

ANGELIC PARACOMPACT SPACES

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Abstract:

In this paper, we introduce the new concept of angelic paracompact spaces and angelic on web paracompact spaces and discuss some of their properties. Further we prove that every relatively countably paracompact subset of $C_p(X)$ is relatively paracompact.

Keywords: Angelic spaces, Paracompact spaces, Web compact spaces, Angelic paracompact spaces, Web-paracompact spaces.

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1. INTRODUCTION

The first attempt to clarify the sequential behavior of the weakly compact subsets in a Banach Space was made by Smulian [11] and Phillips [10]. Their results are based on an argument of metrizable of weakly compact subsets (see Floret [5], pp.29-30). Smulian showed in [11] that a relatively compact subset is relatively sequentially compact for the weak topology of a Banach Space. He also proved that the concepts of relatively countably compact and relatively sequentially compact coincide if the weak*-dual is separable. Grothendieck [6] extended the spaces of continuous functions on compact spaces endowed with the pointwise convergence topology. Bourgain et al.'s notation of angelic space [1] and some of its consequences provide us with the necessary tools for providing those results in clear-cut way (see, Orihuela [9]).

In this paper all topological spaces are regular and Hausdorff. Following Bourgain et al. [1] we call a space X *angelic* if for every relatively countably compact subset A of X the following hold :

- (a) A is relatively compact.
- (b) If $x \in \bar{A}$, then there is a sequence in A that converges to x .

The introduction of the angelic space was a step in the study of compactness.

2. PRELIMINARIES

Definition 2.1: A relatively compact [9] subspace Y of a topological space X is a subset whose closure is compact.

Definition 2.2: A subset M of a topological space X is said to be *countably compact*[9] (briefly, NK), or *relatively countably compact*[9] (briefly, RNK), if every sequence in M has an adherent point in M , or in X respectively.

Definition 2.3: A topological space X is *paracompact* [8], if every open cover has a locally finite open refinement.

Definition 2.4: A topological space will be called a *web-compact space*[9] if there is a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X such that, if we denote by $C_{a_1, a_2, \dots, a_n} = \bigcup \{A_\beta : \beta \in \Sigma, \beta_j = a_j, j=1, 2, \dots, n\}$ for every $\alpha = (a_n)$ in Σ and n in $\mathbb{N}$, The following two conditions are satisfied:

- (i) $\overline{\bigcup \{A_\alpha : \alpha \in \Sigma\}} = X$,
- (ii) If $\alpha = (a_n) \in \Sigma$ and $x_n \in C_{a_1, a_2, \dots, a_n}$, $n=1, 2, \dots$ then the sequence (x_n) has an adherent point in X .

Definition 2.5: Let Z be a topological space, let X be a set and $A \subset Z^X$. If $(f_n) \subset A$ and $(x_m) \subset X$, it is said that (f_n) has the *interchangeable double limit property* [9] (in Z) with (x_m) if $\lim_n \lim_m f(x_m) = \lim_m \lim_n f_n(x_m)$ whenever all the limits involved exist.

Definition 2.6: *Angelic Lemma 5*: Let X and Y be topological spaces, let X be regular and let $\phi: X \rightarrow Y$ be continuous and injective. If $A \subset X$ is RNK and for all $B \subset \phi(A)$ the sequential closure of B is closed, that is, $\bar{B} = \{y \in Y : \text{there is } (y_n) \text{ in } B \text{ with } \lim_n y_n = y\}$, then $\phi(\bar{A})$ is closed in Y and the restriction of ϕ on $\bar{A}$ is a homeomorphism.

3. ANGELIC PARACOMPACT SPACES

Definition 3.1: Relatively Paracompact

A relatively paracompact subspace Y of a topological space X is a subset whose closure is compact and every open cover has a locally finite open refinement.

Definition 3.2: Relatively Countably Paracompact

A subset M of a topological space X is said to be relatively countably paracompact, if every sequence in M has an adherent point in X and every open cover has a locally finite open refinement.

Definition 3.3: Angelic Paracompact Space

A Hausdorff topological space X is called Angelic paracompact space if

- (i) Every relatively countably paracompact subset A of X is relatively paracompact.
- (ii) For any point x in the closure of a relatively paracompact subset A of X , there exists a sequence in A , converging to x .

Example 3.4: The space R^n is angelic paracompact.

Let $X = R^n$. Let $\mathcal{A}$ be an open covering of X , Let $B_0 = \emptyset$ and for each positive integer m . Let B_m denote the open ball of radius m centered at the origin.

