge\nu_S$ for all $x\in X$ and $A\subseteq T\Leftrightarrow\tilde{\mu}_A\le\tilde{\mu}_T,\nu_A\ge\nu_T$ for all $x\in X$. Since

$$(\tilde{\mu}_S\tilde{\cap}\tilde{\mu}_T)(x)=rmin\{\tilde{\mu}_S(x),\tilde{\mu}_T(x)\}=\begin{cases}\tilde{\mu}_T(x) & \text{if } \tilde{\mu}_S(x)\ge\tilde{\mu}_T(x)\\ \tilde{\mu}_S(x) & \text{if } \tilde{\mu}_T(x)\ge\tilde{\mu}_S(x)\end{cases}$$

$$(\nu_S\cap\nu_T)(x)=max\{\nu_S(x),\nu_T(x)\}=\begin{cases}\nu_S(x) & \text{if } \nu_S(x)\ge\nu_T(x)\\ \nu_T(x) & \text{if } \nu_T(x)\ge\nu_S(x)\end{cases}$$

Hence, $\tilde{\mu}_A(x)\le(\tilde{\mu}_S\tilde{\cap}\tilde{\mu}_T)(x)$ and $\nu_A(x)\ge(\nu_S\cap\nu_T)(x)$ for all $x\in X$.

Therefore $S\cap T$ is a cubic *S*-extension of *X*.

Remark 4.2: The union of two cubic *S*-extensions of a cubic subalgebra of *X* may not a cubic *S*-extension of *X* as shown in the following example.

Example 4.2: Consider a *CI*-algebra $X=\{0,a,b,c,d\}$ with the following Cayley table.

*	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	b	1

Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in *X* by

$$\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.7,0.8] & [0.6,0.75] & [0.5,0.65] & [0.3,0.4] \end{pmatrix}$$

and
$$\nu_A = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.5 & 0.6 & 0.7 \end{pmatrix}.$$

Then it is easy to verify that A is a cubic fuzzy subalgebra of X. Let $S = (\tilde{\mu}_S, \nu_S)$ and $T = (\tilde{\mu}_T, \nu_T)$ be two cubic subsets of X defined by

$$\tilde{\mu}_S = \begin{pmatrix} 0 & a & b & c \\ [0.75,0.85] & [0.72,0.80] & [0.6,0.7] & [0.4,0.5] \end{pmatrix}$$

and
$$\nu_S = \begin{pmatrix} 1 & a & b & c \\ 0.15 & 0.3 & 0.5 & 0.6 \end{pmatrix}$$

$$\tilde{\mu}_T = \begin{pmatrix} 1 & a & b & c \\ [0.8,0.9] & [0.63,0.77] & [0.65,0.75] & [0.7,0.8] \end{pmatrix}$$

and
$$\nu_T = \begin{pmatrix} 1 & a & b & c \\ 0.1 & 0.5 & 0.4 & 0.3 \end{pmatrix}.$$

Now $S \cup T = (\tilde{\mu}_{(S\cup T)}, \nu_{(S\cup T)})$, where

$$\tilde{\mu}_{(S\cup T)} = \begin{pmatrix} 1 & a & b & c \\ [0.8,0.9] & [0.72,0.8] & [0.65,0.75] & [0.7,0.8] \end{pmatrix}$$

and
$$\nu_{(S\cup T)} = \begin{pmatrix} 0 & a & b & c \\ 0.1 & 0.3 & 0.4 & 0.3 \end{pmatrix}.$$

Then S and T are both cubic S-extension of A. Also $S \cup T$ is cubic extension of A .Also $S \cup T$ is cubic extension of A . But $S \cup T$ is not cubic S-extension of A. Since
$\tilde{\mu}_{(S\cup T)}(c*a) = \tilde{\mu}_{(S\cup T)}(b) = [0.65, 0.75] \leq [0.7, 0.8] = rmin\{[0.72, 0.80], [[0.7, 0.8]\}$
$= rmin\{\tilde{\mu}_{(S\cup T)}(a), \tilde{\mu}_{(S\cup T)}(c)\}$ and $\nu_{(S\cup T)}(c*a) = \nu_{(S\cup T)}(b) = 0.4 \not\leq 0.3 = max\{0.3, 0.3\}$
$= max\{\nu_{(S\cup T)}(a), \nu_{(S\cup T)}(c)\}$.

Definition 4.3: For a cubic fuzzy subset $A = (\tilde{\mu}_A, \nu_A)$ of X, let $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$ with $\tilde{t} \geq \tilde{\alpha}$. Let

$$U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t}) = \{x \in X \mid \tilde{\mu}_A(x) \geq \tilde{t} - \tilde{\alpha}\},$$
$$L_\beta(\nu_A; s) = \{x \in X \mid \nu_A(x) \leq s + \beta\}.$$

Theorem 4.7: If A is a cubic subalgebra of X, then $U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $L_\beta(\nu_A; s)$ are subalgebra of X for all $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$.

Proof: Let $x, y \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$. Therefore $\tilde{\mu}_A(x) \geq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(y) \geq \tilde{t} - \tilde{\alpha}$. Now

$$\begin{aligned} \tilde{\mu}_A(x*y) &\geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \\ &\geq rmin\{\tilde{t} - \tilde{\alpha}, \tilde{t} - \tilde{\alpha}\} \\ &= \tilde{t} - \tilde{\alpha} \end{aligned}$$

$$\Rightarrow \tilde{\mu}_A(x*y) \geq \tilde{t} - \tilde{\alpha}$$

which implies $x*y \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$.

Let $x, y \in L_\beta(\nu_A; s)$. Therefore $\nu_A(x) \leq s + \beta$ and $\nu_A(y) \leq s + \beta$. Now

$$\begin{aligned} \nu_A(x*y) &\leq max\{\nu_A(x), \nu_A(y)\} \\ &\leq max\{s + \beta, s + \beta\} = s + \beta \end{aligned}$$

$\Rightarrow \nu_A(x*y) \le s+\beta$

which implies $x*y \in L_\beta(\nu_A;s)$.

Remark 4.3: In the above theorem if *A* is not a cubic subalgebra of *X*, then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are not subalgebra of *X* as seen in the following example.

