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INTRODUCING THE UPADHYAYA INTEGRAL TRANSFORM

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Abstract:
Through this introductory paper we announce to the worldwide mathematics community a new type of integral transform, which we call the Upadhyaya Integral Transform or, the Upadhyaya transform (UT), in short. The new transform which we propose to proclaim through this paper, is, in fact a generalized form of the celebrated Laplace transform. The power of this generalization is that this most general form of the Laplace transform generalizes and unifies, besides the classical Laplace transform and the Laplace-Carson transform, most of the very recently introduced integral transforms of this category like, the Sumudu transform, the Elzaki transform, the Kashuri and Fundo transform, the Mahgoub transform, the ZZ-transform, the Sadik transform, the Kamal transform, the Natural transform, the Mohand transform, the Aboodh transform, the Ramadan Group transform, the Shehu transform, the Sawi transform, the Tarig transform, the Yang transform, etc. We develop the general theory of the Upadhyaya transform in a way which exactly parallels the existing theory of the classical Laplace transform and also provide a number of possible generalizations of this transform and thus we prepare a firm ground for future researches in this field by employing this most generalized, versatile and robust form of the classical Laplace transform – the Upadhyaya transform – in almost all the areas wherever, the classical Laplace transform and its various aforementioned variants are currently being employed for solving the vast multitude of problems arising in the areas of applied mathematics, mathematical physics and engineering sciences and other possible fields of study inside and outside the realm of mathematics.

Keywords: Upadhyaya transform, Laplace transform, Laplace-Carson transform, Sumudu transform, Elzaki transform, Kashuri and Fundo transform, Mahgoub transform, ZZ- transform, Sadik transform, Kamal transform, Natural transform, Mohand transform, Aboodh transform, Ramadan Group transform, Shehu transform, Sawi transform, Tarig transform, Yang transform.

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1. INTRODUCTION

The celebrated Laplace transform was introduced by the French mathematician and physicist Pierre Simon Laplace (1749–1827). This classical integral transform together with another celebrated transform the Fourier transform (due to Joseph Fourier (1768–1830)) forms the foundation stone of Operational Calculus, a branch of mathematics which has very powerful applications not only in applied mathematics but also in other branches of science like physics, electrical and mechanical engineering, astronomy, heat transfer, etc. The integral transforms play a pivotal role in finding the solutions of the initial value problems and the initial-boundary value problems which are frequently encountered in engineering sciences. Ever since the British electrical engineer Oliver Heaviside (1850 –1925) first used the Laplace transform to solve ordinary differential equations encountered by him in his study of electrical circuits, the subject of integral transforms caught sight of mathematicians and researchers and continues to be a hot field of mathematical research even today. The utility of the operational calculus methods which solely rest upon the foundation pillars of integral transforms needs no introduction given the undisputed fact that these methods have found vast applications in the fields of ordinary and partial differential equations, integral equations, calculus of finite differences, problems in probability and statistics, the fractional integrals and fractional derivatives. It is well known that the Laplace transform is a special case of the celebrated Fourier transform in the sense that it can be derived from the latter for a special class of exponentially decaying functions which are restricted in their definition to the positive part of the real axis. The Laplace transform of many functions including the elementary functions and higher transcendental functions have been calculated and tabulated in a number of authoritative standard works of mathematics, a few of them are [1 – 5]. The theory of the classical Laplace transform can be found in numerous works. We only mention three here, just only for the sake of reference [6,7,8]. The books mentioned in [7,8] are very recent contributions to existing vast literature on the subject. Many extensions of the classical Laplace transform have also appeared in the literature. One very recent extension is the study of Kim and Kim [9], called the degenerate Laplace transform. Their study was extended by this author in a series of very recent papers [10-13]. We may also mention that the classical Laplace transform in terms of real symmetric positive definite matrices and complex Hermitian positive definite matrices can be found in the work of Mathai [14] and in some references mentioned therein. This book of Mathai also contains Laplace type integrals for many hypergeometric functions of matrix arguments with real positive definite matrices and Hermitian positive definite matrices as arguments. The author's doctoral dissertation [15] also contains Laplace type integrals for many multiple hypergeometric functions of matrix arguments which are proved by using the Mathai's matrix transform technique as discussed by Mathai in [14]. In developing these results, the results for the Laplace transforms of the hypergeometric functions of one and more scalar variables have played a crucial role. Thus, in short, we emphasize here that the utility of the classical Laplace transform in mathematics remains much more even today after about two centuries when they were first employed as effective tools for solving problems arising in applied mathematics, physics and engineering (see, [7], p.1). This underlines the need of continuous research in this field and enhancement of our existing knowledge about this celebrated integral transform.

In this paper we proclaim a new generalization of the classical Laplace Transform which we call the Upadhyaya Integral Transform or the Upadhyaya Transform (UT) for brevity. This generalization is a powerful and versatile generalization and unification of almost all the variants of the classical Laplace transform that are extant in the current mathematical research literature. We mention that the Upadhyaya transform, which we introduce in this paper includes the classical Laplace transform, the Laplace – Carson transform, the Sumudu transform and most of the very recently introduced variants of the classical Laplace transform, namely, the Elzaki transform, the Kashuri and Fundo transform, the Mahgoub transform, the ZZ – transform, the Sadik transform, the Kamal transform, the Natural transform, the Mohand transform, the Aboodh transform, the Ramadan Group transform, the Shehu transform, the Sawi transform, the Tarig transform and the Yang transform. The scheme of the paper is as follows: in the first section of the paper we give the definition of the classical Laplace transform as our preliminary definition. In the second section of the paper we state our definition of the Upadhyaya transform. In the third section of the paper we provide the mathematical basis for the Upadhyaya transform and thus give the complex inversion formula for the Upadhyaya transform. In the fourth section we show how the different variants of the classical Laplace transforms as mentioned above are special cases of the Upadhyaya transform. In the fifth section we develop the theory of the Upadhyaya transform parallel to the existing theory of the Laplace transform and also calculate the Upadhyaya transforms of some frequently used elementary functions. The sixth section is devoted to the $n-$dimensional generalization of the Upadhyaya transform and we point out some special cases of this generalization there, which currently exist in the literature. The seventh section provides the degenerate extension, while the eighth section provides the modified degenerate extension of the Upadhyaya transform. The matrix generalization of the Upadhyaya transform to the cases of real symmetric positive definite matrix arguments and the Hermitian positive definite matrix arguments are discussed respectively in the ninth and tenth sections of the paper.

1.1 Preliminary Definitions

Definition 1.1: Functions of Exponential Order: (see, for example, [7, p. 150]) A function $F(t)$ is said to be of exponential order $a(>0)$ on $0 \le t < \infty$ if there exists a positive constant K such that for all $t > T$

$$\left|F(t)\right| \le Ke^{at}. \tag{1.1}$$

Definition 1.2: The Laplace transform: (see, for example, [6, (1), p.1]). Let $F(t)$ be a real valued function defined on $(-\infty,\infty)$ such that $F(t)=0\ \forall t<0$. The Laplace transform of $F(t)$, represented by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t);s\} = f(s) = \int_0^\infty e^{-st}F(t)\,dt \tag{1.2}$$

where, the parameter s is a real or complex number and $\mathcal{L}$ denotes the Laplace transform operator. Originally Laplace used this transform around 1780's while working in probability theory [26] (see also, [27] and [28]).

