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ALPHA BETA AND GAMMA PRODUCT OF FUZZY MATRICES

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Abstract:
In this paper, the basic properties of Alpha, Beta and Gamma product of fuzzy matrices are discussed.

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2010 Mathematics Subject Classification- 03E72, 05C72.

1. INTRODUCTION

In 1965 Lotfi A. Zadeh[8] introduced the notion of fuzzy set. Fuzzy Graph theory was introduced by Rosenfeld[6]. Alpha, Beta and Gamma Product of fuzzy graphs are defined by the authors [4,5]. The concept of Alpha, Beta and Gamma product of fuzzy graphs is finding applications in the determination of the intelligence of the students, combinatorics and networking, etc. as discussed by the authors in their works [4] and [5]. In this paper we discuss the basic properties of Alpha, Beta and Gamma product of fuzzy matrices by using MATLAB.

2. PRELIMINARIES

Definition 2.1: ([6]) A fuzzy graph is a pair $G:(\sigma,\mu)$ where σ is a fuzzy subset of V, μ is a symmetric fuzzy relation on σ. The elements of V are called the nodes (or) vertices of G and the pair of vertices as edges in G. A fuzzy graph $G:(\sigma,\mu)$ is a set with two functions $\sigma: V \to [0,1]$ and $\mu: E \to [0,1]$ such that $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

Definition 2.2: ([4]) The alpha product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \underset{\alpha}{\times} G_2 = (\sigma_1 \underset{\alpha}{\times} \sigma_2, \mu_1 \underset{\alpha}{\times} \mu_2)$ on $G^* = (V, E)$ where $V = V_1 \times V_2$

and $E = \left\{ ((u_1,u_2),(v_1,v_2)) \middle| \begin{array}{c} u_1 = v_1, u_2v_2 \in E_2 \text{ (Or)} \\ u_2 = v_2, u_1v_1 \in E_1 \text{ (Or)} \\ u_1v_1 \in E_1, u_2v_2 \notin E_2 \text{ (Or)} \\ u_1v_1 \notin E_1, u_2v_2 \in E_2 \end{array} \right\}$

with $\sigma_1 \underset{\alpha}{\times} \sigma_2(u_1,u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2), \forall (u_1,u_2) \in V_1 \times V_2$

$$\left(\mu_1 \underset{\alpha}{\times} \mu_2\right)((u_1,u_2),(v_1,v_2)) = \left\{ \begin{array}{ll} \sigma_1(u_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 = v_1, u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu_1(u_1v_1), & \text{if } u_2 = v_2, u_1v_1 \in E_1 \\ \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1v_1), & \text{if } u_1v_1 \in E_1, u_2v_2 \notin E_2 \\ \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2), & \text{if } u_1v_1 \notin E_1, u_2v_2 \in E_2 \end{array} \right\}$$

Definition 2.3: ([5]) The beta product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \underset{\beta}{\times} G_2 = (\sigma_1 \underset{\beta}{\times} \sigma_2, \mu_1 \underset{\beta}{\times} \mu_2)$ on $G^* = (V,E)$ where $V = V_1 \times V_2$ and

$E = \left\{ ((u_1,u_2),(v_1,v_2)) \middle| \begin{array}{c} u_1 \neq v_1, u_2v_2 \in E_2 \text{ (Or)} \\ u_2 \neq v_2, u_1v_1 \in E_1 \text{ (Or)} \\ u_1v_1 \in E_1, u_2v_2 \in E_2 \end{array} \right\}$

with $\sigma_1 \underset{\beta}{\times} \sigma_2(u_1,u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2), \forall (u_1,u_2) \in V_1 \times V_2$

$$\left(\mu_1 \underset{\beta}{\times} \mu_2\right)((u_1,u_2),(v_1,v_2)) = \left\{ \begin{array}{ll} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 \neq v_1, u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1v_1), & \text{if } u_2 \neq v_2, u_1v_1 \in E_1 \\ \mu_1(u_1v_1) \wedge \mu_2(u_2v_2), & \text{if } u_1v_1 \in E_1, u_2v_2 \in E_2 \end{array} \right\}$$