Example 4.3: Consider *CI* -algebra *X* as in Example 2.1. Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in *X* by

$$\tilde{\mu}_A=\begin{pmatrix} 1 & a & b & c \\ [0.2,0.4] & [0.5,0.7] & [0.6,0.75] & [0.7,0.8] \end{pmatrix}$$

and
$$\nu_A=\begin{pmatrix} 1 & a & b & c \\ 0.6 & 0.3 & 0.4 & 0.5 \end{pmatrix}.$$

Since $\tilde{\mu}_A(a*a)=\tilde{\mu}_A(1)=[0.2,0.4]\le[0.5,0.7]=rmin\{\tilde{\mu}_A(a),\tilde{\mu}_A(a)\}$ and $\nu_A(b*b)=\nu_A(1)=0.6\not\le 0.4=max\{\nu_A(b),\nu_A(b)\}$. Therefore $A=(\tilde{\mu}_A,\nu_A)$ is not a cubic subalgebra of *X*. Let $\tilde{\alpha}=[0.1,0.2]\in D[0,0.2], \beta=0.25\in[0,0.3]$ and $\tilde{t}=[0.5,0.8], s=0.3$. Then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})=\{a,b,c\}$ and $L_\beta(\nu_A;s)=\{a,b,c\}$. Since $a*a=1\notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $a*a=1\notin L_\beta(\nu_A;s)$. Therefore both $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are not a subalgebra of *X*.

Theorem 4.8: For $\tilde{\alpha}\in D[0,\overline{T}], \beta\in[0,\tau]$ let $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ be the cubic $(\hat{\alpha},\beta)$ translation of $A=(\tilde{\mu}_A,\nu_A)$. Then, the following assertions are equivalent:

(i) $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ is a cubic subalgebra of *X*.

(ii) $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are subalgebra of *X* for $\tilde{t}\in Im(\tilde{\mu}_A), s\in Im(\nu_A)$ with $\tilde{t}\ge\hat{\alpha}$.

Proof: Assume that $A^T_{(\tilde{\alpha},\beta)}$ is a cubic subalgebra of *X*. Then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy subalgebra of *X* and $(\nu_A)^T_\beta$ is a doubt fuzzy subalgebra [5] of *X*. Let $x,y\in X$ such that $x,y\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $\tilde{t}\in Im(\tilde{\mu}_A)$ with $\tilde{t}\ge\hat{\alpha}$. Then $\tilde{\mu}_A(x)\ge\tilde{t}-\tilde{\alpha}$ and $\tilde{\mu}_A(y)\succ\tilde{t}-\tilde{\alpha}$. That is $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)=\tilde{\mu}_A(x)+\tilde{\alpha}\ge\tilde{t}$ and $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)=\tilde{\mu}_A(x)+\tilde{\alpha}\ge\tilde{t}$. Since $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy subalgebra of *X*, therefore, we have $\tilde{\mu}_A(x*y)+\tilde{\alpha}=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)\ge rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\ge\tilde{t}$, which implies $\tilde{\mu}_A(x*y)\ge\tilde{t}-\tilde{\alpha}$ so that $x*y\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$.

Again let $x,y\in X$ such that $x,y\in L_\beta(\nu_A;s)$ and $s\in Im(\nu_A)$. Then $\nu_A(x)\le s+\beta$ and $\nu_A(y)\le s+\beta$ i.e., $(\nu_A)^T_\beta(x)=\nu_A(x)-\beta\le s$ and $(\nu_A)^T_\beta(y)=\nu_A(x)-\beta\le s$. Since $(\nu_A)^T_\beta$ is a doubt fuzzy subalgebra of *X*, it follows that

$$\nu_A(x*y)-\beta=(\nu_A)^T_\beta(x*y)\le max\{(\nu_A)^T_\beta(x),(\nu_A)^T_\beta(y)\}\le s.$$

That is $\nu_A(x*y)\le s+\beta$. So that $x*y\in L_\beta(\nu_A;s)$. Therefore $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are subalgebras of *X*.

Conversely, suppose that $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L(\nu_A;s)$ are subalgebra of X for $\tilde{t}\in Im(\tilde{\mu}_A), s\in Im(\nu_A)$ with $\tilde{t}\ge\hat{\alpha}$. To prove that $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ is a cubic subalgebra of *X*. Suppose $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ is not a cubic subalgebra of *X*, then at least one of $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)\prec rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$ and $(\nu_A)^T_\beta(x*y)>max\{(\nu_A)^T_\beta(x),(\nu_A)^T_\beta(y)\}$ must hold. If $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)\prec rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$ holds, then choose $\tilde{t}\in D[0,1]$ such that such that

$$(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a*b)\prec\tilde{t}\le rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(b)\}$$

for some $a,b\in X$ then $\tilde{\mu}_A(a)\geq\tilde{t}-\tilde{\alpha}$ and $\tilde{\mu}_A(y)\geq\tilde{t}-\tilde{\alpha}$ but $\tilde{\mu}_A(a*b)\leq\tilde{t}-\tilde{\alpha}$. This shows that $a\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $b\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ but $a*b\notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. This is a contradiction, and therefore $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)\geq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$ for all $x,y\in X$.

Again if $(\nu_A)^T_\beta(x*y)>max\{(\nu_A)^T_\beta(x),(\nu_A)^T_\beta(y)\}$ holds, then there exist $c,d\in X$ such that $(\nu_A)^T_\beta(c*d)>\delta\geq max\{(\nu_A)^T_\beta(c),(\nu_A)^T_\beta(d)\}$ where $\delta\in[0,1]$. Then $\nu_A(c)\leq\delta+\beta$ and $\nu_A(d)\leq\delta+\beta$ but $\nu_A(c*d)\geq\delta+\beta$. Hence $c\in L_\beta(\nu_A;s)$ and $d\in L_\beta(\nu_A;s)$ but $c*d\notin L_\beta(\nu_A;s)$. This is impossible and therefore $(\nu_A)^T_\beta(x*y)\leq max\{(\nu_A)^T_\beta(x),(\nu_A)^T_\beta(y)\}$ for all $x,y\in X$. Consequently $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ is a cubic subalgebra of X.

Theorem 4.9: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of X and let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. If $\tilde{\alpha}\geq\tilde{\gamma},\beta\geq\delta$ then the cubic $(\tilde{\alpha},\beta)$ translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of *A* is a cubic *S*-extension of the cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}=((\tilde{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of *A*.

Proof: It is straightforward.

For every cubic subalgebra $A=(\tilde{\mu}_A,\nu_A)$ of X and $\hat{\gamma}\in D[0,\overline{T}],\delta\in[0,\tau]$, cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}=((\tilde{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of *A* is a cubic subalgebra of *X*. If $B=(\tilde{\mu}_B,\nu_B)$ is a cubic *S*-extension of $A^T_{(\tilde{\gamma},\delta)}$, then there exists $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ such that $\tilde{\alpha}\geq\tilde{\gamma},\beta\geq\delta$ and $A^T_{(\tilde{\alpha},\beta)}\subset B$ that is $\tilde{\mu}_B(x)\geq(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ and $\nu_B(x)\leq(\nu_A)^T_\beta$ for all $x\in X$. Hence, we have the following theorem.