2. THE UPADHYAYA INTEGRAL TRANSFORM (or, THE UPADHYAYA TRANSFORM)

For the sake of most general situations, we first define three complex parameters $\lambda_1,\lambda_2,\lambda_3$ as below:

$$\lambda_1 = \frac{\left(\sum_{i=0}^{r_1} a_i z_1^i\right)^{m_1}}{\left(\sum_{i=0}^{r_2} b_i z_2^i\right)^{m_2}},\ \lambda_2 = \frac{\left(\sum_{i=0}^{r_3} c_i z_3^i\right)^{m_3}}{\left(\sum_{i=0}^{r_4} d_i z_4^i\right)^{m_4}},\ \lambda_3 = \frac{\left(\sum_{i=0}^{r_5} k_i z_5^i\right)^{m_5}}{\left(\sum_{i=0}^{r_6} l_i z_6^i\right)^{m_6}} \tag{2.1}$$

where, m_i and r_i are nonnegative integers $(i=1,2,\ldots,6)$ and a_i,b_i,c_i,d_i,k_i,l_i are complex constants. Depending upon the situation at hand, we can also choose the parameters $\lambda_1,\lambda_2,\lambda_3$ and the constants a_i,b_i,c_i,d_i,k_i,l_i to be real numbers also. The above choice of the parameters ensures the broadest form of generalization of the classical Laplace transform so that it encompasses almost all the existing variants of the Laplace transform that are currently in vogue in the mathematical research literature or may appear in the future.

Definition 2.1: The Upadhyaya Integral Transform: Suppose the real valued function $F(t)$ belongs to the set of functions defined by

$$A = \left\{F(t) : \exists M, \eta_1 \text{ and /or } \eta_2 > 0, \left|F(t)\right| < Me^{|t|/\eta_j},\ \text{if } t \in (-1)^j \times [0,\infty),\ j=1,2.\right\} \tag{2.2}$$

where, the constant M is finite while η_1 and η_2 may not exist simultaneously. We define the function $F(t)$ such that $F(t)=0$ for $t<0$. The *Upadhyaya transform* (UT) of the function $F(t)$ is defined as

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathrm{u}(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt \tag{2.3}$$

provided this integral converges and the parameters $\lambda_1,\lambda_2,\lambda_3$ are given by (2.1). Here, we call $\mathcal{U}$ the *Upadhyaya Transform Operator.*

3. THEORETICAL BASIS OF THE UPADHYAYA TRANSFORM

It is well known that the celebrated Fourier integral formula simultaneously provided not only the definitions of the Laplace transform as well as the inverse Laplace transform but also granted a method to evaluate the inverse Laplace transform of a function by means of evaluating a complex contour integral by the use of the Cauchy's residue theorem and by suitably deforming the contour of integration in the complex plane. The Upadhyaya transform being a generalized form of the Laplace transform, therefore, we venture now to provide a mathematical basis for the Upadhyaya transform proclaimed above by us.

The Fourier integral formula for a function $f_1(x)$ defined on $-\infty < x < \infty$ is given by (see [7, (3.2.1), p. 144])

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk\int_{-\infty}^{\infty} e^{-ikt}f_1(t)\,dt. \tag{3.1}$$

If we choose in the above equation $f_1(x)=0$ in $-\infty<x<0$ and put $f_1(x)=e^{-cx}F(\lambda_3 x)H(x)$, where, c is a positive fixed number, λ_3 given by (2.1) and $H(x)$ is the Heaviside unit step function defined by (see, [7], (2.3.8), p. 22 and (2.3.9), p.23)

$$H(x)=\begin{cases}1, & x>0\\ 0, & x<0\end{cases} \tag{3.2}$$

and in the general form by

$$H(x-a)=\begin{cases}1, & x>a\\ 0, & x<a\end{cases} \tag{3.3}$$

for a fixed real number a. Now the Fourier integral formula (3.1) yields,

$$e^{-cx}F(\lambda_3 x)H(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}dk\int_{-\infty}^{\infty}e^{-ikt}e^{-ct}F(\lambda_3 t)H(t)\,dt \tag{3.4}$$

Applying the definition of $H(x)$ from (3.2) in (3.4) gives

$$F(\lambda_3 x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(c+ik)x}dk\int_{0}^{\infty}e^{-(c+ik)t}F(\lambda_3 t)\,dt \tag{3.5}$$

We substitute $c+ik=\lambda_2$ (so that, $idk=d\lambda_2$, with $i=\sqrt{-1}$) then (3.5) renders

$$F(\lambda_3 x)=\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}e^{\lambda_2 x}\frac{d\lambda_2}{i}\int_{0}^{\infty}e^{-\lambda_2 t}F(\lambda_3 t)\,dt \tag{3.6}$$

Or, for the parameter λ_1 defined by (2.1), (3.6) can be rewritten as

$$F(\lambda_3 x)=\frac{1}{2\pi i\lambda_1}\int_{c-i\infty}^{c+i\infty}e^{\lambda_2 x}d\lambda_2\left[\lambda_1\int_{0}^{\infty}e^{-\lambda_2 t}F(\lambda_3 t)\,dt\right].$$

Noting that the expression within the square brackets of the last equation is the Upadhyaya transform $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$ of the function $F(t)$ ((2.3) above), we at once get the following *complex inversion formula for the Upadhyaya transform* from the above equation as

$$F(\lambda_3 x)=\mathcal{U}^{-1}\{\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)\}=\frac{1}{2\pi i\lambda_1}\int_{c-i\infty}^{c+i\infty}e^{\lambda_2 x}\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)\,d\lambda_2. \tag{3.7}$$

We call the operator $\mathcal{U}^{-1}$ in the above equation the *inverse Upadhyaya transform operator*. It may be noted from (3.7) that instead of recovering the original function $F(x)$ from it we get the function $F(\lambda_3 x)$. This is due to the fact that in the definition of the Upadhyaya transform the argument of the function is multiplied by λ_3. The original function $F(x)$ can be easily recovered from the function $F(\lambda_3 x)$ given by (3.7) by the simple application of the transformation $x\to x/\lambda_3$ to it.

4. RELATION OF THE UPADHYAYA TRANSFORM TO THE LAPLACE TRANSFORM AND ITS OTHER EXTANT VARIANTS

We now show how the Laplace transform and its other variants which are currently available in the research literature of mathematics follow as the special cases of the Upadhyaya transform defined by (2.3) supra.