Definition 2.4: ([5]) The gamma product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G_1 \underset{\gamma}{\times} G_2$ $=(\sigma_1 \underset{\gamma}{\times} \sigma_2, \mu_1 \underset{\gamma}{\times} \mu_2)$ on $G^*:(V,E)$ where $V = V_1 \underset{\gamma}{\times} V_2$ and

$$E = \left\{ ((u_1,u_2),(v_1,v_2)) \middle| \begin{array}{c} u_1 = v_1, u_2v_2 \in E_2 \text{ (Or)} \\ u_2 = v_2, u_1v_1 \in E_1 \text{ (Or)} \\ u_1 \neq v_1, u_2v_2 \in E_2 \text{ (Or)} \\ u_2 \neq v_2, u_1v_1 \in E_1 \text{(Or)} \\ u_1v_1 \in E_1, u_2v_2 \in E_2 \end{array} \right\}$$

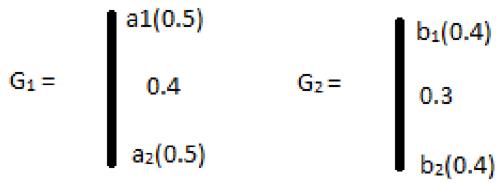
with $\sigma_1 \underset{\gamma}{\times} \sigma_2(u_1,u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$, $\forall (u_1,u_2) \in V_1 \underset{\gamma}{\times} V_2$

$$\left(\mu_1 \underset{\gamma}{\times} \mu_2\right)((u_1,u_2),(v_1,v_2)) = \left\{ \begin{array}{ll} \sigma_1(u_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 = v_1, u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu_1(u_1v_1), & \text{if } u_2 = v_2, u_1v_1 \in E_1 \\ \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 \neq v_1, u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1v_1), & \text{if } u_2 \neq v_2, u_1v_1 \in E_1 \\ \mu_1(u_1v_1) \wedge \mu_2(u_2v_2), & \text{if } u_1v_1 \in E_1, u_2v_2 \in E_2 \end{array} \right\}$$

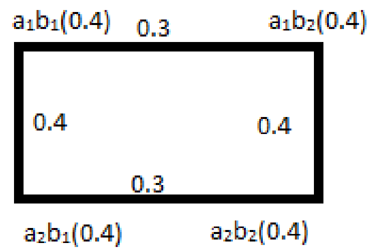
3. DEFINITIONS OF ALPHA, BETA AND GAMMA PRODUCT OF FUZZY MATRICES

Definition 3.1: Alpha Product of Fuzzy Matrix $A = \left[a_i b_j\right], i = 1,\ldots,n_1 \ j = 1,\ldots,n_2$ is a symmetric matrix which is defined by $[aibj] = \left\{ \begin{array}{l} \sigma_1 \times \sigma_2, i = j \\ \mu_1 \times \mu_2, i \neq j \end{array} \right\}$. Here n_1 and n_2 are number of vertices of the two Fuzzy Graphs.

Example:



Alpha Product of Two Fuzzy Graphs is

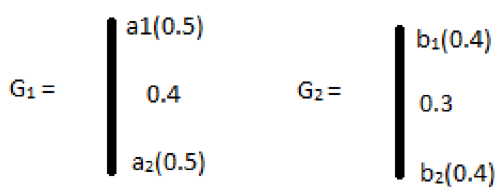


Alpha Product of Fuzzy Matrix is given by

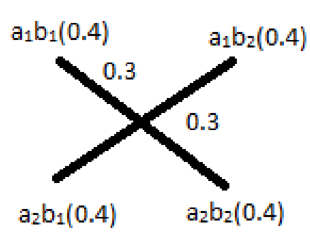
$$\begin{array}{c} \\ a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{array} \begin{array}{c} \begin{array}{cccc} a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2 \end{array} \\ \begin{bmatrix} 0.4 & 0.3 & 0.4 & 0 \\ 0.3 & 0.4 & 0 & 0.4 \\ 0.4 & 0 & 0.4 & 0.3 \\ 0 & 0.4 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

Definition 3.2: Beta Product of Fuzzy Matrix $A = \left[a_i b_j\right], i = 1,\ldots,n_1 \ \ j = 1,\ldots,n_2$ is a symmetric matrix which is defined by $\left[aibj\right] = \begin{Bmatrix} \sigma_1 \times \sigma_2, i = j \\ \mu_1 \times \mu_2, i \neq j \end{Bmatrix}$. Here n_1 and n_2 are number of vertices of the two Fuzzy Graphs.