Given m , let us choose finitely many elements of $\mathcal{A}$ cover $\bar{B}_m$ and intersect each one with open set $X - \bar{B}_{m-1}$. Let this finite collection of open sets be denoted by $\mathcal{C}_m$. Then the collection $\mathcal{C} = \bigcup \mathcal{C}_m$ is a refinement of $\mathcal{A}$. It is clearly locally finite, for the open set B_m intersects only finitely many elements of $\mathcal{C}$, namely those elements belonging to the collection $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$. Finally $\mathcal{C}$ covers X . For, given x , let m be the smallest integer such that $x \in \bar{B}_m$. Then $x \in \mathcal{C}_m$, by definition. Here every open set containing x contains at least one point of A . The point x is an adherent point for A iff x is in the closure of A . Hence every relatively countably paracompact subset of Y is relatively paracompact. For any point x in the closure of a relatively paracompact subset A of X , there exists a sequence in A , converging to x . Hence the space R^n is an angelic paracompact.

Theorem 3.5: Every angelic paracompact space is normal.

Proof: Let X be an angelic paracompact space and let A, B be closed (disjoint) sets in X . Since X is regular, for each point x of A , an open neighbourhood U_x can be chosen such that $\bar{U}_x \subseteq X - B$. Similarly, for each point $y \in X - A$, an open neighbourhood V_y can be chosen such that $\bar{V}_y \subseteq X - A$. Let $\mathcal{G}^1 = \{U_x / x \in A\} \cup \{V_y / y \in X - A\}$ be an open covering of X and X is paracompact. Hence $\mathcal{G}^1$ has a locally finite open refinement $\mathcal{G}$ (which is also an open covering of X). Let $\mathcal{G}$ be denoted by $\{G_\alpha / \alpha \in I\}$. Assume that $G_\alpha \cap A \neq \emptyset$. Since for $y \in X - A$, $\bar{V}_y \subseteq X - A$. That is $\bar{V}_y \cap A = \emptyset$, $G_\alpha \not\subseteq \bar{V}_y$ for any $y \in X - A$. Hence, since $\mathcal{G}$ is a refinement of $\mathcal{G}^1$, for some $x \in A$, $G_\alpha \subseteq U_x$. Further, since $\bar{U}_x \subseteq X - B$, $\bar{G}_\alpha \subseteq X - B$. Thus if $G_\alpha \cap A \neq \emptyset$, $\bar{G}_\alpha \subseteq X - B$. Now let $C = \{\alpha \in I / G_\alpha \cap A \neq \emptyset\}$. Since $\mathcal{G}$ is a covering of X , $\mathcal{G}$ covers A . By definition of C , then $\bigcup_{\alpha \in C} G_\alpha \supseteq A$. Since $\mathcal{G}$ is locally finite, $\bigcup_{\alpha \in C} \bar{G}_\alpha \subseteq X - B$. By the result, "If $\mathcal{M}$ is a locally finite system in a space S , then $\overline{\bigcup \mathcal{M}} = \bigcup \bar{\mathcal{M}}$ ", $\bigcup_{\alpha \in C} \bar{G}_\alpha = \overline{\bigcup_{\alpha \in C} G_\alpha}$ and consequently, $X - \bigcup_{\alpha \in C} \bar{G}_\alpha$ is open. Since $\bigcup_{\alpha \in C} \bar{G}_\alpha \subseteq X - B$, $X - \bigcup_{\alpha \in C} \bar{G}_\alpha \supseteq B$. Thus $\bigcup_{\alpha \in C} I$ and $X - \bigcup_{\alpha \in C} \bar{G}_\alpha$ are disjoint open sets and so X is normal.

Theorem 3.6: Every closed subspace of an angelic paracompact space is angelic paracompact.

Proof : Let Y be a closed subspace of the angelic paracompact space X . Then the subspace topology of Y is regular and Hausdorff. If $A \subseteq Y$ is relatively countably paracompact in Y , then A is relatively countably paracompact in X , so $\bar{A}$, the closure of A in X is relatively paracompact. Now if $x \in \bar{A}$, then there is a sequence $\{x_n\}_{n=1}^\infty$ in A converging to x . But $\{x_n\}_{n=1}^\infty$ must have a limit point in Y , since Y is closed. Thus $\bar{A} \subseteq Y$, which is a closure of A in Y . Hence A is relatively paracompact and any point of $\bar{A}$ is the limit of a sequence in A . Therefore Y is angelic paracompact.