4.1 The Laplace transform

The classical Laplace transform defined by (1.2) can be easily seen to follow from (2.3) if we choose $\lambda_1=1=\lambda_3$ and $\lambda_2=s$ in (2.3). From (2.1) this can be achieved in many ways by suitably choosing the values of the non-negative integers and the complex constants. The details being trivial we do not mention them here. Thus we have,

$$\mathcal{U}\{F(t);1,s,1\}=\mathrm{u}(1,s,1)=\mathcal{L}\{F(t)\}=f(s).$$

4.2 The Laplace – Carson transform

This transform is sometimes also referred to as the '$s-$multiplied' form of the Laplace transform and it is defined by

$$\mathcal{L}_{\mathrm{LC}}\{F(t);s\}=f_{\mathrm{LC}}(s)=s\int_0^\infty e^{-st}F(t)\,dt \tag{4.1}$$

for the same conditions as are mentioned with the definition of the Laplace transform in (1.2) above. The subscript 'LC' attached to $\mathcal{L}$ and f on the left hand side in (4.1) is used by us here to signify that it is the Laplace – Carson transform of $F(t)$ (see [49], see also, [30, (1.5), p.168]). It is easy to infer from (2.3) that for the choice of parameters $\lambda_1=\lambda_2=s$ and $\lambda_3=1$ in it, we obtain (4.1) which gives us that

$$\mathcal{U}\{F(t);s,s,1\}=\mathrm{u}(s,s,1)=\mathcal{L}_{\mathrm{LC}}\{F(t)\}=f_{\mathrm{LC}}(s).$$

We find it worth mentioning here that the Laplace – Carson transform perhaps appeared in the mathematical literature around 1948 as remarked by Elzaki et al. [16, p. 2]. Deakin [17] tells us that this transform was earlier more preferred to the usual definition (1.2) of the Laplace transform than at the present times because this form was more conveniently suited for working with 'the Older Heaviside Operational Calculus' by making a reference to his work [18]. He also points out that by the time he wrote this letter (1995-1997) to the Editor of the International Journal of Mathematical Education in Science and Technology the Laplace – Carson transform retained 'a considerable place in the Russian literature' by referring to the work of Ditkin and Prudnikov [19]!

4.3 The Sumudu Transform

The Sumudu transform was introduced by Watugala [20] in 1993 to solve differential equations and some problems in control engineering. It was defined for the functions $F(t)$ belonging to the set A defined by (2.2) as below:

$$\mathcal{S}\{F(t);u\}=\mathbb{S}\{u\}=\frac{1}{u}\int_0^\infty e^{-t/u}F(t)\,dt,\quad u\in(-\eta_1,\eta_2) \tag{4.2}$$

In the literature it also appears in slightly another form as given below, which is obtained by rewriting (4.2) by setting $v=t/u$ to get

$$\mathcal{S}\{F(t);u\}=\mathbb{S}\{u\}=\int_0^\infty e^{-v}F(uv)\,dv,\quad u\in(-\eta_1,\eta_2) \tag{4.3}$$

Soon after the above paper of Watugala [20], another paper by Weerakoon [21] followed where she applied the Sumudu transform to solve partial differential equations. It can easily be inferred from (2.3) and (4.2) that when we choose $\lambda_1=\lambda_2=\frac{1}{u}$ and $\lambda_3=1$ in (2.3) we obtain (4.2) thereby implying the relation

$$\mathcal{U}\left\{F(t);\frac{1}{u},\frac{1}{u},1\right\}=\mathrm{u}\left(\frac{1}{u},\frac{1}{u},1\right)=\mathcal{S}\{F(t);u\}=\mathbb{S}\{u\}.$$

Similarly, the choice $\lambda_1=\lambda_2=1$ and $\lambda_3=u$ in (2.3) gives (4.3), thence implying that

$$\mathcal{U}\{F(t);1,1,u\}=\mathrm{u}(1,1,u)=\mathcal{S}\{F(t);u\}=\mathbb{S}\{u\}.$$

Watugala [20] proclaimed that the Sumudu transform (4.2) introduced by him was a 'new integral transform'. But this claim of Watugala [20] was refuted by Deakin [17] who pointed out that the Sumudu transform defined by (4.2) could be arrived at from the definition of the Laplace – Carson transform (4.1) 'by means of the trivial change of the variable' $s=\frac{1}{u}$. He concluded that the Sumudu transform "is nothing other than a thinly disguised form of the familiar Laplace transform. Indeed it already has some currency in this form [22, section 2.1]. All the properties demonstrated for the 'Sumudu transform' in [20, 21] may very readily be deduced from the corresponding properties for the standard Laplace transform, and this is an exercise I leave to the reader". Deakin [17] also remarked that 'The origin of the name 'Sumudu transform' is nowhere given either in [20] or [21], nor can I enlighten readers myself. (No author named Sumudu has ever been noticed by *Mathematical Reviews*).' Similar comments are made by Al-Omari and Agarwal [23, p.17] where they write: "While we are in agreement with most of the claims expounded by Watugala, the transform is not so new as proclaimed; the Sumudu transform is connected to the s-multiplied Laplace transform. But, this, however, in no way diminishes its importance or usefulness; Belgacem et al. (2003)" [24]. The present author would like to remark here that over the past twenty six years the integral transform named Sumudu transform has been applied to study a wide variety of phenomenon by a number of different authors and it has gained wide acceptance in the mathematical research literature and many research papers written on this topic and its applications have now found place in the records of the world's two topmost and the most prestigious research databases of mathematics, viz. the Mathematical Reviews/MathSciNet of the American Mathematical Society, USA and the zbMATH (formerly, Zentralblatt MATH), Germany of the European Mathematical Society, for both of whom this author is a permanent reviewer. The author would like to mention here only just two papers, the reference numbers [23] and [25], in this connection only for instance. In the light of this discussion and the aforementioned remarks of the very respected learned authors of the reference nos. [17] and [23], as far as this author is aware of at present, now, this author would like to share here his personal experience and motivation for writing this paper. Most humbly he would like to inform the respected readers of this paper that the Founder Editor-in-Chief and the current Managing Editor of this Journal, Prof. Dr. A.K. Sharma, who himself is also a great teacher for this author as well as one of his greatest mentors, had entrusted the responsibility of reviewing a manuscript for possible publication in this journal sometimes back. While going through that manuscript this author was introduced for the first time to one of the variants of Laplace transform introduced recently by an author whose results and definition were made use of in that manuscript. It was just out of curiosity that this author chose to pursue the field of study that he was exposed to for the first time. This led to this author's first time introduction with the numerous other variants of the classical Laplace transform that have appeared in the literature so far and are mentioned in the abstract of this paper. Having felt the need that all these various varieties of new Laplace transforms can be suitably combined together and fused into a more compact form which is a generalization of the classical Laplace transform, this author decided to introduce this most versatile and robust generalization of the classical Laplace transform, which he prefers to call the *Upadhyaya Integral Transform*, or the *Upadhyaya Transform* (UT) for brevity, in the memory of his late father Dr. Urba Datt Upadhyaya, who being a teacher and researcher himself, had initiated this author into the study of mathematics during this author's early formative years and who had always encouraged and inspired this author to keep contributing something new to this subject till he breathed his last in the year 2013. The only reason for this disclosure is the inspiration drawn by this author from reading the work of the very learned and this author's very respected mathematician Prof. M.A.B. Deakin [17] of Monash University, Clayton, Australia and the very learned and respected authors of the reference no. [23]. This author would also most humbly like to request the respected readers of this paper that as they go through the details of the theory of the Upadhyaya transform as developed in the next section of this paper, they will notice that all the results established here are parallel to the corresponding results of the existing theory of the Laplace transform and the above quoted remarks of the respected Prof. Deakin [17] in the context of the Sumudu transform of Watugala [20] may also apply very well here and most of the respected readers and critics of this paper may naturally ask the question 'Why, a new integral transform with a new name is introduced whose kernel is essentially the same – the exponential function – which is also the kernel of the classical Laplace transform?' This author's only humble response to all such queries of the learned readers and critics of this paper is that this author is very much hopeful that in the coming years the readers, the researchers from allied branches of study like applied physics, engineering and the mathematicians worldwide would apply the results developed here to almost all the phenomena wherever the various types of integral transforms mentioned in the abstract of this paper have been very fruitfully applied during the recent years. It is only after going through the vast number of research papers that have appeared in the topics concerning the various integral transforms as mentioned in the abstract and introductory section of this paper, the present author decided to put forth this unification and generalization of all these integral transforms as mentioned supra. Now, concluding these pertinent remarks, we proceed ahead to mention the other integral transforms which follow as the special cases of the Upadhyaya transform.