Theorem 4.10: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of *X* and let $\tilde{\gamma}\in D[0,\overline{T}],\delta\in[0,\tau]$. For every cubic *S*-extension $B=(\tilde{\mu}_B,\nu_B)$ of the cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}=((\tilde{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of *A*, there exists $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ such that $\tilde{\alpha}\geq\tilde{\gamma},\beta\geq\delta$ and *B* is an cubic *S*-extension of the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of *A*.

Let us illustrate above Theorem with the help of the following example.

Example 4.4: Consider a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in *CI*- X as in Example 2.1,

$$\tilde{\mu}_A=\begin{pmatrix}1 & a & b & c\\ [0.6,0.7] & [0.15,0.25] & [0.4,0.5] & [0.3,0.4]\end{pmatrix}$$

and
$$\nu_A=\begin{pmatrix}1 & a & b & c\\ 0.2 & 0.7 & 0.4 & 0.6\end{pmatrix}.$$

Then it is easy to verify that $A=(\tilde{\mu}_A,\nu_A)$ is a cubic subalgebra of *X*. Here $\overline{T}=1-sup\{\overline{\mu}_A(x)\mid x\in X\}=1-0.7=0.3$ and $\tau=inf\{\nu_A(x)\mid x\in X\}=0.2$. Let $\tilde{\gamma}=[0.05,\ 0.1]\in D[0,\overline{T}]$ and $\beta=0.1\in[0,\tau]$. Then the $(\tilde{\gamma},\delta)$-translation $((\tilde{\mu}_A)^T_{[0.05,0.1]},(\nu_A)^T_{0.1})$ of the cubic set $A=(\tilde{\mu}_A,\nu_A)$ is given by

$$(\tilde{\mu}_A)^T_{[0.05,0.1]}=\begin{pmatrix}1 & a & b & c\\ [0.65,0.80] & [0.2,0.35] & [0.45,0.6] & [0.35,0.5]\end{pmatrix}$$

and
$$(\nu_A)^T_{0.1}=\begin{pmatrix}1 & a & b & c\\ 0.1 & 0.6 & 0.3 & 0.5\end{pmatrix}.$$

Let $B=(\tilde{\mu}_B,\nu_B)$ be a cubic subset of *X* defined by

$$\tilde{\mu}_B=\begin{pmatrix}1 & a & b & c\\ [0.75,0.91] & [0.35,0.50] & [0.51,0.75] & [0.45,0.61]\end{pmatrix}$$

and
$$\nu_B = \begin{pmatrix} 1 & a & b & c \\ 0.01 & 0.52 & 0.2 & 0.42 \end{pmatrix}$$

Then B is a cubic S-extension of $(\tilde{\gamma}, \delta)$-translation $A^T_{(\tilde{\gamma},\delta)}$ of A and it can be easily verified that B is a cubic subalgebra of X. But B is not a cubic $(\tilde{\alpha}, \beta)$-translation of A for all $\hat{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$. If we take $\hat{\alpha} = [0.1, 0.2]$ and $\beta = 0.15$ then $\tilde{\alpha} \geq \tilde{\gamma}, \beta \geq \delta$ and the cubic $(\tilde{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ of A is given by
$$(\tilde{\mu}_A)^T_{[0.1,0.2]} = \begin{pmatrix} 1 & a & b & c \\ [0.7,0.9] & [0.25,0.45] & [0.5,0.7] & [0.4,0.6] \end{pmatrix}$$
and
$$(\nu_A)^T_{0.15} = \begin{pmatrix} 1 & a & b & c \\ 0.05 & 0.55 & 0.25 & 0.45 \end{pmatrix}.$$
Since $\tilde{\mu}_B(x) \geq (\hat{\mu}_A)^T_{\tilde{\alpha}}(x)$ and $\nu_B(x) \leq (\nu_A)^T_{\beta}$, therefore $A^T_{(\tilde{\alpha},\beta)} \subseteq B$ i.e., B is an cubic S-extension of the cubic $(\tilde{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A.

Theorem 4.11: $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X, then the following assertions are equivalent:
(i) A is a cubic subalgebra of X.
(ii) For all $\gamma \in (0,1]$, A^M_γ is a cubic subalgebra of X.

Proof: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X. Therefore, $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.
$$\begin{aligned}(\tilde{\mu}_A)^M_\gamma(x * y) = \tilde{\mu}_A(x * y).\gamma &\geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}.\gamma \\ &= rmin\{\tilde{\mu}_A(x).\gamma, \tilde{\mu}_A(y).\gamma\} \\ &= rmin\{(\tilde{\mu}_A)^M_\gamma(x), (\tilde{\mu}_A)^M_\gamma(y)\}\end{aligned}$$
$$\begin{aligned}(\nu_A)^M_\gamma(x * y) = \nu_A(x * y).\gamma &\leq max\{\nu_A(x), \nu_A(y)\}.\gamma \\ &= max\{\nu_A(x).\gamma, \nu_A(y).\gamma\} \\ &= max\{\nu_A(x).\gamma, \nu_A(y).\gamma\} \\ &= max\{(\nu_A)^M_\gamma(x), (\nu_A)^M_\gamma(y)\}\end{aligned}$$
Hence A^M_γ is a cubic subalgebra of X.

Conversely let $\gamma \in (0,1]$ be such that $A^M_\gamma = ((\tilde{\mu}_A)^M_\gamma, (\nu_A)^M_\gamma)$ is a cubic subalgebra of X. Then for all $x, y \in X$, we have
$$\begin{aligned}\tilde{\mu}_A(x * y).\gamma = (\tilde{\mu}_A)^M_\gamma(x * y) &\geq rmin\{(\tilde{\mu}_A)^M_\gamma(x), (\tilde{\mu}_A)^M_\gamma(y)\} \\ &= rmin\{\tilde{\mu}_A(x).\gamma, \tilde{\mu}_A(y).\gamma\} \\ &= rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}.\gamma\end{aligned}$$
$$\begin{aligned}\nu_A(x * y).\gamma = (\nu_A)^M_\gamma(x * y) &\leq max\{(\nu_A)^M_\gamma(x), (\nu_A)^M_\gamma(y)\} \\ &= max\{\nu_A(x).\gamma, \nu_A(y).\gamma\} \\ &= max\{\nu_A(x), \nu_A(y)\}.\gamma\end{aligned}$$
Therefore $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$ since $\gamma \neq 0$. Hence, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X.

5. TRANSLATIONS OF CUBIC FILTERS

Theorem 5.1: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of *X* and let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. Then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic filter of *X*.