Example:



Beta Product of Two Fuzzy Graphs is

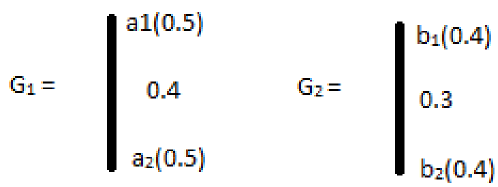


Beta Product of Fuzzy Matrix is given by

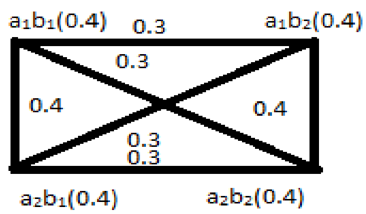
	a_1b_1	a_1b_2	a_2b_1	a_2b_2
a_1b_1	0.4	0	0	0.3
a_1b_2	0	0.4	0.3	0
a_2b_1	0	0.3	0.4	0
a_2b_2	0.3	0	0	0.4

Definition 3.3: Gamma Product of Fuzzy Matrix $A=\left[a_i b_j\right], i=1,\ldots,n_1 \ j=1,\ldots,n_2$ is a symmetric matrix which is defined by $\left[aibj\right]=\begin{Bmatrix}\sigma_1\times\sigma_2, i=j\\ \mu_1\times\mu_2, i\neq j\end{Bmatrix}$. Here n_1 and n_2 are number of vertices of the two Fuzzy Graphs.

Example:



Gamma Product of Two Fuzzy Graphs is



Gamma Product of Fuzzy Matrix is given by

	a_1b_1	a_1b_2	a_2b_1	a_2b_2
a_1b_1	0.4	0.3	0.4	0.3
a_1b_2	0.3	0.4	0.3	0.4
a_2b_1	0.4	0.3	0.4	0.3
a_2b_2	0.3	0.4	0.3	0.4

4. PROPERTIES OF ALPHA, BETA AND GAMMA PRODUCT OF FUZZY MATRICES

Property 4.1: All Alpha, Beta and Gamma Product of Fuzzy matrices are symmetric matrices.
Proof: Let A and B be two symmetric Matrices. That is $A=A^T$ and $B=B^T$.
To prove $(A\times B)^T=A\times B$. $(A\times B)^T=A^T\times B^T=A\times B$.

Property 4.2: Every Alpha, Beta and Gamma Product of Fuzzy matrices are non singular.

Proof: We know that an $n\times n$ matrix A is called non singular if the equation $Ax=0$ has only the zero solution $x=0$. By using MATLAB, the property of nonsingularity over Alpha, Beta and Gamma product of fuzzy matrices are clarified.

Example:

Consider the Alpha Product of Fuzzy Matrix,

```
>> A=[0.4 0.3 0.4 0;0.3 0.4 0 0.4;0.4 0 0.4 0.3;0 0.4 0.3 0.4];

>>syms x1 x2 x3 x4

>> X=[x1;x2;x3;x4];

>>eq=A*X==0;

>>solx=solve(eq);

>>x1=solx.x1

x1 = 0

>>x2=solx.x2

x2 =0

>>x3=solx.x3

x3 =0

>>x4=solx.x4

x4 =0
```

Consider the Beta product of Fuzzy matrix

```
>> A=[0.4 0 0 0.3;0 0.4 0.3 0;0 0.3 0.4 0;0.3 0 0 0.4];

>>null(A)

ans = Empty matrix:
```

$$\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Property 4.3: Every Alpha, Beta and Gamma Product of fuzzy matrices are invertible.

Proof: We know that by the Invertible matrix theorem, if A be an $n\times n$ matrix with real elements. Then A is invertible if and only if $AX=0$ has the only zero solution $x=0$. From property 2 it is clear that for alpha, beta and gamma product, $AX=0$ has a zero solution . Hence every Alpha, Beta and Gamma Product of fuzzy matrices are invertible. Another proof of this property is that since as the Alpha, Beta and Gamma product of fuzzy matrices are non singular, hence $|A\times B|\neq 0$.Thus Alpha, Beta and Gamma product of fuzzy matrices are invertible.