4.4 The Natural Transform

The Natural transform was introduced by Khan and Khan [29] in 2008. For the functions $F(t)$ belonging to the set A defined by (2.2) the natural transform is defined by (see also, [29, (1), p.127])

$$\mathbb{N}\left[F(t)H(t);s,u\right]=\mathbb{N}^{+}\left[F(t);s,u\right]=R^{+}(s,u)=\int_0^\infty e^{-st}F(ut)\,dt,\quad (u,s)\in(-\eta_1,\eta_2) \tag{4.4}$$

(see, [31, (3.2), p.80] and [30, (3), p.4]). A simple comparison between (2.3) and (4.4) leads us to the conclusion that the choice $\lambda_1=1,\lambda_2=s$ and $\lambda_3=u$ in (2.3) gives (4.4) and we thus have

$$\mathcal{U}\{F(t);1,s,u\}=\mathrm{u}(1,s,u)=\mathbb{N}^{+}\left[F(t);s,u\right]=R^{+}(s,u).$$

4.5 The Elzaki Transform

This transform was introduced in 2011 by Elzaki [32]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Elzaki transform is defined by (see [32, (2), p. 57])

$$E\left[F(t);v\right]=T(v)=v\int_0^\infty e^{-t/v}F(t)\,dt,\quad v\in(\eta_1,\eta_2) \tag{4.5}$$

By using the change of variable $t=vu$ in (4.5), it may also be rewritten as (see, [16, (2-2), p.2])

$$E\left[F(t);v\right]=T(v)=v^2\int_0^\infty e^{-t}F(vt)\,dt,\quad v\in(\eta_1,\eta_2) \tag{4.6}$$

With the choice $\lambda_1=v,\lambda_2=\frac{1}{v}$ and $\lambda_3=1$ in (2.3) we get (4.5) thereby implying that

$$\mathcal{U}\left\{F(t);v,\frac{1}{v},1\right\}=\mathfrak{u}\left(v,\frac{1}{v},1\right)=E\left[F(t);v\right]=T(v).$$

Similarly, the choice of $\lambda_1=v^2,\lambda_2=1$ and $\lambda_3=v$ in (2.3) we arrive at (4.6) which is suggestive of the relation

$$\mathcal{U}\left\{F(t);v^2,1,v\right\}=\mathfrak{u}\left(v^2,1,v\right)=E\left[F(t);v\right]=T(v).$$

4.6 The Aboodh Transform

K.S. Aboodh [33] introduced this transform in 2013. For the functions $F(t)$ belonging to the set A defined in (2.2) the Aboodh transform is defined by (see [33, (2), p.36])

$$A\left[F(t);v\right]=K(v)=\frac{1}{v}\int_0^\infty e^{-vt}F(t)\,dt,\quad \eta_1\le v\le\eta_2. \tag{4.7}$$

If we put $\lambda_1=\frac{1}{v},\lambda_2=v$ and $\lambda_3=1$ in (2.3) we at once arrive at (4.7) thus implying the relation

$$\mathcal{U}\left\{F(t);\frac{1}{v},v,1\right\}=\mathfrak{u}\left(\frac{1}{v},v,1\right)=A\left[F(t);v\right]=K(v).$$

4.7 The Kashuri and Fundo Transform

This transform was also introduced in the year 2013 by Kashuri and Fundo [34]. For a function $F(t)$ belonging to the set A defined by (2.2) the Kashuri and Fundo transform is defined by

$$\mathcal{K}\left[F(t);v\right]=\mathfrak{k}(v)=\frac{1}{v}\int_0^\infty e^{-t/v^2}F(t)\,dt,\quad v\in(-\eta_1,\eta_2). \tag{4.8}$$

where, $\left|F(t)\right|\le Me^{|t|/\eta_j^2}$, $j=1,2$ (see [28, (6), p. 6] and [27, (2.11), p. 266]). A plain comparison of (2.3) with (4.8) shows that the choice of $\lambda_1=\frac{1}{v},\lambda_2=\frac{1}{v^2}$ and $\lambda_3=1$ in (2.3) leads us to (4.8) thereby showing that

$$\mathcal{U}\left\{F(t);\frac{1}{v},\frac{1}{v^2},1\right\}=\mathfrak{u}\left(\frac{1}{v},\frac{1}{v^2},1\right)=\mathcal{K}\left[F(t);v\right]=\mathfrak{k}(v).$$

4.8 The ZZ Transform

The ZZ transform was introduced in 2016 by Zain Ul Abadin Zafar [35]. For the function $F(t)$ in the set A defined by (2.2) the ZZ transform is defined by (see [35, (1), p. 1605])

$$H\left[F(t);u,s\right]=Z(u,s)=s\int_0^\infty e^{-st}F(ut)\,dt. \tag{4.9}$$

We can at once observe that for the choice of $\lambda_1=s,\lambda_2=s$ and $\lambda_3=u$ in (2.3) gives (4.9) thus giving us the relation

$$\mathcal{U}\left\{F(t);s,s,u\right\}=\mathfrak{u}(s,s,u)=H\left[F(t);u,s\right]=Z(u,s).$$

4.9 The Ramadan Group Transform

The Ramadan Group Transform was introduced in 2016 by Raslan et al. [36]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Ramadan Group Transform is defined by (see [27, (2.16), p. 267])

$$RG\left[F(t);v,s\right]=U(v,s)=\int_0^\infty e^{-st}F(vt)\,dt,\quad v\in(\eta_1,\eta_2). \tag{4.10}$$

We see that the choice of $\lambda_1=1,\lambda_2=s$ and $\lambda_3=v$ in (2.3) leads us to (4.10) thus establishing that

$$\mathcal{U}\{F(t);1,s,v\}=\mathrm{u}(1,s,v)=RG\left[F(t);v,s\right]=U(v,s).$$

It is pertinent to mention here that the Ramadan Group transform is exactly the same as the Natural transform introduced in 2008 (see also [28, p.9]).

