

DETOUR ECCENTRIC DOMINATION IN GRAPHS

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Abstract:

In this paper, the detour eccentric point set and detour eccentric dominating sets are defined. The detour eccentric domination numbers are also defined. The detour eccentric domination numbers for some standard graphs are determined. Also some theorems related to detour eccentric domination number are stated and proved.

Keywords: Dominating set, Domination number, Detour eccentric set, Detour eccentric dominating set, Detour eccentric domination number.

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1. INTRODUCTION

The domination in graphs starts from 1960. O. Ore [11] defined the terms dominating set and domination number of a graph in 1962. T.W. Haynes, S.T. Hedetniemi and P.J. Slater [7] discussed many dominating parameters. In 2010 T.N. Janakiraman, M. Bhanumathi and S. Muthammai [8] defined eccentric domination in graphs. M. Bhanumathi and S. Muthammai [3] studied eccentric domination in trees and various bounds of eccentric domination in graph. Detour domination number was discussed by J. John and N. Arianayagam [9] in the year 2017. Papers [8,9] are motivated us to study the detour eccentric domination in graphs.

In this paper, the concept of detour eccentric dominating set and its numbers are introduced. The detour eccentric domination numbers are obtained for various standard graphs. The bounds for detour eccentric domination number are established.

2. PRELIMINARIES

In this section some basic definitions such as graph, distance between two vertices in a graph, detour distance, detour eccentric distance, detour radius, detour diameter, central vertex, peripheral vertex, dominating set,

detour dominating set etc. are given. In this paper only non trivial simple connected graphs are considered and for all the other undefined terms we refer the interested reader to [4,5].

Definition 2.1:[7] For any graph $G = (V, E)$ the set of vertices are denoted by $V(G)$ and the set of edges denoted by $E(G)$. The *order* of G is the cardinality $|V|$ of the vertex set V and *size* of G is the cardinality $|E|$ of the edge set E . The *distance* $d(u, v)$ is the length of the shortest path joining between u and v . If $d(u, v) = 1$, then u and v are said to be *adjacent*.

Definition 2.2:[8] For any vertex u of G , the *eccentricity* of u is $e(u) = \max \{d(u, v) : v \in V\}$. A vertex v of G is called an *eccentric vertex* of u if $D(u, v) = e_D(u)$. The *radius* R and the *diameter* D of G are defined by $R = \text{rad}(G) = \min \{e(v) : v \in V\}$ and diameter D of G are defined by $D = \text{diam}(G) = \max \{e(v) : v \in V\}$. For any connected graph G , $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$. The vertex v in G is a *central vertex* if $e(v) = \text{rad}(G)$. The center $C(G)$ is the set of all central vertices. The central sub graph $\langle C(u) \rangle$ is induced by the center. v is a *peripheral vertex* if $e(v) = \text{diam}(G)$. The periphery $P(G)$ is the set of all peripheral vertices. The peripheral sub graph $\langle P(G) \rangle$ is induced by the periphery.

Definition 2.3: [10] Let G be a connected graph with vertices u and v , the *detour distance* $D(u, v)$ is the length of the longest $u - v$ path in G . For any vertex u of G , the *detour eccentricity* of u is $e_D(u) = \max \{D(u, v) : v \in V\}$. A vertex v of G is called a *detour eccentric vertex* of u such that $D(u, v) = e_D(u)$. The *detour radius* R_D and *detour diameter* D_D of G are defined by R_D and detour diameter D_D of G are defined by $R_D = \text{rad}_D(G) = \min \{e_D(v) : v \in V\}$ and $D_D = \text{diam}_D(G) = \max \{e_D(v) : v \in V\}$ respectively. The vertex v in G is a *detour central vertex* if $e_D(v) = \text{rad}_D(G)$. The *detour center* $C_D(G)$ is the set of all detour central vertices. The detour central sub graph $\langle C_D(G) \rangle$ is induced by the center. v is a *detour peripheral vertex* if $e_D(v) = \text{diam}_D(G)$. The *detour periphery* $P_D(G)$ is the set of all detour peripheral vertices. The detour peripheral sub graph $\langle P_D(G) \rangle$ is induced by the periphery.