Example:

Consider theAlpha product of fuzzy matrix

```
>> A=[0.4 0.3 0.4 0;0.3 0.4 0 0.4;0.4 0 0.4 0.3;0 0.4 0.3 0.4];

>> u=det(A)
```

u = -0.0495

Consider the Beta product of fuzzy matrix

```
>> A=[0.4 0 0 0.3;0 0.4 0.3 0;0 0.3 0.4 0;0.3 0 0 0.4];
>> U=det(A)
U = 0.0049
```

Consider the Gamma product of fuzzy matrix

```
>> A=[0.5 0.3 0.4 0.3;0.3 0.5 0.3 0.4;0.4 0.3 0.5 0.3;0.3 0.4 0.3 0.5];
>> U=det(A)
U = 0.0045
```

Property 4.4: Beta and Gamma product of fuzzy matrices are singular if $\sigma_1 \times \sigma_2(a_i b_j)$ and $\mu_1 \times \mu_2(a_i b_j, a_i b_j)$ are constant.

Proof: Let $G_1 \times G_2$ be the Gamma product of fuzzy graph having 4 vertices and 4 edges. If all the vertices and edges of $G_1 \times G_2$ having the membership value 'a'. Then the Beta and Gamma products of the fuzzy matrices are given by

Beta product of matrix

$$\begin{bmatrix} a & 0 & 0 & a \\ 0 & a & a & 0 \\ a & 0 & 0 & a \\ 0 & a & a & 0 \end{bmatrix}$$

Gamma Product of matrix

$$\begin{bmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{bmatrix}$$

Here we see that the rows of the two matrices are identical . Hence we get the result.

Corollary: If the Gamma product of fuzzy graph is a regular fuzzy graph having constant vertices then the Gamma product of fuzzy matrix is singular .

Example:

Consider the Beta product of fuzzy matrix

```
>> A=[0.3 0 0 0.3;0 0.3 0.3 0;0 0.3 0.3 0;0.3 0 0 0.3];
>> U=det(A)
U = 0
```

Consider the Gamma Product of fuzzy matrix

```
>> A=[0.4 0.3 0.4 0.3;0.3 0.4 0.3 0.4;0.4 0.3 0.4 0.3;0.3 0.4 0.3 0.4];
>> U=det(A)
U = 0
```

Property 4.5: All Alpha, Beta and Gamma products of Fuzzy matrices are linearly independent.

Proof: Let A be any $n\times n$ square matrix. A linear relation between the vectors can be written as

$$A\begin{bmatrix} a_1 \\ a_2 \\ - \\ - \\ a_n \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & - & - & z_1 \\ x_2 & y_2 & - & - & z_2 \\ - & - & - & - & - \\ - & - & - & - & - \\ x_n & y_n & - & - & z_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ - \\ - \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ - \\ - \\ 0 \end{bmatrix}$$

Now multiply both sides by Adjugate of A,we get

$$\mathrm{Adj}(A)A\begin{bmatrix} a_1 \\ a_2 \\ - \\ - \\ a_n \end{bmatrix} = \mathrm{Adj}(A)\begin{bmatrix} x_1 & y_1 & - & - & z_1 \\ x_2 & y_2 & - & - & z_2 \\ - & - & - & - & - \\ - & - & - & - & - \\ x_n & y_n & - & - & z_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ - \\ - \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ - \\ - \\ 0 \end{bmatrix}$$

$$\mathrm{Det}(A)I_3\begin{bmatrix} a_1 \\ a_2 \\ - \\ - \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ - \\ - \\ 0 \end{bmatrix}$$ =. If the determinant of A is not equal to zero. Then

$$\begin{bmatrix} a_1 \\ a_2 \\ - \\ - \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ - \\ - \\ 0 \end{bmatrix}.$$

Thus the matrix A is linearly independent. From property 2, all Alpha, Beta and Gamma product of fuzzy matrices are nonsingular. Hence their determinants are not equal to zero. Thus all the vectors are linearly independent.

Property 4.6: Rank of nonsingular Alpha ,Beta and Gamma Product of Fuzzy matrices are $|V_1|\times|V_2|$.

Proof: Let A be any $n\times n$ nonsingular square symmetric matrix of any Alpha, Beta or Gamma product of fuzzy matrix. We know that all Alpha, Beta and Gamma products of fuzzy matrix are linearly independent. Thus the rank of the matrix is n ,(i.e.) $|V_1|\times|V_2|$.