Theorem 3.7: Let (Z, d) be an Angelic paracompact metric space and let A be a set of functions from X into Z . We suppose that for every $\alpha = (a_n) \in \Sigma$ and every sequence (x_n) in X , that is, eventually in every set $C_{a_1, a_2, \dots, a_n}$ for $n=1, 2, \dots$ we have for every sequence in A the interchangeable double limit property with (x_n) in Z . Then for every f in the closure of A in the product space Z^X there is a sequence (f_n) in A such that (f_n) converges pointwise to f on X .

Proof : Step I: Given functions $g_1, g_2, \dots, g_n$ in Z^X , $\varepsilon > 0$ and a subset C of X , there is a finite subset L of C such that $\min_{y \in L} \max_{k \leq n} \{d(g_k(x), g_k(y))\} \leq \varepsilon$ for every $x \in C$. Indeed, it is enough to consider the mapping G from C into Z^n defined by $G(x) = (g_1(x), g_2(x), \dots, g_n(x))$ and to use the compactness of Z^n .

Step II: The interchangeable double limit property together with step I will now enable us to find a sequence of functions (f_n) in A that converges pointwise to f on X . The idea is to construct a countable subset L in X together with a sequence of functions (f_m) in A satisfying $\lim_n f_m(y) = f(y)$ for every y belonging to L . We shall do it in such a way that for every x in X we shall have enough points of the countable set L 'close to x ' such that we shall obtain the convergence of the sequence (f_m) at the point x by iteration of limits.

Since S is countable there is a bijection $\psi: \mathbb{N} \rightarrow S$. For every positive integer n . Let D_n be equal to $C_{\psi(n)}$ and let f_1 be equal to f . By Step I there is a finite subset $L_1^1 \subset D_1$ such that $\min_{y \in L_1^1} \{d(f_1(x), f_1(y))\} \leq 1$ for every $x \in D_1$. But f is in the closure of A , so there is $f_2 \in A$ such that $\max_{y \in L_1^1} \{d(f_2(y), f(y))\} \leq 1/2$.

Proceeding by recurrence, for every positive integer n , we find finite subsets $L_n^i \subset D_i$ for $i \leq n$ and functions f_i , $i=1, 2, \dots, n+1$ such that $\min_{y \in L_n^i} \max_{k \leq n} \{d(f_k(x), f_k(y))\} \leq 1/n$ for every $x \in D_i$ and $\max \{d(f_{n+1}(y), f(y)) : y \in \bigcup \{L_j^i : i \leq j \leq n\}\} \leq 1/(n+1)$.

Step: III The sequence $\{f_n : n=1, 2, \dots\}$ selected above clearly satisfies $\lim_n f_n(y) = f(y)$ for every $y \in \bigcup \{L_j^i : i \leq j=1, 2, \dots\}$.

We are now going to see that $\lim_n f_n(x) = f(x)$ whatever x in X we take. Indeed, let us take $x \in X$ and $\alpha \in \Sigma, \alpha = (a_n)$, such that $x \in A_\alpha$. We set $P = \psi^{-1}(\{(a_1, a_2, \dots, a_n) : n=1, 2, \dots\})$, which is an infinite subset of positive integers because ψ is a bijection. The point x clearly belongs to every D_p for $p \in P$ and so, given $p \in P$ and $n \geq p$, by step II there is $y_{n,p} \in L_n^p$ such that $\max_{k \leq n} \{d(f_k(x), f_k(y_{n,p}))\} \leq 1/n$. we put $y_p = y_{p,p}$ and we have $\max_{k \leq p} \{d(f_k(x), f_k(y_p))\} \leq 1/p$. (1)

Since $y_p = y_{p,p} \in L_p^p \subset D_p$ for every $p \in P$, the sequence $\{y_p : p \in P\}$ is eventually in every $C_{a_1, a_2, \dots, a_n}$ for $n=1, 2, \dots$. Indeed, let m be a positive integer and P_j be equal to $\psi^{-1}((a_1, a_2, \dots, a_j))$ for $j=1, 2, \dots, m$. If $p \in P$ and $p \neq p_j$, $j=1, 2, \dots, m$, we have $y_p \in D_p = C_{a_1, a_2, \dots, a_{m_0}}$ with $m_0 > m$ and the conclusion follows when we bear in mind that $C_{a_1, a_2, \dots, a_{m_0}} \subset C_{a_1, a_2, \dots, a_m}$.