Proof. Here $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ and let $x,y\in X$. Then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(1)=\tilde{\mu}_A(1)+\tilde{\alpha}\geq\tilde{\mu}_A(x)+\tilde{\alpha}=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$ and $(\nu_A)^T_{\beta}(1)=\nu_A(1)-\beta\leq\nu_A(x)-\beta=(\nu_A)^T_{\beta}(x)$ for all $x\in X$. Also

$$
\begin{aligned}
(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)&=\tilde{\mu}_A(y)+\tilde{\alpha}\\
&\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}+\tilde{\alpha}\\
&=rmin\{\tilde{\mu}_A(x*y)+\tilde{\alpha},\tilde{\mu}_A(x)+\tilde{\alpha}\}\\
&=rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)\}\\
(\nu_A)^T_{\beta}(y)&=\nu_A(y)-\beta\\
&\leq max\{\nu_A(x*y),\nu_A(x)\}-\beta\\
&=max\{\nu_A(x*y)-\beta,\nu_A(x)-\beta\}\\
&=max\{(\nu_A)^T_{\beta}(x*y),(\nu_A)^T_{\beta}(x)\}\quad\text{for all}\quad x,y\in X.
\end{aligned}
$$

Therefore the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic filter of *X*.

Theorem 5.2: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subset of *X* such that the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic filter of *X* for some $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. Then $A=(\tilde{\mu}_A,\nu_A)$ is a cubic filter of *X*.

Proof: We have $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ and let $x,y\in X$, we have

$$
\begin{aligned}
\tilde{\mu}_A(1)+\tilde{\alpha}&=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(1)\\
&\geq(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)\\
&=\tilde{\mu}_A(x)+\tilde{\alpha}\\
\nu_A(1)-\beta&=(\nu_A)^T_{\beta}(1)\\
&\leq(\nu_A)^T_{\beta}(x)\\
&=\nu_A(x)-\beta\quad\text{for all}\quad x,y\in X,
\end{aligned}
$$

which implies that $\tilde{\mu}_A(1)\geq\tilde{\mu}_A(x)$ and $\nu_A(1)\leq\nu_A(x)$. Again

$$
\begin{aligned}
\tilde{\mu}_A(y)+\tilde{\alpha}&=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\\
&=rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)\}\\
&=rmin\{\hat{\mu}_A(x*y)+\hat{\alpha},\tilde{\mu}_A(x)+\hat{\alpha}\}\\
&\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}+\tilde{\alpha}\\
\nu_A(y)-\beta&=(\nu_A)^T_{\beta})(y)\\
&\leq max\{(\nu_A)^T_{\beta}(x*y),(\nu_A)^T_{\beta}(x)\}\\
&=max\{\nu_A(x*y)-\beta,\nu_A(x)-\beta\}\\
&=max\{\nu_A(x*y),\nu_A(x)\}-\beta\quad\text{for all}\quad x,y\in X,
\end{aligned}
$$

from where follows that $\tilde{\mu}_A(y)\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}$ and $\nu_A(y)\leq max\{\nu_A(x*y),\nu_A(x)\}$. Hence $A=(\tilde{\mu}_A,\nu_A)$ is a cubic filter of X.

Theorem 5.3: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subset of X and $0\le\overline{\alpha}\le\overline{T}$ where $\tilde{\alpha}=[\underline{\alpha},\overline{\alpha}]\in D[0,\overline{T}]$, $\beta\in[0,\tau]$ and $\gamma\in(0,1]$. Then $A=(\tilde{\mu}_A,\nu_A)$ is cubic filter of X iff $A^{MT}_{(\tilde{\alpha},\beta;\gamma)}$ is cubic filter of X.

Proof. It follows from Theorem 5.1 and Theorem 5.2.

Lemma 5.1: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of X. If $x\le y$, then $\tilde{\mu}_A(y)\ge\tilde{\mu}_A(x)$ and $\nu_A(y)\le\nu_A(x)$ that is $\tilde{\mu}_A$ is preserving and ν_A is order order-reversing.
Proof. Let $x,y\in X$ be such that $x\le y$. Then $x*y=1$ and so

$$\tilde{\mu}_A(y)\ge rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}=rmin\{\tilde{\mu}_A(1),\tilde{\mu}_A(x)\}=\tilde{\mu}_A(x)$$
$$\nu_A(y)\le max\{\nu_A(x*y),\nu_A(x)\}=max\{\nu_A(1),\nu_A(x)\}=\nu_A(x)$$

This completes the proof.

Theorem 5.4: Let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ and $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of X. If X be a BE-algebra, then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X.
Proof. Since X is a BE-algebra therefore $1=x*1=x*(y*y)=y*(x*y)$ for all $x,y\in X$ i.e $y\le x*y$. Hence, by Lemma 5.1, we have $\tilde{\mu}_A(x*y)\ge\tilde{\mu}_A(y)$ and $\nu_A(x*y)\le\nu_A(y)$. Now

$$\begin{aligned}(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y)&=\tilde{\mu}_A(x*y)+\tilde{\alpha}\\&\ge\tilde{\mu}_A(y)+\tilde{\alpha}\\&\ge rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}+\tilde{\alpha}\\&\ge rmin\{\tilde{\mu}_A(y),\tilde{\mu}_A(x)\}+\tilde{\alpha}\\&=rmin\{\tilde{\mu}_A(x)+\tilde{\alpha},\tilde{\mu}_A(y)+\tilde{\alpha}\}\\&=rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\end{aligned}$$

$$\begin{aligned}(\nu_A)^T_{\beta}(x*y)&=\nu_A(x*y)-\beta\\&\le\nu_A(y)-\beta\\&\le max\{\nu_A(x*y),\nu_A(x)\}-\beta\\&\le max\{\nu_A(y),\nu_A(x)\}-\beta\\&=max\{\nu_A(x)-\beta,\nu_A(y)-\beta\}\\&=max\{(\nu_A)^T_{\beta}(x),(\nu_A)^T_{\beta}(y)\}\quad\text{for all}\quad x,y\in X.\end{aligned}$$

Hence the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X.