4.10 The Mahgoub Transform (in fact, The Laplace-Carson Transform)

This transform was introduced in 2016 by Mahgoub [37]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Mahgoub transform is defined by (see, [37, (2), p. 392])

$$M\left[F(t);v\right]=H(v)=v\int_0^\infty e^{-vt}F(t)\,dt,\quad v\in[\eta_1,\eta_2]. \tag{4.11}$$

We remark here that the Mahgoub transform is not at all a new transform but it is exactly the same as the Laplace –Carson transform mentioned in the subsection 4 (see, (4.1), perhaps the learned author was not aware of the Laplace-Carson transform). As in the case of the Laplace- Carson transform, by setting $\lambda_1=v,\lambda_2=v$ and $\lambda_3=1$ in (2.3) , we can at once obtain (4.11) which shows that

$$\mathcal{U}\{F(t);v,v,1\}=\mathrm{u}(v,v,1)=M\left[F(t);v\right]=H(v).$$

4.11 The Kamal Transform

A.K.H. Seedeg [38] introduced the Kamal transform in 2016. For the functions $F(t)$ belonging to the set A defined by (2.2) the Kamal transform is defined by (see, [38, (2), p. 452])

$$K\left[F(t);v\right]=G(v)=\int_0^\infty e^{-t/v}F(t)\,dt,\quad v\in[\eta_1,\eta_2]. \tag{4.12}$$

The choice of the parameters $\lambda_1=1,\lambda_2=\frac{1}{v}$ and $\lambda_3=1$ in (2.3) gives us (4.12) thus establishing that

$$\mathcal{U}\left\{F(t);1,\frac{1}{v},1\right\}=\mathrm{u}\left(1,\frac{1}{v},1\right)=K\left[F(t);v\right]=G(v).$$

4.12 The Mohand Transform

M.M.A. Mahgoub [39] introduced this transform in 2017, while in 2016 he had introduced the Mahgoub (in fact, he had reintroduced only the earlier well – known version of the Laplace-Carson transform) transform [37]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Mohand transform is defined by (see, [39, (2), p. 114])

$$M\left[F(t);v\right]=R(v)=v^2\int_0^\infty e^{-vt}F(t)\,dt,\quad v\in[\eta_1,\eta_2]. \tag{4.13}$$

We can see that the parameters $\lambda_1=v^2,\lambda_2=v$ and $\lambda_3=1$ in (2.3) generates (4.13), to prove that

$$\mathcal{U}\{F(t);v^2,v,1\}=\mathrm{u}(v^2,v,1)=M\left[F(t);v\right]=R(v).$$

We mention here that for setting the value of the parameter λ_1 in (2.1) equal to v^2 we can also have a choice $m_1=1=m_2,z_1=v,a_2=1,a_0=a_1=a_3=a_4=\ldots=a_{r_1}=0$ and $b_0=1,b_i=0\quad(i=1,\ldots,r_2)$ besides other possible choices also. The reader would now be able to see that how our expressions for defining the parameters $\lambda_1,\lambda_2,\lambda_3$ in (2.1) are justified for giving the definition of the Upadhyaya transform in (2.3) so that it generalizes the Laplace transform of one variable defined by (1.2) in the broadest possible sense.

4.13 The Tarig Transform

The Tarig transform was introduced by T.M. Elzaki and S.M. Elzaki [40-42] in 2011. For the functions $F(t)$ belonging to the set A defined by (2.2) the Tarig transform is defined by (see, [41, (1), p. 3230] and [43, (1), p. 13982])

$$T\left[F(t);u\right]=E_1[u]=\frac{1}{u}\int_0^{\infty}e^{-t/u^2}F(t)\,dt,\quad u\neq 0 \tag{4.14}$$

For the choice of the parameters $\lambda_1=\frac{1}{u},\lambda_2=\frac{1}{u^2}$ and $\lambda_3=1$ in (2.3) generates (4.13), to prove that

$$\mathcal{U}\left\{F(t);\frac{1}{u},\frac{1}{u^2},1\right\}=\mathfrak{u}\left(\frac{1}{u},\frac{1}{u^2},1\right)=T\left[F(t);u\right]=E_1[u].$$

We can also see that for the choice of the parameter λ_2 equal to $\frac{1}{u^2}$ in the Tarig transform we can see from (2.1) that the choice $m_3=m_4=1,c_0=1,c_i=0(i=1,\ldots,r_3),\quad z_4=u,d_2=1,d_0=d_1=d_3=d_4=\ldots=d_{r_4}=0$. Similarly, we can argue for the choice of the parameter λ_1 equal to $\frac{1}{u}$ in the Tarig transform by setting the values of the various constants and other quantities involved in (2.1). There can also be other choices for the values of the various constants and quantities in (2.1) to achieve these desired values of λ_1 and λ_2 for the Tarig transform. *Our aim for this discussion here is only to highlight the importance of (2.1) where the parameters λ_1,λ_2 and λ_3 are defined in such a manner which ensures that our definition (2.3) includes almost all the known and yet unknown variants of the classical Laplace transform defined by (1.2).*

4.14 The Sadik Transform

The Sadik transform was defined in 2018 by S.A. Shaikh [44]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Sadik transform is defined by (see, [44, (1), p. 101])

$$S\left[F(t);v^{\alpha},\beta\right]=\mathcal{F}\left(v^{\alpha},\beta\right)=\frac{1}{v^{\beta}}\int_0^{\infty}e^{-tv^{\alpha}}F(t)\,dt \tag{4.15}$$

where, v is a complex variable, α is any nonzero real number and β is any real number. We can see that the choice of the parameters $\lambda_1=\frac{1}{v^{\beta}},\lambda_2=v^{\alpha}$ and $\lambda_3=1$ in (2.3) gives us (4.15) to show that

$$\mathcal{U}\left\{F(t);\frac{1}{v^{\beta}},v^{\alpha},1\right\}=\mathfrak{u}\left(\frac{1}{v^{\beta}},v^{\alpha},1\right)=S\left[F(t);v^{\alpha},\beta\right]=\mathcal{F}\left(v^{\alpha},\beta\right).$$

As remarked in the case of the Tarig and Mohand transforms above, we can see that we can suitably choose the values of the various constants and other quantities in (2.1) to obtain the desired values of the parameters $\lambda_1,\lambda_2,\lambda_3$ so that (4.15) becomes a special case of (2.3). When α in (4.15) becomes any nonzero real number (i.e. other than a positive integer) we can remove the restriction that m_3 is a nonnegative integer. In that case we can say that $m_3=\alpha$ in (2.1) is any nonzero real number (positive, negative, rational or irrational, as the case may be) and can take in it $z_3=v,c_1=1,c_0=c_2=c_3=\ldots=c_{r_3}=0,m_4=1,d_0=1,d_i=0\left(i=1,\ldots,r_4\right)$ to make $\lambda_2=v^{\alpha}$. Similar suitable choices of the constants and other quantities involved in (2.1) can be made to give $\lambda_1=\frac{1}{v^{\beta}}$.

4.15 The Yang Transform

The Yang transform was introduced in 2016 by X.-J. Yang [45]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Yang transform is defined by (see, [45, (6), p. S640])

$$Y\left[F(t);\varpi\right]=\Phi[\varpi]=\int_0^{\infty}e^{-t/\varpi}F(t)\,dt \tag{4.16}$$

We can easily see that the choice $\lambda_1=1,\lambda_2=\frac{1}{\varpi}$ and $\lambda_3=1$ for the parameters in (2.3) leads us to (4.16) thereby showing that

$$\mathcal{U}\left\{F(t);1,\frac{1}{\varpi},1\right\}=\mathrm{u}\left(1,\frac{1}{\varpi},1\right)=Y\left[F(t);\varpi\right]=\Phi[\varpi].$$

4.16 The Shehu Transform

This transform was introduced very recently in this year itself (2019) by Shehu Maitama and Weidong Zhao [30]. For the functions $F(t)$ belonging to the set A defined by (2.2) the Shehu transform is defined by (see, [30, (2.1), p. 170])

$$\mathfrak{S}\left[F(t);s,u\right]=V(s,u)=\int_0^\infty e^{-st/u}F(t)\,dt \tag{4.17}$$