We shall now use the permutability of limits to conclude that $\lim_n f_n(x) = f(x)$. It will be enough to prove that $f(x)$ is the only adherent point of the sequence $(f_n(x))$ in Z , since Z is Angelic paracompact. Let y be an adherent point of $(f_n(x))$ and (n_k) be a strictly increasing sequence of positive integers such that $\lim_k f_{n_k}(x) = y$. Since P is infinite, it is a confinal subset in $\mathbf{N}$ and there is a strictly increasing sequence (P_j) in P . Property (1) ensures that $\lim_k f_k(y_{p_j}) = f_k(x)$ for $k=1, 2, \dots$. Thus we have $y = \lim_k f_{n_k}(x) = \lim_k \lim_j f_{n_k}(y_{p_j}) = \lim_j \lim_k f_{n_k}(y_{p_j}) = \lim_j f(y_{p_j}) = \lim_j f_i(y_{p_i}) = f_i(x) = f(x)$.

4. ANGELIC ON WEB – PARACOMPACT SPACES

Definition 4.1: An angelic space will be called a *Web-paracompact space* if there is a subset Σ of $\mathbf{N}^{\mathbf{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X such that, if we denote by $C_{a_1, a_2, \dots, a_n} = \bigcup \{A_\beta : \beta \in \Sigma, \beta = (b_m), b_j = a_j, j=1, 2, \dots, n\}$ for every $\alpha = (a_n)$ in Σ and n in $\mathbf{N}$, The following conditions are satisfied:

- (i) $\overline{U\{A_\alpha : \alpha \in \Sigma\}} = X$
- (ii) If $\alpha = (a_n) \in \Sigma$ and $x_n \in C_{a_1, a_2, \dots, a_n}$, $n=1, 2, \dots$ then the sequence (x_n) has a limit point in X .
- (iii) every open cover has a locally finite open refinement.

Theorem 4.2: Let X be a Web-paracompact space. The space $C_p(X)$ is angelic.

Proof : Let $\overline{\mathbf{R}}$ be the compactification of $\mathbf{R}$ with the two points $+\infty$ and $-\infty$. The inclusion mapping from $C_p(X, \mathbf{R})$ into $C_p(X, \overline{\mathbf{R}})$ is continuous and injective. The angelic lemma ensures that it is sufficient to prove the theorem for the space $C_p(X, Z)$, where Z is a compact metric space. Let $\{A_\alpha : \alpha \in \Sigma\}$ the family of subsets of X giving it a Web-paracompact structure. Let Y be equal to the union of the sets of the family $\{A_\alpha : \alpha \in \Sigma\}$. We consider the mapping $\phi : C_p(X, Z) \rightarrow Z^{\mathbf{N}}[T_p]$ defined by restriction on Y , $\phi(f) := f|_Y$, that is continuous and injective because of the density of Y in X . Let A be a relatively countably paracompact subset of $C_p(X, Z)$. Every sequence in A has the interchangeable double limit property with every sequence in X having limit point in X . Therefore $\phi(A)$ is a set of functions that satisfies the conditions of the Theorem 3.7 in $Z^{\mathbf{N}}$. Thus for any B contained in $\phi(A)$ we have that $\overline{B} = \{f \in Z^{\mathbf{N}} : \text{there is a sequence } (f_n) \text{ in } B \text{ with } \lim_n f_n = f \text{ in } T_p\}$.

By Angelic lemma, $\phi(\overline{A})$ is closed in $Z^{\mathbf{N}}$ and so paracompact and that the restriction of ϕ on $\overline{A}$ is a homeomorphism. Thus the closure of A in $C_p(X, Z)$ is paracompact. That is A is relatively paracompact and every point in $\overline{A}$ has a convergent sequence in A . Hence $C_p(X)$ is angelic.

Corollary 4.3: If X is a web paracompact space and Z is a metric space, the space $C_p(X, Z)$ of continuous functions from X into Z endowed with the pointwise convergence topology is an angelic space.

Proof: This follows from the Theorem 4.2.

Definition 4.4: Let X be an arbitrary topological space and let $\{X_i : i \in I\}$ be the family of all the subspaces of X which are web-paracompact. We shall say that X is a $\mathcal{W}_p$ -space, if any function from X into $\mathbf{R}$ is continuous if and only if its restriction on every X_i is continuous.

Theorem 4.5: Let X be a $\mathcal{W}_p$ -space. Then every relatively countably paracompact subset of $C_p(X)$ is relatively paracompact.

Proof: Let A be a relatively countably paracompact subset of $C_p(X)$. Then the closure B of A in $\mathbf{R}^X$ is paracompact and from Theorem 3.7, together with the fact of X being a $\mathcal{W}_p$ -space, ensures that B is contained in $C_p(X)$, and so A is relatively countably paracompact in $C_p(X)$.

5. CONCLUSION

Our main result includes Angelic paracompactness in topological spaces and we also deduced the property of angelic in web-paracompact spaces.

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