Example 2.1: Consider the graph given in Fig.1, $D(v_1, v_4)$ is 5, $e_D(v_1) = e_D(v_2) = e_D(v_4) = e_D(v_5) = e_D(v_6) = 5$ and $e_D(v_3) = 4$. Hence $\text{rad}_D(G) = 4$, $\text{diam}_D(G) = 5$, $\{v_3\}$ is a central vertex and is called detour centre. The set $\{v_1, v_2, v_4, v_5, v_6\}$ is the set of detour peripheral vertices and is called detour periphery.

Definition 2.4: For a vertex v , each vertex at a detour distance $e_D(v)$ from v is a detour eccentric vertex of v . *Detour eccentric set* of a vertex v is defined as $E_D(v) = \{u \in V / D(u, v) = e_D(v)\}$.

Example 2.2: Consider the graph given in Fig.1, $E_D(v_1) = \{v_4\}$ and $E_D(v_3) = \{v_1, v_2, v_5, v_6\}$ are detour eccentric sets of v_1 and v_3 respectively.

Definition 2.5: The *neighborhood* $N(u)$ of a vertex u is the set of all vertices adjacent to u in G . $N[v] = N(v) \cup \{v\}$ is called the *closed neighborhood* of v .

Definition 2.6: [7] A set $D \subseteq V$ is said to be a *dominating set* in G , if every vertex in $V - D$ is adjacent to some vertex in D . The minimum cardinality taken over all the dominating sets is called the domination number of G and is denoted by $\gamma(G)$. γ -set is called the dominating set with minimum cardinality.

Definition 2.7: [10] A set $D \subseteq V$ is said to be a *detour dominating set* in G , if D is both a dominating set and a detour set. The minimum cardinality taken over all detour dominating sets is called the detour domination number of G and is denoted by $\gamma_D(G)$.

Result 1: [2] For $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil n/3 \rceil$.

Result 2: [10] For any connected graph G , $\text{rad}_D(G) \leq \text{diam}_D(G) \leq 2\text{rad}_D(G)$.

3. DETOUR ECCENTRIC DOMINATION

In this section we define the detour eccentric dominating set and its numbers and provide a suitable example. The detour eccentric point set and its numbers are also defined. The relations between the domination numbers, the detour dominating number and the detour eccentric domination numbers are obtained.

Definition 3.1: A dominating set $D \subseteq V$ of G is a *detour eccentric dominating set*, if for every $v \in V - D$, there exists at least one detour eccentric vertex u of v in D . A detour eccentric dominating set D is a *minimal detour eccentric dominating set* if there exists a subset $D' \subset D$ which is not a detour eccentric dominating set. The minimum cardinality of a minimal detour eccentric dominating set of D is called the *detour eccentric domination number* and is denoted by γ_{Ded} . The maximum cardinality of a minimal detour eccentric dominating set of D is called the *Upper detour eccentric domination number* and is denoted by $\Gamma_{Ded}(G)$.

Example 3.1: Consider the graph given in Fig. 1.

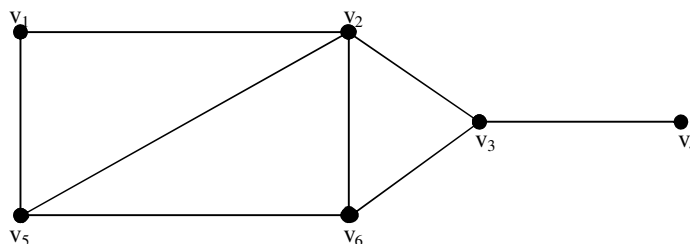


Figure 1:

In this graph, $E_D(v_1) = \{v_4\}$, $E_D(v_2) = \{v_4\}$, $E_D(v_3) = \{v_1, v_2, v_5, v_6\}$, $E_D(v_4) = \{v_1, v_2, v_5, v_6\}$, $E_D(v_5) = \{v_4\}$ and $E_D(v_6) = \{v_4\}$. Here $D_1 = \{v_2, v_3, v_4\}$, $D_2 = \{v_1, v_2, v_5, v_6\}$ etc., are all detour eccentric dominating sets. D_1 is a minimum detour eccentric dominating set but D_2 is a minimal detour eccentric dominating set. The detour eccentric domination number is $\gamma_{Ded}(G) = 3$ and the upper detour eccentric domination number is $\Gamma_{Ded} = 4$. $S = \{v_1, v_3\}$ is a minimum dominating set and the domination number is $\gamma(G) = 2$. $V - D_1 = \{v_1, v_5, v_6\}$ is not a detour eccentric dominating set, since detour eccentric vertex of v_2 is v_4 which is not in $V - D$.

