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Odd vertex magic total labeling of the extended comb graph *

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Abstract Let G be a simple finite graph with n vertices and m edges. A vertex magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $\{1, 2, 3, \ldots, m+n\}$ with the property that for every v in $V(G)$, $f(v) + \Sigma f(uv) = k$ for some constant k, where the sum is taken over all edges incident with v.The parameter k is called the *magic constant* for f. Nagaraj et al. (C. T. Nagaraj, C. Y. Ponnappan and G. Prabakaran, Odd vertex magic total labeling of trees, International Journal of Mathematics Trends and Technology, 52(6), 2017, 374–379) introduced the concept of odd vertex magic total labeling. A vertex magic total labeling is called an *odd vertex magic total labeling* if $f(V(G)) = \{1, 3, 5, \ldots, 2n-1\}$. A graph G is called an odd vertex magic if there exists an odd vertex magic total labeling for G. In this paper we prove that the extended comb graph $EC\,(t, k)$ for $k = 2$ admits an odd vertex magic total labeling when t is odd and the extended comb graph $EC\,(t, k)$, $k = 2$ with an additional edge admits an odd vertex magic total labeling when t is even.

Key words Magic labeling, vertex magic total labeling, odd vertex magic total labeling, extended comb graph.

2020 Mathematics Subject Classification 05C78.

1 Introduction

In Graph Theory, graph labeling is the assignment of labels, generally integers, to the edges or vertices or both of a graph, subject to certain constraints. Graph labelings trace their origin to the labelings presented by Alexander Rosa [7] in 1967. Rosa had identified three types of labelings: α-labelings, β-labelings and ρ-labelings. Later, β-labelings were renamed as graceful labeling by Golomb [2]. Several graph labeling techniques have been introduced and are being studied. One of these graph labeling techniques is the magic labeling introduced by Sedlek [8] in 1963 influenced by the magic squares in Number Theory. But Kotzig and Rosa [3] defined the magic labeling of a graph as a bijection $f : V \cup E \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ such that for all the edges uv of G, the quantity $f(u) + f(v) + f(uv)$ is a constant called the magic constant. MacDougall et al. [4] introduced the concept of *Vertex Magic Total Labeling* (VMTL) and defined it as a bijection f from $V(G) \cup E(G)$ to

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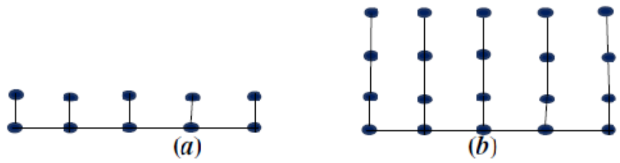

Fig. 1: (a) Comb graph $P_5 \bigodot K_1$, (b) Extended comb graph $EC(5, 3)$.

the integers $\{1, 2, 3, \ldots, m + n\}$ with $|V(G)| = n$ and $|E(G)| = m$ and the property that for every vertex v in $V(G)$, $f(v) + \Sigma f(uv) = k$, for some constant k, where the sum is taken over all edges incident with v. The parameter k is called the magic constant for f.
Nagaraj et al. [6] introduced the concept of an odd vertex magic total labeling. A vertex magic total labeling f is called an *odd vertex magic total labeling* if $f(V(G)) = \{1, 3, 5, \ldots, 2n - 1\}$. A graph G is called an odd vertex magic if there exists an odd vertex magic total labeling of G. Nagaraj et al. [5] proved that the path P_n and the cycle C_n are odd vertex magic if and only if n is odd. The incoherent union of r copies of cycles of length s, rC_s admits odd vertex magic total labeling if and only if r and s are odd. The (s, t) kite graph admits odd vertex magic total labeling if and only if $(s + t)$ is odd. For an exhaustive survey on odd vertex magic labeling we refer to Gallian [1].
In this paper we prove that the extended comb graph $EC(t, k), k = 2$ admits an odd vertex magic total labeling when t is odd and the extended comb graph $EC(t, k), k = 2$ with an additional edge admits an odd vertex magic total labeling when t is even.

2 Preliminary definitions

In this section we give a few definitions which are required for proving the main results in Section 3 of the paper.

Definition 2.1. The comb graph $P_n \bigodot K_1$ is obtained by joining a single edge to each vertex of the path graph P_n on n vertices. A comb graph is shown in the part (a) of Fig. 1.