Theorem 5.5: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of X such that the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ of A is a cubic filter of X for all $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. If $x*(y*z)=1$ for all $x,y,z\in X$, then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)\ge rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$ and $(\nu_A)^T_{\beta}(z)\le max\{(\nu_A)^T_{\beta}(x),(\nu_A)^T_{\beta}(y)\}$

Proof: Let $x,y,z\in X$, be such that $x*(y*z)=1$. Then,

$$\begin{aligned}(\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)&\ge rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y*z),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\\&\ge rmin\{rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*(y*z)),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)\},(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\\&=rmin\{rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(1),(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)\},(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}\end{aligned}$$

$$= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \quad \text{since} \quad (\tilde{\mu}_A)^T_{\tilde{\alpha}}(1) \geq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$$
$$= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$$

$$\begin{aligned}(\nu_A)^T_\beta(z) &\leq max\{(\nu_A)^T_\beta(y*z), (\nu_A)^T_\beta(y)\}\\ &\leq max\{max\{(\nu_A)^T_\beta(x*(y*z)), (\nu_A)^T_\beta(x)\}, (\nu_A)^T_\beta(y)\}\\ &\leq max\{max\{(\nu_A)^T_\beta(1), (\nu_A)^T_\beta(x)\}, (\nu_A)^T_\beta(y)\}\\ &= max\{(\nu_A)^T_\beta(x), (\nu_A)^T_\beta(y)\} \quad \text{since} \quad (\nu_A)^T_\beta(1) \leq (\nu_A)^T_\beta(x)\\ &= max\{(\nu_A)^T_\beta(x), (\nu_A)^T_\beta(y)\}.\end{aligned}$$

Theorem 5.6: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic filter of X such that the cubic $(\tilde{\alpha}, \beta)$ -translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ of *A* is a cubic filters of X for all $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$. Then

(i) $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*z) \geq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*(y*z)), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$
$(\nu_A)^T_\beta(x*z) \leq max\{(\nu_A)^T_\beta(x*(y*z)), (\nu_A)^T_\beta(y)\}$

(ii) $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \leq (\tilde{\mu}_A)^T_{\tilde{\alpha}}((x*y)*y)$
$(\nu_A)^T_\beta(x) \geq (\nu_A)^T_\beta((x*y)*y)$

(iii) $(\tilde{\mu}_A)^T_{\tilde{\alpha}}((a*(b*x))*x) \geq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(b)\}$
$(\nu_A)^T_\beta((a*(b*x))*x) \leq max\{(\nu_A)^T_\beta(a), (\nu_A)^T_\beta(b)\}$.

Theorem 5.7. The intersection and union of any two $(\hat{\alpha}, \beta)$ -translations of the cubic filters of *X* is also a cubic filter of *X*.

Proof: Same as Theorem 4.4.

Definition 5.1: Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be two cubic subsets of *X*. Then, *B* is called a cubic *F*-extension(Filter extension) of *A* if the following assertions are valid:
(i) *B* is a cubic extension of *A*.
(ii) If *A* is a cubic filter of *X*, then *B* is a cubic filter of *X*.

Theorem 5.8: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic filter of X and $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$. Then the cubic $(\tilde{\alpha}, \beta)$ -translation $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic *F*-extension of *A*.

Proof. Here $A = (\tilde{\mu}_A, \nu_A)$ and $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$. Now $\tilde{\mu}_A(x) + \tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$, which implies $\tilde{\mu}_A(x) \leq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$. Again $\nu_A(x) - \beta = (\nu_A)^T{}_\beta(x)$ which implies $\nu_A(x) \geq (\nu_A)^T_\beta(x)$. Therefore $A \subseteq A^T_{(\tilde{\alpha},\beta)}$. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic extension of *A*.

Since *A* is a cubic ideal of *X*, therefore $A^T_{(\tilde{\alpha},\beta)}$ of *A* is a cubic ideal of *X*. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic *F*-extension of *A*.

Remark 5.1: The converse of above theorem is not necessarily true as can be seen by the following example.

Example 5.1: Consider a *CI* -algebra *X* as in Example 2.1. Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in *X* by

$$\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.7, 0.8] & [0.4, 0.7] & [0.3, 0.6] & [0.1, 0.5] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.3 & 0.5 & 0.8 \end{pmatrix}.$$

Then $A=(\tilde{\mu}_A,\nu_A)$ is a cubic filter of *X*. Let $B=(\hat{\mu}_B,\nu_B)$ be an cubic subset of *X* defined by

$$\tilde{\mu}_B=\begin{pmatrix} 1 & a & b & c \\ [0.72,0.92] & [0.43,0.77] & [0.36,0.68] & [0.12,0.54] \end{pmatrix}$$

and
$$\nu_B=\begin{pmatrix} 1 & a & b & c \\ 0.18 & 0.27 & 0.4 & 0.63 \end{pmatrix}.$$

Then $B=(\hat{\mu}_B,\nu_B)$ is a cubic *F*-extension of *A*. But it is not the cubic $(\hat{\alpha},\beta)$ -translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ of *A* for all $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$.

Theorem 5.9: The intersection of any two cubic *F*-extensions of a cubic filter of *X* is also a cubic *F*-extension of *X*.

Proof: Same as Theorem 4.6.

Remark 5.2: The union of two cubic *F*-extensions of a cubic filter of *X* may not a cubic filter of *X*.

Definition 5.2: For a cubic fuzzy subset $A=(\tilde{\mu}_A,\nu_A)$ of *X*, let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ and with $\tilde{t}\geq\tilde{\alpha}$, suppose

$$U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})=\{x\in X\mid \mu_A\geq\tilde{t}-\tilde{\alpha}\}$$
$$L_{\beta}(\nu_A;s)=\{x\in X\mid \nu_A\leq s+\beta\}.$$

Theorem 5.10: If *A* is a cubic filter of *X*, then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are filters of *X* for all $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$.

Proof.: Let $x\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Therefore $\tilde{\mu}_A(x)\geq\tilde{t}-\tilde{\alpha}$. Now $\tilde{\mu}_A(1)\geq\tilde{\mu}_A(x)\geq\tilde{t}-\tilde{\alpha}\Rightarrow\tilde{\mu}_A(1)\geq\tilde{t}-\tilde{\alpha}$, which implies that $1\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Again let $x\in L_{\beta}(\nu_A;s)$. Therefore $\nu_A(x)\leq s+\beta$. Now $\nu_A(1)\leq\nu_A(x)\leq s+\beta\Rightarrow\nu_A(1)\leq s+\beta$, which implies that $1\in L_{\beta}(\nu_A;s)$.

Let $x*y,x\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Therefore $\mu_A(x*y)\geq\tilde{t}-\tilde{\alpha}$ and $\mu_A(x)\geq\tilde{t}-\tilde{\alpha}$. Now

$$\begin{aligned}\tilde{\mu}_A(y)&\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}\\&\geq rmin\{\tilde{t}-\tilde{\alpha},\tilde{t}-\tilde{\alpha}\}\\&=\tilde{t}-\tilde{\alpha}\end{aligned}$$
$$\Rightarrow\tilde{\mu}_A(y)\geq\tilde{t}-\tilde{\alpha},$$

which shows that $y\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$.