Remark 3.1:

- (i) For any connected graph G , $\gamma(G) \leq \gamma_{Ded}(G)$.
- (ii) Every superset of a detour eccentric dominating set is a detour eccentric dominating set.
- (iii) For any connected graph, $\gamma_{Ded} \leq \Gamma_{Ded}$.
- (iv) The complement of a detour eccentric dominating set need not be a detour dominating set.
- (v) Every detour eccentric dominating set is a dominating set but the converse is not true.
- (vi) If $diam_D(G) = rad_D(G)$, then $\gamma_{Ded}(G) = \gamma(G)$.
- (vii) If a connected graph G has more than one pendent vertex, then the detour eccentric dominating set contains at least two pendent vertices.
- (viii) If G is a disconnected graph, then $\gamma_{Ded}(G) = \gamma(G)$.

Definition 3.2: Let $S \subseteq V(G)$. Then S is said to be a *detour eccentric point set* of G . If for every $v \in V - S$, S has at least one vertex u such that $u \in E(v)$. A detour eccentric point set S of G is a *minimal detour eccentric point set*, if no proper subset S' of S is a detour eccentric point set of G . The minimum cardinality of a minimal detour eccentric point set of S is called the *detour eccentric number* and is denoted by $e_{De}(G)$. Let D be a minimum dominating set of a graph G and S be a minimum detour eccentric point set of G . Then clearly $D \cup S$ is a *detour eccentric dominating set* of G .

Observation 3.1: For any connected graph, $\gamma_{Ded}(G) \leq \gamma(G) + e_{De}(G)$.

Observation 3.2: For any tree T with $|V(T)| \geq 3$, $\gamma_{Ded}(T) \leq n - \Delta(T) + 2$.

Observation 3.3: If G is disconnected then $\gamma(G) = \gamma_{Ded}(G)$, since vertices from different components are detouring eccentric to each other.

Observation 3.4: For any graph, $1 \leq \gamma_{Ded}(G) \leq n$.

The bounds are sharp, since $\gamma_{Ded}(G) = 1$ if and only if $G = K_n$ and $\gamma_{Ded}(G) = n$ if and only if $G = \overline{K_n}$.

4. BOUND ON DETOUR ECCENTRIC DOMINATION

In this chapter, the detour eccentric domination numbers are obtained for various standard graphs and theorems related to the detour eccentric domination numbers are stated and proved. The detour eccentric domination numbers of some standard classes of graphs are given in the following theorem.

Theorem 4.1:

- (i) $\gamma_{\text{Ded}}(K_n) = 1$.
- (ii) $\gamma_{\text{Ded}}(K_{1, n}) = 2, n \geq 2$.
- (iii) $\gamma_{\text{Ded}}(K_{m, n}) = 2$.
- (iv) $\gamma_{\text{Ded}}(W_n) = 2$, for $n \geq 4$.
- (v) $\gamma_{\text{Ded}}(C_n) = \lceil n/3 \rceil$ for $n \geq 3$, where $\lceil x \rceil$ is a least integer greater than x .

Proof :

- (i) If $G = K_n$ then $\text{rad}_D(G) = \text{diam}_D(G) = n - 1$. Hence any vertex $u \in V(G)$ dominates all other vertices and is also a detour eccentric point of other vertices. Hence, $\gamma_{\text{Ded}}(K_n) = 1$.
- (ii) If $G = K_{1, n}$ and let $D = \{u, v\}$ and v be a central vertex. Then the central vertex dominates all vertices which are in $V - D$ and u is a detour eccentric point of all vertices in $V - D$. Hence, $\{u, v\}$ is a minimum eccentric dominating set and $\gamma_{\text{Ded}}(K_{1, n}) = 2, n \geq 2$.
- (iii) If $G = (K_{m, n})$, $V(G) = V_1 \cup V_2$, $|V_1| = m$ and $|V_2| = n$ such that each element of V_1 is adjacent to every vertex of V_2 and vice versa. Let $D = \{u, v\}, u \in V_1$ and $v \in V_2$. D is a minimal dominating set and every singleton set $\{u\}$ is a detour set. Hence, D is a minimum detour eccentric set and hence $\gamma_{\text{Ded}}(K_{m, n}) = 2$.
- (iv) If $G = W_n$, for $n \geq 3$, and let u be a central vertex of G , then u is a detour eccentric point of all other vertices of G and $\{u\}$ is also a dominating set. Hence $\{u\}$ is a minimum detour eccentric dominating set. Therefore $\gamma_{\text{Ded}}(W_n) = 1$ for $n \geq 4$.
- (v) If $G = C_n$, for $n \geq 3$, each vertex u of C_n has exactly two detour eccentric vertices which are the neighbors of u and also dominates the neighbors. Hence the dominating set is also a detour eccentric dominating set. Therefore by the Result 1, $\gamma_{\text{Ded}}(C_n) = \gamma(G) = \lceil n/3 \rceil$.