Definition 2.2. An extended comb graph $EC(t, k)$ is a graph obtained by joining a path P_k on k vertices to each of the pendant vertices of the comb graph. An extended comb graph is shown in the part (b) of Fig. 1.

3 Main results

In this section we prove the our main results of this paper in the form of two theorems.

Theorem 3.1. *The extended comb graph $EC(t, k)$ for $k = 2$ admits odd vertex magic total labeling if t is odd.*

Proof. Let $EC(t, k)$, $k = 2$ be the extended comb graph and t be an odd positive integer. Let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively and $|V(G)| = 3t$ and $|E(G)| = 3t - 1$. The vertices of the extended comb graph are described from the bottom to the top and from the left to the right as $u_1, u_2, \ldots, u_t, v_1, v_2, \ldots, v_t, w_1, w_2, \ldots, w_t$ as shown in Fig. 2.The edges between the vertices u_i and u_{i+1} are denoted by x_i for $1 \leq i \leq t - 1$. The edges between the vertices u_i and v_i are denoted by y_i and the edges between v_i and w_i are denoted by z_i for $1 \leq i \leq t$ as shown in Fig. 2.
Consider a function $f : V \cup E \to \{1, 2, 3, \ldots, 6t - 1\}$. The labelings for the vertices of G are defined as follows:
For $1 \leq i \leq t$,

$$f(u_i) = \begin{cases} 2n - 1, \text{if } i = 1, \\ 3i - 4, \text{if } i \text{ is odd}, \\ 3i - 3, \text{if } i \text{ is even}, \end{cases} \tag{3.1}$$

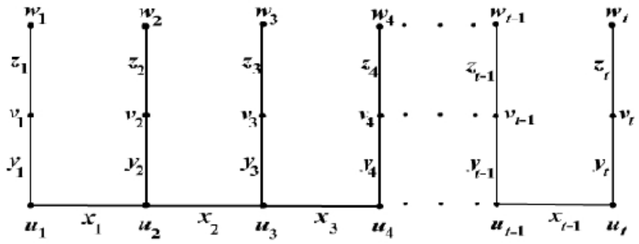

Fig. 2: The extended comb graph $EC(t, 2)$.

$$f(v_i) = 2n - 6i + 1, \tag{3.2}$$

$$f(w_i) = \begin{cases} 2n - 3i, & \text{if } i \text{ is odd,} \\ 2n - 3i - 1, & \text{if } i \text{ is even.} \end{cases} \tag{3.3}$$

The labelings for the edges of G are as follows:
For $1 \le i \le t - 1$,

$$f(x_i) = \begin{cases} n - 3i, & \text{if } i \text{ is odd,} \\ 2n - 3i, & \text{if } i \text{ is even.} \end{cases} \tag{3.4}$$

For $1 \le i \le t$,

$$f(y_i) = \begin{cases} 3i - 1, & \text{if } i \text{ is odd,} \\ 3i - 2, & \text{if } i \text{ is even,} \end{cases} \tag{3.5}$$

$$f(z_i) = \begin{cases} n + 3i - 2, & \text{if } i \text{ is odd,} \\ n + 3i - 1, & \text{if } i \text{ is even.} \end{cases} \tag{3.6}$$

We compute the magic constant based on the above vertex and edge labels as follows:

Case 1: For the vertex u_1,

$$k = f(u_1) + f(u_1 v_1) + f(u_1 u_2) = 2n - 1 + 3 - 1 + n - 3 = 9t - 2.$$

Case 2: For the vertex u_t,

$$k = f(u_t) + f(u_t v_t) + f(x_{t-1}) = 3t - 4 + 3t - 1 + 2n - 3(t - 1) = 9t - 2.$$

Case 3: For the vertices u_i, $2 \le i \le t - 1$,

Case 3.1: When i is odd,

$$k = f(u_i) + f(x_i) + f(x_{i+1}) + f(y_i) = 3i - 4 + n - 3i + 2n - 3(i - 1) + 3i - 1 = 9t - 2.$$

Case 3.2: When i is even,

$$k = f(u_i) + f(x_i) + f(x_{i+1}) + f(y_i) = 3i - 3 + n - 3(i - 1) + 2n - 3i + 3i - 2 = 9t - 2.$$