Now let $x*y,x\in L_{\beta}(\nu_A;s)$. Therefore $\nu_A(x*y)\leq s+\beta$ and $\nu_A(x)\leq s+\beta$. Thus

$$\begin{aligned}\nu_A(y)&\leq max\{\nu_A(x*y),\nu_A(x)\}\\&\leq max\{s+\beta,s+\beta\}=s+\beta\\&\Rightarrow\nu_A(y)\leq s+\beta\end{aligned}$$

whence follows that $y\in L_{\beta}(\nu_A;s)$. Hence $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are filters of X.

Remark 5.3: In above Theorem if *A* is not a cubic filter of *X*, then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are not filters of *X* as seen in the following example.

Example 5.2: Consider a *CI*-algebra X as in Example 2.1. Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in *X* by

$$\tilde{\mu}_A=\begin{pmatrix} 1 & a & b & c \\ [0.2,0.4] & [0.5,0.7] & [0.6,0.75] & [0.7,0.8] \end{pmatrix}$$

and
$$\nu_A=\begin{pmatrix} 0 & a & b & c \\ 0.6 & 0.3 & 0.4 & 0.5 \end{pmatrix}.$$

Since $\tilde{\mu}_A(a*a)=\tilde{\mu}_A(1)=[0.2,0.4]\leq[0.5,0.7]=rmin\{\tilde{\mu}_A(a),\tilde{\mu}_A(a)\}$ and $\nu_A(b*b)$

$= \nu_A(1) = 0.6 \geq 0.4 = max\{\nu_A(b), \nu_A(b)\}$. Therefore $A = (\tilde{\mu}_A, \nu_A)$ is not a cubic filter of *X*. Let $\tilde{\alpha} = [0.1, 0.15] \in D[0, 0.2], \beta = 0.25 \in [0, 0.3]$ and $\tilde{t} = [0.4, 0.6], s = 0.3,$. Then $U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t}) = \{a, b, c\}$ and $L_\beta(\nu_A; s) = \{a, b, c\}$. Since $a * a = 1 \notin U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $a * a = 1 \notin L_\beta(\nu_A; s)$. Therefore both $U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $L_\beta(\nu_A; s)$ are not filters of X.

Theorem 5.11: For $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$, let $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ be the cubic $(\tilde{\alpha}, \beta)$ -translation of $A = (\tilde{\mu}_A, \nu_A)$. Then the following assertions are equivalent:

(i) $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ is a cubic filter of *X*.

(ii) $U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $L_\beta(\nu_A; s)$ are filters of *X* for $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$ with $\tilde{t} \geq \hat{\alpha}$.

Proof: Assume that $A^T_{(\tilde{\alpha},\beta)}$ is a cubic filter of *X*. Then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy filter of *X* and $(\nu_A)^T_\beta$ is a doubt fuzzy filter of *X*. Let $x \in X$ such that $x \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $\tilde{t} \in Im(\tilde{\mu}_A)$ with $\tilde{t} \geq \hat{\alpha}$. Then $\tilde{\mu}_A(x) \geq \tilde{t} - \tilde{\alpha}$. That is $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha} \geq \tilde{t}$. Since $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy filter of *X*, therefore, we have $\tilde{\mu}_A(1) + \tilde{\alpha} \geq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(1) \geq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha} \geq \tilde{t}$. That is $\tilde{\mu}_A(1) \geq \tilde{t} - \tilde{\alpha}$, so that $1 \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$.

Let $x, y \in X$ such that $x * y, y \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $\tilde{t} \in Im(\tilde{\mu}_A)$ with $\tilde{t} \geq \tilde{\alpha}$. Then $\tilde{\mu}_A(x * y) \geq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(y) \geq \tilde{t} - \tilde{\alpha}$. That is $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x * y) = \tilde{\mu}_A(x * y) + \tilde{\alpha} \geq \tilde{t}$ and $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y) = \tilde{\mu}_A(y) + \tilde{\alpha} \geq \tilde{t}$. Since $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy filter of X, therefore, we have $\tilde{\mu}_A(x) + \tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \geq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x * y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \geq \tilde{t}$. That is $\tilde{\mu}_A(x) \geq \tilde{t} - \tilde{\alpha}$, so that $x \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$.

Again let $x \in X$ such that $x \in L_\beta(\nu_A; s)$ and $s \in Im(\nu_A)$. Then $\nu_A(x) \leq s + \beta$ i.e., $(\nu_A)^T_\beta(x) = \nu_A(x) - \beta \leq s$. Since $(\nu_A)^T_\beta$ is a doubt fuzzy filter of *X* it follows that $\nu_A(1) - \beta = (\nu_A)^T_\beta(1) \leq (\nu_A)^T_\beta(x) = \nu_A(x) - \beta \leq s$. That is $\nu_A(1) \leq s + \beta$, so that $1 \in L_\beta(\nu_A; s)$.

Again let $x, y \in X$ such that $x * y, y \in L_\beta(\nu_A; s)$ and $s \in Im(\nu_A)$. Then $\nu_A(x * y) \leq s + \beta$ and $\nu_A(y) \leq s + \beta$ i.e., $(\nu_A)^T_\beta(x * y) = \nu_A(x) - \beta \leq s$ and $(\nu_A)^T_\beta(y) = \nu_A(x) - \beta \leq s$. Since $(\nu_A)^T_\beta$ is a doubt fuzzy filter of *X*, it follows that $\nu_A(x) - \beta = (\nu_A)^T_\beta(x) \leq max\{(\nu_A)^T_\beta(x * y), (\nu_A)^T_\beta(y)\} \leq s \Rightarrow$ $\nu_A(x) \leq s + \beta$. So that $x * y \in L_\beta(\nu_A; s)$. Therefore $U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $L_\beta(\nu_A; s)$ are filters of X.

Conversely, $U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $L_\beta(\nu_A; s)$ are filters of *X* for $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$ with $\tilde{t} \geq \hat{\alpha}$. If possible, let $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ be not a cubic filter of *X* then there exists some $a \in X$ such that $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(1) < \tilde{t} \leq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(a)$ i.e., $\tilde{\mu}_A(1) + \tilde{\alpha} < \tilde{t} \leq \tilde{\mu}_A(a) + \tilde{\alpha}$ then $\tilde{\mu}_A(a) \geq \tilde{t} - \tilde{\alpha}$ but $\tilde{\mu}_A(1) < \tilde{t} - \tilde{\alpha}$. This shows that $a \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ but $1 \notin U_\alpha(\mu_A; \tilde{t})$. This is a contradiction and therefore we must have $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(1) \geq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$ for all $x, y \in X$. Again if there exist $a, b \in X$ such that $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a) < \tilde{t} \leq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a * b), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(b)\}$ then $\tilde{\mu}_A(a * b) \geq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(b) \geq \tilde{t} - \tilde{\alpha}$ but $\tilde{\mu}_A(a) \leq \tilde{t} - \tilde{\alpha}$. This shows that $a * b \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ and $b \in U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$ but $a \notin U_{\tilde{\alpha}}(\tilde{\mu}_A; \tilde{t})$. This is again a contradiction and therefore $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) > rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x * y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$ for all $x, y \in X$.