Example:

Alpha Product of Fuzzy Matrix

```
>> A=[0.4 0.3 0.4 0;0.3 0.4 0 0.4;0.4 0 0.4 0.3;0 0.4 0.3 0.4];
>> U=rank(A)
  U =  4
```

Beta Product of Fuzzy Matrix

```
>> A=[0.4 0 0 0.3;0 0.4 0.3 0;0 0.3 0.4 0;0.3 0 0 0.4];
>> U=rank(A)
   U = 4
```

Gamma Product of Fuzzy Matrix

```
>> A=[0.5 0.3 0.4 0.3;0.3 0.5 0.3 0.4;0.4 0.3 0.5 0.3;0.3 0.4 0.3 0.5];
>> U=rank(A)
  U =  4
```

Property 4.7: Any Alpha, Beta and Gamma Product of Fuzzy Matrices with distinct eigen values are diagonalizable.

Proof: Let *A* be any Alpha, Beta (or) Gamma product of fuzzy matrix with distinct eigen values. Suppose $n=2$, let λ_1 , λ_2 be the distinct eigen values and v_1, v_2 be the eigen vectors. If a matrix has distinct eigen values, then the eigen vectors are linearly independent. Suppose the eigen vectors are linearly dependent.

Then $c_1v_1+c_2v_2=0$ (1)

with c_1, c_2 not both zero.

Multiplying both sides by A, we have $c_1\lambda_1v_1+c_2\lambda_2v_2=0$ (2)

Multiplying first equation by λ_1 we get $c_1\lambda_1v_1+c_2\lambda_1v_2=0$ (3)

Subtracting (3) from (2) gives $c_2(\lambda_2-\lambda_1)v_2=0$. We know that λ_1 and λ_2 are distinct eigen values of A. This implies that $c_2=0$. In the same way we can conclude that $c_1=0$. which is a contradiction .Hence any Alpha, Beta and Gamma product of Fuzzy matrices with distinct eigen values are diagonalizable.

Example:

Consider Alpha Product of Fuzzy Matrix

```
>> A=[0.4 0.3 0.4 0;0.3 0.4 0 0.4;0.4 0 0.4 0.3;0 0.4 0.3 0.4];
>> [V,D]=eig(A)
            V =  -0.5000    0.5000   -0.5000    0.5000
                  0.5000    0.5000    0.5000    0.5000
                  0.5000   -0.5000   -0.5000    0.5000
                 -0.5000   -0.5000    0.5000    0.5000
```

D = -0.3000	0	0	0
0	0.3000	0	0
0	0	0.5000	0
0	0	0	1.1000

Here V denotes eigen vectors and D gives the diagonalizable eigen values

Property 4.8: The sum of absolute eigen values of Alpha, Beta and Gamma Product of Fuzzy Matrices is atl east $\sqrt{2mn}$, where n and m are, the number of vertices and edges of Alpha, Beta and Gamma Product of Fuzzy Matrices.

Consider the Alpha Product of Fuzzy Matrix

```
>> A=[0.4 0.3 0.4 0;0.3 0.4 0 0.4;0.4 0 0.4 0.3;0 0.4 0.3 0.4];

>> [V,D]=eig(A)

V =  -0.5000   0.5000  -0.5000   0.5000
      0.5000   0.5000   0.5000   0.5000
      0.5000  -0.5000  -0.5000   0.5000
     -0.5000  -0.5000   0.5000   0.5000

D = -0.3000        0         0         0
          0   0.3000         0         0
          0        0    0.5000         0
          0        0         0    1.1000

>> A=[0.4 0.3 0.4 0;0.3 0.4 0 0.4;0.4 0 0.4 0.3;0 0.4 0.3 0.4];

>> U=sumabs(eig(A))

U =2.2000
```

Here $n=4$ and $m=4$, $\sqrt{2nm}=\sqrt{32}=5.6$.

2.2 is less than 5.6.

Consider Gamma Product of Fuzzy Matrix

```
>> A=[0.5 0.3 0.4 0.3;0.3 0.5 0.3 0.4;0.4 0.3 0.5 0.3;0.3 0.4 0.3 0.5];

>> D=eig(A)

D =  0.1000
     0.1000
     0.3000
     1.5000

>> U=sumabs(eig(A))
```

U =2.0000

Here $n = 4, m = 6$ and $\sqrt{2\text{nm}} = \sqrt{48} = 6.9,$

2 is less than 6.9.

5. CONCLUSION

In this paper the basic properties of Alpha, Beta and Gamma products of fuzzy matrices related to linear algebra are discussed. The authors would discuss the concepts of energy and the upper bound of energy for the Alpha, Beta and Gamma products of fuzzy graphs in a future work.

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