Case 4: For the vertices v_i, $1 \le i \le t$,

Case 4.1: When i is odd,

$$k = f(v_i) + f(u_i v_i) + f(v_i w_i) = 2n - 6i + 1 + 3i - 1 + n + 3i - 2 = 9t - 2.$$

Case 4.2: When i is even,

$$k = f(v_i) + f(u_i v_i) + f(v_i w_i) = 2n - 6i + 1 + 3i - 2 + n + 3i - 1 = 9t - 2.$$

Case 5: For the vertices w_i, $1 \le i \le t$,

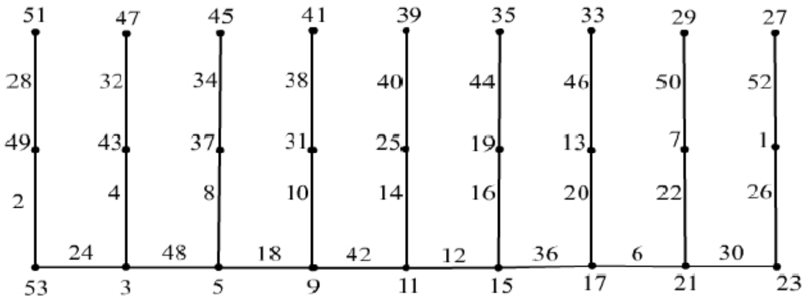

Fig. 3: The odd vertex magic total labeling of $EC(9, 2)$ with $k = 79$.

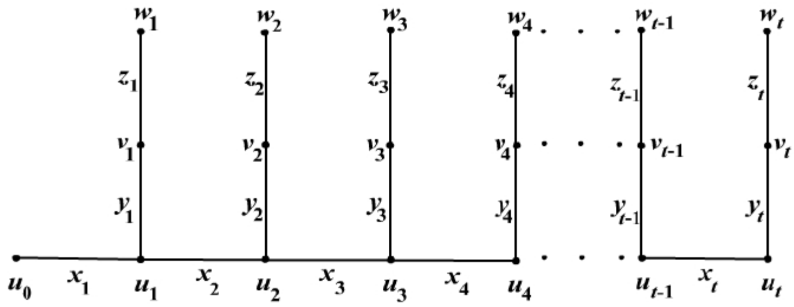

Fig. 4: The extended comb graph $EC(t, 2)$ with an additional edge.

Case 5.1: When i is odd,

$$k = f(w_i) + f(v_i w_i) = 2n - 3i + n + 3i - 2 = 9t - 2.$$

Case 5.2: When i is even,

$$k = f(w_i) + f(v_i w_i) = 2n - 3i - 1 + n + 3i - 1 = 9t - 2.$$

Therefore f is an odd vertex magic total labeling for $EC(t, k), k = 2$ with magic constant $k = 9t - 2$. An illustration of odd vertex magic total labeling of extended comb graph $EC(9, 2)$ is given in Fig. 3. □

Theorem 3.2. *Let G be an extended comb graph $EC(t, k)$, $k = 2$ with an additional edge when t is even. Then G admits an odd vertex magic total labeling.*

Proof. Let $EC(t, k)$, $k = 2$ be an extended comb graph with an additional edge when t is even. Let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively and $|V(G)| = 3t+1$ and $|E(G)| = 3t$. The vertices of the extended comb graph are described from the bottom to the top and from the left to the right as $u_0, u_1, u_2, \ldots, u_t, v_1, v_2, \ldots, v_t, w_1, w_2, \ldots, w_t$ as shown in Fig. 4. The edges between the vertices u_{i-1} and u_i are denoted by x_i for $1 \le i \le t$. The edges between the vertices u_{i-1} and v_i are denoted by y_i and the edges between v_i and w_i are denoted by z_i for $1 \le i \le t$ as shown in Fig. 4.
Consider a function $f : V \cup E \to \{1, 2, 3, \ldots, 6t + 1\}$. The labelings for the vertices of G are as follows:

For $0 \le i \le t$,

$$f(u_i) = \begin{cases} 2n-1, & \text{if } i = 0, \\ 3i-2, & \text{if } i \text{ is odd}, \\ 3i-1, & \text{if } i \text{ is even}. \end{cases} \tag{3.7}$$