It is easy to discern that the choice $\lambda_1=1,\lambda_2=\frac{s}{u}$ and $\lambda_3=1$ for the parameters in (2.3) produces (4.17) to give the relation

$$\mathcal{U}\left\{F(t);1,\frac{s}{u},1\right\}=\mathrm{u}\left(1,\frac{s}{u},1\right)=\mathfrak{S}\left[F(t);s,u\right]=V(s,u).$$

4.17 The Sawi Transform

Mahgoub [46], who had earlier introduced the Mahgoub transform and the Mohand transform as discussed above, introduced this transform in this very year, i.e. 2019. For the functions $F(t)$ belonging to the set A defined by (2.2) the Sawi transform is defined by (see, [46, (2), p. 81])

$$S\left[F(t);v\right]=R(v)=\frac{1}{v^2}\int_0^\infty e^{-t/v}F(t)\,dt,\quad t\in\left[\eta_1,\eta_2\right]. \tag{4.18}$$

It may be quickly noted that the Sawi transform introduced in 2019 turns out to be a special case of the Sadik transform introduced in 2018, as (4.18) follows from (4.15) for $\beta=2$ and $\alpha=-1$. A simple inspection tells us that the choice $\lambda_1=\frac{1}{v^2},\lambda_2=\frac{1}{v}$ and $\lambda_3=1$ for the parameters in (2.3) produces (4.18) to show that

$$\mathcal{U}\left\{F(t);\frac{1}{v^2},\frac{1}{v},1\right\}=\mathrm{u}\left(\frac{1}{v^2},\frac{1}{v},1\right)=S\left[F(t);v\right]=R(v).$$

4.18 The Barne's Polynomial Integral Transform

This Polynomial Integral Transform was given by Benedict Barnes [47] in 2016. He defined it like this: if $F(x)$ be a function defined for $x\geq 1$, then the Barne's Polynomial integral transform of $F(x)$ is defined by (see [47, Theorem 1, p.142])

$$\mathcal{B}\left[F(x);s\right]=\mathfrak{F}(s)=\int_1^\infty x^{-s-1}F(\ln x)\,dx \tag{4.19}$$

provided the integral converges. *We can see that this integral transform is nothing but only a simple transformation of the classical Laplace transform (1.2) under the mere change of variable* $t=\ln x$. Similar remarks about this integral transform are made by Nuruddeen et al. [28] when they write " …Barnes (2016) [47] developed the polynomial integral transform, which we see it as a mere Laplace transform in another transformation" (see [28, p.2]) and also, " … the polynomial integral transforms by B. Barnes (2016) which we see it as just mere Laplace transform in different transformation as he admitted too in his article" (see [28, p. 12]). We remark that for the same change of variable, i.e. $t=\ln x$, the Upadhyaya transform (2.3) renders the following form

$$\mathcal{U}\left\{F(\ln x);\lambda_1,\lambda_2,\lambda_3\right\}=\mathrm{u}\left(\lambda_1,\lambda_2,\lambda_3\right)=\lambda_1\int_1^\infty x^{-\lambda_2-1}F\left(\lambda_3\ln x\right)dx \tag{4.20}$$

provided the integral converges. The comparison between (4.20) and (4.19) at once shows that the Barne's Polynomial integral transform (4.19) also follows as a special case of the Upadhyaya transform (or more appropriately, we may call it the *Upadhyaya Polynomial Transform* (UPT) in order to keep the mathematical terminology identical with that of B. Barne's [47]) (4.20) when we set $\lambda_1=1,\lambda_2=s,\lambda_3=1$ in it to deduce that

$$\mathcal{U}\{F(\ln x);1,s,1\} = \mathrm{u}(1,s,1) = \mathcal{B}[F(x);s] = \mathfrak{F}(s).$$

4.19 The Atangana-Kilicman Integral Operator Transform (AKIOT)

Abdon Atangana and Adem Kilicman [48] introduced the Atangana-Kilicman Integral Operator Transform (AKIOT) in 2013 and gave its properties and applied it to solve certain types of fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs). They defined it as follows:

If $F(x)$ be a function which is continuous in the open interval $(0,\infty)$, whose Laplace transform is differentiable n times, then, the Atangana-Kilicman Integral Operator Transform (AKIOT) of $F(x)$ of *order* n is defined by

$$M_n[F(x);s] = M_n(s) = \int_0^\infty e^{-sx} x^n F(x)\,dx \tag{4.21}$$

(see [48, (3), p.2]). We observe that the Upadhyaya transform (2.3) under the transformation $t \to t^n F(t)$ transforms into the following form (which we may call the *Upadhyaya Integral Operator Transform* (UIOT) in order to make the terminology identical with that used by Atangana and Kilicman [48]) of $F(x)$ of *order* n is defined by :

$$\mathcal{U}\{t^n F(t);\lambda_1,\lambda_2,\lambda_3,n\} = \mathrm{u}(\lambda_1,\lambda_2,\lambda_3,n) = \lambda_1\lambda_3^n \int_0^\infty e^{-\lambda_2 t} t^n F(\lambda_3 t)\,dt \tag{4.22}$$

Now a simple comparison of (4.22) with (4.21) shows that for the choice of the parameters $\lambda_1 = 1, \lambda_2 = s$, $\lambda_3 = 1, n \to n$ and the change of the dummy variable $t \to x$ in (4.22) leads one to (4.21) thence showing that the AKIOT of $F(x)$ of order n follows as a special case of UIOT of order n as

$$\mathcal{U}\{x^n F(x);1,s,1,n\} = \mathrm{u}(1,s,1,n) = M_n[F(x);s] = M_n(s).$$

4.20 The Hankel Transform

The well known Hankel transform is defined by (see [30, (1.6), p.168] or, [7, (7.2.8), p.345])

$$H_v[F(r);s] = \mathcal{F}_v(s) = \int_0^\infty rJ_v(sr)F(r)\,dr \tag{4.23}$$

where, denotes the Bessel function of the first kind of order v with $v \geq -\frac{1}{2}$. If we choose the function $F(t)$ in the definition of the Upadhyaya transform (2.3) as $tJ_v(st)F(t)$ then we get

$$\mathcal{U}\{tJ_v(st)F(t);\lambda_1,\lambda_2,\lambda_3,v\} = \mathrm{u}(\lambda_1,\lambda_2,\lambda_3,v) = \lambda_1\int_0^\infty e^{-\lambda_2 t}.(\lambda_3 t).J_v(s\lambda_3 t)F(\lambda_3 t)\,dt \tag{4.24}$$

It can be seen that (4.23) follows from (4.24) for the change of the dummy variable $t \to r$ in the latter with the choice of parameter values $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 1, v = v$ to yield the Hankel transform (4.23) as the following special case of (4.24)

$$\mathcal{U}\{tJ_v(st)F(t);1,0,1,v\} = \mathrm{u}(1,0,1,v) = H_v[F(r);s] = \mathcal{F}_v(s).$$

4.21 The $\mathcal{L}_2-$Transform of Yürekli and Sadek

Yürekli and Sadek [52] introduced the $\mathcal{L}_2-$ transform in 1991 which is defined as (see also, [53, (1.6), p. 2 of 21])

$$\mathcal{L}_2\{f(x);y\} = \tilde{F}(y) = \int_0^\infty xe^{-x^2y^2} f(x)\,dx, \Re(y) > 0. \tag{4.25}$$