Theorem 4.2:

A detour eccentric dominating set D is a minimal detour eccentric dominating set if and only if each vertex $u \in D$ one of the following conditions holds:

- (i) u is an isolated vertex of D or u has no detour eccentric vertex in D .
- (ii) There exists some $v \in V - D$ such that $N(v) \cap D = \{u\}$ or $E_D(v) \cap D = \{u\}$.

Proof : Assume that D is a minimal detour eccentric dominating set of G . Then for every vertex $u \in D$, $D - \{u\}$ is not a detour eccentric dominating set. That is there exists some vertex v in $(V - D) \cup \{u\}$ which is not dominated by any vertex in $D - \{u\}$ or there exists v in $(V - D) \cup \{u\}$ such that v has no detour eccentric vertex in $D - \{u\}$.

Case (i): Suppose $u = v$, then u is an isolate of D or u has no detour eccentric vertex in D .

Case (ii): Suppose $v \in V - D$.

- (a) If v is not dominated by $D - \{u\}$, but is dominated by D , then V is adjacent only to u in D , that is $N(V) \cap D = \{u\}$.
- (b) Suppose v has no detour eccentric point in $D - \{u\}$ but v has a detour eccentric point in D . Then u is the only detour eccentric point of v in D that is, $E_D(v) \cap D = \{u\}$.

Conversely, suppose that D is a detour eccentric dominating set and for each $u \in D$ one of the conditions holds, we show that D is a minimal detour eccentric dominating set.

Suppose that D is not a minimal detour eccentric dominating set, that is, there exists a vertex $u \in D$ such that $D - \{u\}$ is a detour eccentric dominating set. Hence, u is adjacent to at least one vertex v in $D - \{u\}$ and u has a detour eccentric vertex in $D - \{u\}$. Therefore, condition (i) does not hold. Also, if $D - \{u\}$ is a detour eccentric dominating set, every element v in $V - D$ is adjacent to at least one vertex in $D - \{u\}$ and v has a detour eccentric vertex in $D - \{u\}$.

Hence, condition (ii) does not hold. This is a contradiction to our assumption that for each $u \in D$, one of the conditions holds.

Theorem: 4.3:

Let n be an even integer. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{Ded}(G) = n/2$.

Proof: Let G be formed from a non trivial complete graph K_n . Then G has at least two components. By the observation 3.3, $(G) = \gamma_{Ded}(G)$. The domination number of G is $n/2$ since G has an even number of vertices. Hence $\gamma(G) = \gamma_{Ded}(G) = n/2$.

Theorem 4.4:

If G is of radius 2, with a unique central vertex u then $\gamma_{Ded}(G) \leq n - \deg(u)$.

Proof: G is of radius 2, with a unique central vertex u then u dominates $N(u)$ and the vertices $V - N(u)$ dominate themselves and each vertex in $N(u)$ has detour eccentric vertices in $V - N(u)$ only. Therefore, $V - N(u)$ is a detour eccentric dominating set of cardinality $n - \deg(u)$, so that $\gamma_{Ded}(G) \leq n - \deg(u)$.

Corollary 4.1: If G is a unicentral tree of detour radius 2, then $\gamma_{Ded}(G) \leq n - \deg(u)$, where u is the central vertex.

Theorem 4.5:

For a tree T , $\gamma(T) \leq \gamma_{Ded}(T) \leq \gamma(T) + 1$.

Proof:

By remark 3.1 (i) $\gamma(T) \leq \gamma_{Ded}(T)$. Let $Diam_D(T)$ be the detour diameter of T . Let $u, v \in V(T)$ such that $e_D(u) = e_D(v) = Diam$ and $D(u, v) = Diam$. Then u is a detour eccentric point of v and v is a detour eccentric point of u . Let D be any minimum dominating set of T . Then $D \cup \{u\}$ or $D \cup \{v\}$ is a detour eccentric dominating set of T . Hence, $\gamma_{Ded}(T) \leq \gamma(T) + 1$.