Again assume that there exists $c \in X$ such that $(\nu_A)^T_\beta(1) > \delta \geq (\nu_A)^T_\beta(c)$ Then $\nu_A(c) \leq \delta + \beta$ and $\nu_A(1) \geq \delta + \beta$. Hence $c \in L_\beta(\nu_A; s)$ but $1 \notin L_\beta(\nu_A; s)$. This is impossible and therefore $(\nu_A)^T_\beta(x) \leq max\{(\nu_A)^T_\beta(x * y), (\nu_A)^T_\beta(y)\}$ for all $x, y \in X$.

Again assume that there exist $c,d\in X$ such that $(\nu_A)^T_\beta(c)>\delta\geq max\{(\nu_A)^T_\beta(c*d),(\nu_A)^T_\beta(d)\}$. Then $\nu_A(c*d)\leq\delta+\beta$ and $\nu_A(d)\leq\delta+\beta$ but $\nu_A(c)\geq\delta+\beta$. Hence $c*d\in L_\beta(\nu_A;s)$ and $d\in L_\beta(\nu_A;s)$ but $c\notin L_\beta(\nu_A;s)$. This is impossible, therefore, $(\nu_A)^T_\beta(x)\leq max\{(\nu_A)^T_\beta(x*y),(\nu_A)^T_\beta(y)\}$ for all $x,y\in X$. Consequently $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ is a cubic filter of *X*.

Theorem 5.12: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of *X* and let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. If $\tilde{\alpha}\geq\tilde{\gamma},\beta\geq\delta$ then the cubic $(\tilde{\alpha},\beta)$ translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of *A* is a cubic *F*-extension of the cubic $(\hat{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}=((\hat{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of *A*.

Proof: It is straightforward.

For every fuzzy ideal $A=(\tilde{\mu}_A,\nu_A)$ of *X* and $\hat{\gamma}\in D[0,\overline{T}],\delta\in[0,\tau]$ the cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}=((\hat{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of *A* is a cubic filter of *X*. If $B=(\tilde{\mu}_B,\nu_B)$ is an cubic *F*-extension of $A^T_{(\tilde{\gamma},\delta)}$, then there exists $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ such that $\tilde{\alpha}\geq\tilde{\gamma},\beta\geq\delta$ and $A^T_{(\tilde{\alpha},\beta)}\subset B$ that is $\tilde{\mu}_B(x)\geq(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$ and $\nu_B(x)\leq(\nu_A)^T_\beta(x)$ for all $x\in X$. Hence, we have the following theorem.

Theorem 5.13: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of *X* and let $\tilde{\gamma}\in D[0,\overline{T}],\delta\in[0,\tau]$. For every cubic *F*-extension $B=(\tilde{\mu}_B,\nu_B)$ of the cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}=((\hat{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of *A* there exists $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ such that $\tilde{\alpha}\geq\tilde{\gamma},\beta\geq\delta$ and *B* is a cubic *F*-extension of the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of *A*.

Theorem 5.14: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of *X*, then the following assertions are equivalent:

(i) *A* is a cubic filter of *X*.

(ii) For all $\gamma\in[0,1]$, A^M_γ is a a cubic filter of *X*.

Proof: Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic filter of *X*. Therefore we have $\tilde{\mu}_A(1)\geq\tilde{\mu}_A(x)$, $\nu_A(1)\leq\nu_A(x)$, $\tilde{\mu}_A(y)\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}$ and $\nu_A(y)\leq max\{\nu_A(x*y),\nu_A(x)\}$ for all $x,y\in X$. Now

$$(\tilde{\mu}_A)^M_\gamma(1)=\tilde{\mu}_A(1).\gamma\geq\tilde{\mu}_A(x).\gamma=(\tilde{\mu}_A)^M_\gamma(x)$$
$$(\nu_A)^M_\gamma(1)=\nu_A(1).\gamma\leq\nu_A(x).\gamma=(\nu_A)^M_\gamma(x)$$
$$\begin{aligned}(\tilde{\mu}_A)^M_\gamma(y)=\tilde{\mu}_A(y).\gamma&\geq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(x)\}.\gamma\\&=rmin\{\tilde{\mu}_A(x*y).\gamma,\tilde{\mu}_A(x).\gamma\}\\&=rmin\{(\tilde{\mu}_A)^M_\gamma(x*y),(\tilde{\mu}_A)^M_\gamma(x)\}\end{aligned}$$
$$\begin{aligned}(\nu_A)^M_\gamma(y)=\nu_A(x).\gamma&\leq max\{\nu_A(x*y),\nu_A(x)\}.\gamma\\&=max\{\nu_A(x*y).\gamma,\nu_A(x).\gamma\}\\&=max\{(\nu_A)^M_\gamma(x*y),(\nu_A)^M_\gamma(x)\}.\end{aligned}$$

Conversely, let $\gamma\in[0,1]$ be such that $A^M_\gamma=((\tilde{\mu}_A)^M_\gamma,(\nu_A)^M_\gamma)$ be a cubic filter of *X*. Then, for all $x,y\in X$, we have

$$\tilde{\mu}_A(1).\gamma=(\tilde{\mu}_A)^M_\gamma(1)\geq(\tilde{\mu}_A)^M_\gamma(x)=\tilde{\mu}_A(x).\gamma$$
$$\nu_A(1).\gamma=(\nu_A)^M_\gamma(1)\leq(\nu_A)^M_\gamma(x)=\nu_A(x).\gamma$$

which implies that $\tilde{\mu}_A(1) \geq \tilde{\mu}_A(x)$ and $\nu_A(1) \leq \nu_A(x)$ for all $x \in X$ since $\gamma \neq 0$. Also,

$$\begin{aligned}\tilde{\mu}_A(y).\gamma = (\tilde{\mu}_A)^M_\gamma(y) &\geq rmin\{(\tilde{\mu}_A)^M_\gamma(x*y), (\tilde{\mu}_A)^M_\gamma(x)\}\\ &= rmin\{\tilde{\mu}_A(x*y).\gamma, \tilde{\mu}_A(x).\gamma\}\\ &= rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}.\gamma\\ \nu_A(y).\gamma = (\nu_A)^M_\gamma(y) &\leq max\{(\nu_A)^M_\gamma(x*y), (\nu_A)^M_\gamma(x)\}\\ &= max\{\nu_A(x*y).\gamma, \nu_A(x).\gamma\}\\ &= max\{\nu_A(x*y), \nu_A(x)\}.\gamma\end{aligned}$$

Therefore, $\tilde{\mu}_A(y) \geq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$ and $\nu_A(y) \leq max\{\nu_A(x*y), \nu_A(x)\}$ for all $x, y \in X$ since $\gamma \neq 0$. Hence, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic filter of *X*.