If we set $\lambda_1 = \frac{1}{2}, \lambda_2 = y^2, \lambda_3 = 1$ and $t = x^2$ in (2.3) and define $F(t) = F(x^2) = \begin{cases} f(x), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$, we obtain

$$\mathcal{U}\left\{F(x^2);\frac{1}{2},y^2,1\right\} = \frac{1}{2}\int_0^\infty e^{-y^2x^2} F(1\cdot x^2)\cdot 2x\,dx = \int_0^\infty xe^{-y^2x^2} f(x)\,dx = \mathcal{L}_2\{f(x);y\} = \tilde{F}(y).$$

It can now be observed from (4.20), (4.22), (4.24) and (4.25) that by a suitable choice of the function $F(t)$ *and the values of the parameters* $\lambda_1, \lambda_2, \lambda_3$ *we can also extract several other known and unknown integral transforms from the basic definition (2.3) of the Upadhyaya transform.*

5. DEVELOPMENT OF THE THEORY OF THE UPADHYAYA INTEGRAL TRANSFORM

We now begin the process of development of the theory of the Upadhyaya integral transform (UT) defined by (2.3) above. Since the UT is one of the broadest possible generalizations of the Laplace transform defined by (1.2), we develop the theory of the UT parallel to the existing theory of the Laplace transform. We point out that all the results which we deduce below in this section reduce to the corresponding known results for the Laplace transform when we set the values of the parameters $\lambda_1, \lambda_2, \lambda_3$ in them as $\lambda_1 = 1, \lambda_2 = s$ and $\lambda_3 = 1$. This is only natural because the Upadhyaya transform (2.3) is a generalization of the Laplace transform (1.2) as seen in the subsection 4.1 earlier. We further record here that as we have shown in the fourth section of this paper that a number of extant variants of the Laplace transforms are special cases of the Upadhyaya transform, therefore with those respective special cases as mentioned there all the results being developed in this section of the paper will reduce to the corresponding known or yet unknown results for these various variants of the Laplace transforms. Hence our results for the Upadhaya transform being developed here may be viewed as the most generalized results developed till date in the existing theory of the Laplace transforms.

5.1 A sufficient condition for the existence of the UT

Theorem 5.1 If the function $F(t)$ is continuous or piecewise continuous in every finite interval $(0,T)$ and of exponential order a for $t \geq T$, then the UT of $F(t)$ defined by (2.3) exists for all λ_2 for which $\Re(\lambda_2 - a\lambda_3) > 0$ where, $\Re(\lambda_2 - a\lambda_3)$ denotes the real part of the complex number $\lambda_2 - a\lambda_3$.

Proof We can write (2.3) as below:

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathrm{u}(\lambda_1,\lambda_2,\lambda_3) = \lambda_1 \int_0^T e^{-\lambda_2 t} F(\lambda_3 t)\,dt + \lambda_1 \int_T^\infty e^{-\lambda_2 t} F(\lambda_3 t)\,dt \qquad (5.1)$$

Since $F(t)$ is continuous or piecewise continuous in every finite interval $(0,T)$, therefore, the first integral on the right hand side of (5.1) exists. We now only need to ensure the existence of the second integral on the right hand side of (5.1) in order that the UT of $F(t)$ exists. For that we observe that

$$\left|\lambda_1 \int_T^\infty e^{-\lambda_2 t} F(\lambda_3 t)\,dt\right| \leq |\lambda_1| \left|\int_T^\infty e^{-\lambda_2 t} F(\lambda_3 t)\,dt\right| \leq |\lambda_1| \left|\int_T^\infty e^{-\lambda_2 t}\right| |F(\lambda_3 t)|\,dt.$$

Since, $F(t)$ is of exponential order a for $t \geq T$, from the Definition 1.1 there exists a positive constant K such that for all $t > T$ such that $|F(t)| \leq Ke^{at}$, which gives

$$\left|\lambda_1 \int_T^\infty e^{-\lambda_2 t} F(\lambda_3 t)\,dt\right| < K|\lambda_1| \left|\int_T^\infty e^{-\lambda_2 t} . e^{a\lambda_3 t}\,dt < K|\lambda_1| \right| \int_0^\infty e^{-(\lambda_2 - a\lambda_3)t}\,dt$$

$$= \frac{K|\lambda_1|}{(\lambda_2 - a\lambda_3)}, \quad \text{if} \quad \Re(\lambda_2 - a\lambda_3) > 0.$$

This shows that the second integral on the right hand side of (5.1) also exists if $\Re(\lambda_2 - a\lambda_3) > 0$.

■

5.2 The Upadhyaya – Laplace Duality

Theorem 5.2 Suppose that for a function $F(t)$ both the Upadhyaya and Laplace transforms exist for the sets of parameters, say, $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_1^{'}, \lambda_2^{'}, \lambda_3^{'}$. Then the following relations hold

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathrm{u}(\lambda_1,\lambda_2,\lambda_3) = \frac{\lambda_1}{\lambda_3} \mathcal{L}\left\{F(t);\frac{\lambda_2}{\lambda_3}\right\} = \frac{\lambda_1}{\lambda_3} f\left(\frac{\lambda_2}{\lambda_3}\right), \quad \Re\left(\frac{\lambda_2}{\lambda_3}\right) > 0. \qquad (5.2)$$

$$\mathcal{L}\left\{F(t);\frac{\lambda_2^{'}}{\lambda_3^{'}}\right\}=f\left(\frac{\lambda_2^{'}}{\lambda_3^{'}}\right)=\frac{\lambda_3^{'}}{\lambda_1^{'}}\mathcal{U}\left\{F(t);\lambda_1^{'},\lambda_2^{'},\lambda_3^{'}\right\}=\frac{\lambda_3^{'}}{\lambda_1^{'}}\mathrm{u}\left(\lambda_1^{'},\lambda_2^{'},\lambda_3^{'}\right),\quad \Re\left(\frac{\lambda_2^{'}}{\lambda_3^{'}}\right)>0. \tag{5.3}$$

Proof To see (5.2) we consider the UT of $F(t)$ and recall (2.3) as

$$\mathcal{U}\left\{F(t);\lambda_1,\lambda_2,\lambda_3\right\}=\mathrm{u}\left(\lambda_1,\lambda_2,\lambda_3\right)=\lambda_1\int_0^\infty e^{-\lambda_2 t}F\left(\lambda_3 t\right)dt$$

in which we substitute $u=\lambda_3 t$ to get

$$\mathcal{U}\left\{F(t);\lambda_1,\lambda_2,\lambda_3\right\}=\mathrm{u}\left(\lambda_1,\lambda_2,\lambda_3\right)=\frac{\lambda_1}{\lambda_3}\int_0^\infty e^{-\lambda_2 u/\lambda_3}F\left(u\right)du$$

$$=\frac{\lambda_1}{\lambda_3}\mathcal{L}\left\{F(t);\frac{\lambda_2}{\lambda_3}\right\}=\frac{\lambda_1}{\lambda_3}f\left(\frac{\lambda_2}{\lambda_3}\right),\text{ if }\Re\left(\frac{\lambda_2}{\lambda_3}\right)>0.$$