Remark 4.1: Every Tree contains at least two pendent (peripheral) vertices.

Theorem: 4.6:

For a tree T with radius greater than two, $\gamma_{Ded}(T) < n - \Delta(T) + 2$.

Proof: Let u be a vertex with maximum degree. Then every dominating set contains u and the domination number must be less than or equal to $n - \Delta$. Every Tree must contain at least two peripheral (pendent) vertices and by Remark 3.1 (vii) every detour eccentric dominating set contains at least two pendent vertices. Hence $\gamma_{Ded}(T) < n - \Delta(T) + 2$.

Theorem: 4.7:

Let T be a tree with diameter greater than two. $\gamma(T) = \gamma_{Ded}(T)$ if and only if T has a γ -set D containing at least two peripheral vertices at a detour distance D to each other.

Proof: Assume $\gamma(T) = \gamma_{Ded}(T)$. Let D be a detour eccentric dominating set with cardinality $\gamma(T) = \gamma_{Ded}(T)$. From the Remark 3.1 (vii) and the Remark 4.1, T has a γ -set D containing at least two peripheral vertices at a detour distance D to each other.

Conversely, assume that D is a γ -set of T containing at least two peripheral vertices u and v at a detour distance D to all other peripheral vertices. Every vertex $x \in V(T)$ has a detour eccentric vertex which is an element of a dominating set D of T . Hence, $\gamma(T) = \gamma_{Ded}(T)$.

Example 4.1: $\gamma(P_4) = 2 = \gamma_{Ded}(P_4)$.

Theorem 4.8:

$\gamma_{Ded}(P_n) = \lceil n/3 \rceil$, if $n = 3k + 1$
 $\gamma_{Ded}(P_n) = \lceil n/3 \rceil + 1$ if $n = 3k$ or $3k + 2$.

Proof: Case (i): $n = 3k$

A detour eccentric dominating set of P_n must contain one of the end vertices. Let $v_1, v_2, v_3, \dots, v_{3k}$ represent the path P_n . $D = \{v_2, v_5, v_8, \dots, v_{3k-1}\}$ is the only γ -set of P_n but not a detour eccentric dominating set. $D' = \{v_1, v_4, v_7, \dots, v_{3k-1}, v_{3k}\}$ is a detour eccentric dominating set since every dominating set contains at least two peripheral vertices and $|D'| = k + 1 = \gamma(P_n) + 1$. Hence, $\gamma_{Ded}(P_{3k}) + 1 = \lceil n/3 \rceil + 1$.

Case (ii): $n = 3k + 1$

$D = \{v_1, v_4, v_7, \dots, v_{3k-2}, v_{3k+1}\}$ is the minimum dominating set P_n . It contains two peripheral vertices. Hence, it is also a detour eccentric dominating set. Therefore, $\gamma_{Ded}(P_n) = \gamma_D(P_n) = \lceil n/3 \rceil$.

Case (iii): $n = 3k + 2$

$D = \{v_2, v_5, v_8, \dots, v_{3k+2}\}$ is a minimum dominating set. It contains end vertices v_{3k+2} and it is not a detour eccentric dominating set and all other minimum dominating sets are also not detour eccentric dominating set. Hence, $D \cup \{v_1\}$ is a minimum detour eccentric dominating set. Therefore, $\gamma_{\text{Ded}}(P_n) = \gamma_D(P_n) + 1 = \lceil n/3 \rceil + 1$.

Theorem: 4.9:

$$\gamma_{\text{Ded}}(\bar{c}_n) = 2, n > 3$$

Proof: If $G = \bar{c}_n$ for $n \geq 6$ for every vertex $u \in V$ of G such that $E_D(u) = \{v : v \in V - \{u\}\}$ and let w be any one the neighbor of u , then $\{u, w\}$ is a dominating set. Hence $\{u, w\}$ is a detour eccentric dominating set. Hence, $\gamma_{\text{Ded}}(\bar{c}_n) = 2$.

Remark 4.2: $\gamma_{\text{Ded}}(\bar{c}_3) = 3$, since $\bar{c}_3$ is totally disconnected graph.

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