REFERENCES

[1]. Ahn, S.S., Kim, Y.H. and Ko, J. Mi. (2014). Cubic subalgebras and filters of CI-algebras, *Honam Mathematical Journal*, 36(1), 43-54.

[2]. Akram, M., Yaqoob, N. and Kavikumar J.(2014). Interval valued $(\tilde{\theta}, \tilde{\delta})$-fuzzy KU-ideals of KU-algebras, *Int. J. Pure Appl. Math.*, 92(3), 335-349.

[3]. Atanassov, K.T. and Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 31(1), 343-349.

[4]. Atanassov, K.T. (1994). Operators over interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 64, 159-174.

[5]. Barbhuiya, S.R. (2016). Fuzzy translations and Fuzzy multiplications of Interval-Valued Doubt Fuzzy Ideals of *BF*-Algebras, *Journal of Information and Optimization Sciences*, 37(1), 23-36.

[6]. Borzooei, R.A., Saeid, A.B., Rezaei, A. and Ameri, R. (2014). Anti fuzzy filters of *CI*-algebras, *Afr. Mat.*, 25(4), 1197-1210.

[7]. Dutta, T.K., Kar, S. and Purkait S. (2015). Interval-valued fuzzy prime and semiprime ideals of a hyper Semiring, *Annals of Fuzzy Mathematics and Informatics*, 9(2), 261-278.

[8]. Hu, Q.P. and Li, X. (1983). On BCH-algebras, *Math. Seminar Notes*, 11(2) part 2, 313-320.

[9]. Imai, Y. and Iseki, K. (1966). On Axiom systems of Propositional Calculi XIV, *Proc. Japan Academy*, 42, 19-22.

[10]. Iseki, K. (1966). An algebra related with a Propositional Calculus (1966), *Proc. Japan Academy*, 42, 26-29.

[11]. Iseki, K. (1975). On some ideals in BCK-algebras, *Math.Seminar Notes*, 3, 65-70.

[12]. Jun, Y.B. (2000). Interval-valued fuzzy subalgebras/ideals in BCK-algebras, *Scientiae Mathematicae*, 3(3), 435-444.

[13]. Jun, Y.B. (2011).Translation of fuzzy ideals in BCK/BCI-Algebras, *Hacettepe Journal of Mathematics and Statistics*, 40(3), 349-358.

[14]. Jun, Y.B., Kim, C.S. and Yang, K.O. (2012). Cubic sets, *Ann. Fuzzy Math. Inform.*, 4(1), 83-98.

[15]. Jun, Y.B., Kim, C.S. and Kang, M.S.(2010). Cubic subalgebras and ideals of BCK/BCI-algebras, *Far East J. Math Sci.*, 44, 239-250.

[16]. Jun, Y.B., Lee, K.J. and Kang, M.S. (2011). Cubic structures applied to ideals of BCI-algebras, *Comput Math. Appl.*, 62(9), 3334-3342.

[17]. Jun, Y.B., Kim, C.S. and Kang, J.G. (2011). Cubic q-ideals of BCI-algebras, *Ann. Fuzzy Math Inform.* 1(1), 25-34.

[18]. Jun, Y.B., Jung, S.T. and Kim, M.S. (2011). Cubic subgroups, *Ann. Fuzzy Math Inform.*, 2(1), 9-15.

[19]. Klaua, D. (1965). Ãœber einen Ansatz zur mehrwertigen Mengenlehre, *Monatsberichte der Deutschen Akademie derWissenschaften zu Berlin*, 7, 859-867.

[20]. Klaua, D. (1966). Ãœber einen zweiten Ansatz zur mehrwertigen Mengenlehre, *Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin*, 8, 161-177.

[21]. Kim, K.H. (2011). A Note on *CI*- algebras, *International Mathematical Forum*, 6(1), 1-5.

[22]. Kim, K.H. and Yon, Y. H. (2007). Dual BCK-algebra and MV –algebra, *Sci. Math. Jpn.*, 66, 247-253.

[23]. Lee, K.J., Jun, Y.B. and Doh, M.I. (2009). Fuzzy translation and Fuzzy multiplications of BCK/BCI-algebras, *Commun. Korean Math.Soc.*, 24(3), 353-360.

[24]. Majumder, S.K. and Sardar, S.K. (2008). Fuzzy Magnified translation on groups, *Journal of Mathematics North Bengal University*, 1(2), 117-124.

[25]. Meng, B.L. (2009). *CI*-algebras, *Sci. Math. Japo. Online*. (e-2009) , 695-701.
[26]. Mostafa, S.M., Abdel Naby, M.A. and Elgendy, O.R. (2011). Fuzzy ideals in *CI*- algebras, *Journal of American Science*, 7(8), 485-488.
[27]. Mostafa, S.M., Abd-Elnaby, M.A. and Elgendy, O.R. (2011). Interval-valued fuzzy *KU*-ideals in *KU*-algebras. *Int.Math. Forum*, 6(64), 3151-3159.
[28]. Neggers, J. and Kim,H.S. (2002). On *B*-algebras, *Math.Vensik*. , 54, 21-29.
[29]. Neggers, J. and Kim,H.S. (1999). On *d*-algebras, *Math Slovaca*, 40(1), 19-26.
[30]. Rosenfeld, A. (1971). Fuzzy subgroups, *J. Math. Anal. Appl.*, 35, 512-517.
[31]. Saeid, A.B. (2006). Interval-valued fuzzy *BG*-algebras, *Kangweon-Kyungki Math.Jour.*, 14(2), 203-215.
[32]. Xi, O.G. (1991). Fuzzy *BCK* algebras, *Math. Japonica*, 36, 935-942.
[33]. Yaqoob, N., Mostafa, S.M. and Ansari, M.A. (2013). On cubic *KU*-ideals of *KU*-algebras, *ISRN Algebra*, Article ID 935905, 10 pages.
[34]. Zadeh, L.A. (1965). Fuzzy sets, *Information and Control*, 8, 338-353.
[35]. Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning-I, *Information Science*, 8, 199-249.