For $1 \le i \le t$,

$$f(v_i) = 2n - 6i + 1, \tag{3.8}$$

$$f(w_i) = \begin{cases} 2n-3i, & \text{if } i \text{ is odd}, \\ 2n-3i-1, & \text{if } i \text{ is even}. \end{cases} \tag{3.9}$$

The labelings for the edges of G are as follows:
For $1 \le i \le t$,

$$f(x_i) = \begin{cases} n-3i+2, & \text{if } i \text{ is odd}, \\ 2n-3i+2, & \text{if } i \text{ is even}. \end{cases} \tag{3.10}$$

$$f(y_i) = \begin{cases} 3i-1, & \text{if } i \text{ is odd}, \\ 3i-2, & \text{if } i \text{ is even}. \end{cases} \tag{3.11}$$

$$f(z_i) = \begin{cases} n+3i-2, & \text{if } i \text{ is odd}, \\ n+3i-1, & \text{if } i \text{ is even}. \end{cases} \tag{3.12}$$

We compute the magic constant based on the above vertex and edge labels as follows:

Case 1: For the vertex u_0,

$$k = f(u_0) + f(u_0u_1) = 2n - 1 + n - 3 + 2 = 9t + 1.$$

Case 2: For the vertex u_t,

$$k = f(u_t) + f(u_tv_t) + f(x_t) = 3t - 1 + 3t - 2 + 2n - 3t + 2 = 9t + 1.$$

Case 3: For the vertices $u_i, 1 \le i \le t$,

Case 3.1: When i is odd,

$$k = f(u_i) + f(x_i) + f(x_{i+1}) + f(y_i) = 3i - 2 + n - 3i + 2 + 2n - 3(i+1) + 2 + 3i - 1 = 9t + 1.$$

Case 3.2: When i is even,

$$k = f(u_i) + f(x_i) + f(x_{i+1}) + f(y_i) = 3i - 1 + n - 3(i+1) + 2 + 2n - 3i + 2 + 3i - 2 = 9t + 1.$$

Case 4: For the vertices $v_i, 1 \le i \le t$,

Case 4.1: When i is odd,

$$k = f(v_i) + f(u_iv_i) + f(v_iw_i) = 2n - 6i + 1 + 3i - 1 + n + 3i - 2 = 9t + 1.$$

Case 4.2: When i is even,

$$k = f(v_i) + f(u_iv_i) + f(v_iw_i) = 2n - 6i + 1 + 3i - 2 + n + 3i - 1 = 9t + 1.$$

Case 5: For the vertices $w_i, 1 \le i \le t$,

Case 5.1: When i is odd,

$$k = f(w_i) + f(v_iw_i) = 2n - 3i + n + 3i - 2 = 9t + 1.$$

Case 5.2: When i is even,

$$k = f(w_i) + f(v_iw_i) = 2n - 3i - 1 + n + 3i - 1 = 9t + 1.$$

Therefore, f is an odd vertex magic total labeling for $EC(t,k)$, $k = 2$ with an additional edge for even t with magic constant $k = 9t + 1$. An illustration of an odd vertex magic total labeling of an extended comb graph $EC(8,2)$ with an additional edge is given in Fig. 5.

□

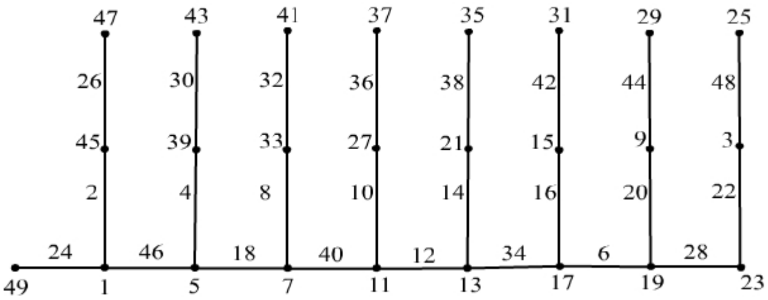

Fig. 5: The odd vertex magic total labeling of $EC(8,2)$ with an additional edge with $k = 73$.

4 Conclusion

In this paper we showed that the extended comb graph $EC\,(t,k)\,, k = 2$ when t is odd and the extended comb graph $EC\,(t,k)\,, k = 2$ with an additional edge when t is even admit odd vertex magic total labeling.

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