For obtaining (5.3) we consider the Laplace transform of $F(t)$ and using (1.2) write

$$\mathcal{L}\left\{F(t);\frac{\lambda_2^{'}}{\lambda_3^{'}}\right\}=f\left(\frac{\lambda_2^{'}}{\lambda_3^{'}}\right)=\int_0^\infty e^{-\lambda_2^{'}t/\lambda_3^{'}}F(t)\,dt$$

wherein we put $t=\lambda_3^{'}v$ to see

$$\mathcal{L}\left\{F(t);\frac{\lambda_2^{'}}{\lambda_3^{'}}\right\}=f\left(\frac{\lambda_2^{'}}{\lambda_3^{'}}\right)=\lambda_3^{'}\int_0^\infty e^{-\lambda_2^{'}v}F\left(\lambda_3^{'}v\right)dv=\frac{\lambda_3^{'}}{\lambda_1^{'}}\left[\lambda_1^{'}\int_0^\infty e^{-\lambda_2^{'}v}F\left(\lambda_3^{'}v\right)dv\right]$$

$$=\frac{\lambda_3^{'}}{\lambda_1^{'}}\mathcal{U}\left\{F(t);\lambda_1^{'},\lambda_2^{'},\lambda_3^{'}\right\}=\frac{\lambda_3^{'}}{\lambda_1^{'}}\mathrm{u}\left(\lambda_1^{'},\lambda_2^{'},\lambda_3^{'}\right),\text{ if }\Re\left(\frac{\lambda_2^{'}}{\lambda_3^{'}}\right)>0.$$

Note that in the last equality above the quantity $\Re\left(\frac{\lambda_2^{'}}{\lambda_3^{'}}\right)>0$ enters automatically as we have assumed that $\mathcal{L}\left\{F(t);\frac{\lambda_2^{'}}{\lambda_3^{'}}\right\}$ exists. ■

As per the convention prevalent in the current mathematical research literature (see, for example, [23, (2.7), pp. 18-19]; [50, Theorem 2.3.1 on p.3, Theorem 3.1.1, Theorem 3.2.1 and Theorem 3.3.1 on p.4, Theorem 3.4.1 and Theorem 4.1.1 on p. 5, Theorem 4.2.1 and Theorem 4.3.1 on p.6 and Theorem 4.4.1 on p.7]; [51, p.1, 2, Section 3 and (3.2) on p. 5, Second Entry of Table 5.1 on p.18]) we call the relations (5.2) and (5.3) above the *Upadhaya-Laplace Duality Relations* (ULD). It is also pertinent to mention here that similar other duality relations between the Upadhyaya transform and other extant variants of the Laplace transforms as mentioned in the preceding section of this paper also exist and they can be established along the similar lines. This will be the subject of one of the future papers of this author.

5.3 Upadhyaya Transforms of Some Elementary Functions

In view of (3.7), if we disallow null functions (see, [6, p.9 and p.42]) we can see that the inverse Upadhyaya transform of a given function is unique. The same can be calculated from (3.7) by finding the sum of residues of the integrand utilizing the Residue Theorem (see [6, (24), p. 143 and Chapter 7]). Now we calculate the Upadhyaya transform of some elementary functions which are often utilized in various branches of mathematics and also occur frequently in the problems of physics and engineering. The Upadhyaya transform of the trigonometric and hyperbolic functions will be given in the sequel.

For finding the UT of t^a where a is any complex number, we have from (2.3),

$$\begin{aligned}\mathcal{U}\left\{t^a;\lambda_1,\lambda_2,\lambda_3\right\}&=\lambda_1\int_0^\infty e^{-\lambda_2 t}\left(\lambda_3 t\right)^a dt=\lambda_1\lambda_3^a\int_0^\infty e^{-\lambda_2 t}t^a dt\\&=\frac{\lambda_1\lambda_3^a\Gamma(a+1)}{\lambda_2^{a+1}},\Re\left(\lambda_2\right)>0,\Re\left(a\right)>-1.\end{aligned}\tag{5.4}$$

From (5.4) follows that

$$\mathcal{U}^{-1}\left\{\frac{1}{\lambda_2^{a+1}};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{t^a}{\lambda_1\lambda_3^a\Gamma(a+1)},\Re(\lambda_2)>0,\Re(a)>-1. \tag{5.5}$$

Some special cases of (5.4) and (5.5) are as follows:
When $a=n,n\in\mathbb{N}$ (the set of natural numbers) it follows from (5.4) and (5.5) at once that

$$\mathcal{U}\{t^n;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1\lambda_3^n n!}{\lambda_2^{n+1}},\Re(\lambda_2)>0 \tag{5.6}$$

and

$$\mathcal{U}^{-1}\left\{\frac{1}{\lambda_2^{n+1}};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{t^n}{\lambda_1\lambda_3^n n!},\Re(\lambda_2)>0. \tag{5.7}$$

When $a=0$ (5.4) and (5.5) give

$$\mathcal{U}\{1;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{\lambda_2},\Re(\lambda_2)>0. \tag{5.8}$$

and

$$\mathcal{U}^{-1}\left\{\frac{1}{\lambda_2};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{1}{\lambda_1},\Re(\lambda_2)>0. \tag{5.9}$$

5.4 The Linearity Property of UT

Theorem 5.3 Let $F_1(t),F_2(t)$ be two functions with UTs $\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)$ and $\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)$ relative to the parameters $\lambda_1,\lambda_2,\lambda_3$ and c_1,c_2 be any constants then

$$\begin{aligned}\mathcal{U}\{c_1F_1(t)+c_2F_2(t);\lambda_1,\lambda_2,\lambda_3\}&=c_1\mathcal{U}\{F_1(t);\lambda_1,\lambda_2,\lambda_3\}+c_2\mathcal{U}\{F_2(t);\lambda_1,\lambda_2,\lambda_3\}\\&=c_1\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)+c_2\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)\end{aligned} \tag{5.10}$$

Proof This result follows immediately from (2.3) by using the property of linearity of integrals. It can be easily extended to the case of more than two functions. ■

5.5 First Shifting (First Translation) of UT

Theorem 5.4 If $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$ then

$$\mathcal{U}\{e^{at}F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathcal{U}\{F(t);\lambda_1,\lambda_2-a\lambda_3,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2-a\lambda_3,\lambda_3) \tag{5.11}$$

Proof Using (2.3) we write the left hand side of (5.11) as

$$\begin{aligned}\mathcal{U}\{e^{at}F(t);\lambda_1,\lambda_2,\lambda_3\}&=\lambda_1\int_0^\infty e^{-\lambda_2 t}\{e^{a\lambda_3 t}F(\lambda_3 t)\}dt=\lambda_1\int_0^\infty e^{-(\lambda_2-a\lambda_3)t}F(\lambda_3 t)\,dt\\&=\mathcal{U}\{F(t);\lambda_1,\lambda_2-a\lambda_3,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2-a\lambda_3,\lambda_3)\end{aligned}$$

■

Example 5.5 As an illustration of the first shifting theorem of UT we find the UT of the function $\frac{t^a e^{bt}}{\Gamma(a+1)},\Re(b)>-1$. From (5.4) and (5.11) follows that

$$\begin{aligned}\mathcal{U}\left\{\frac{t^a e^{bt}}{\Gamma(a+1)};\lambda_1,\lambda_2,\lambda_3\right\}&=\mathcal{U}\left\{\frac{t^a}{\Gamma(a+1)};\lambda_1,\lambda_2-b\lambda_3,\lambda_3\right\}\\&=\frac{\lambda_1\lambda_3^a}{(\lambda_2-b\lambda_3)^{a+1}},\Re(\lambda_2-b\lambda_3)>0,\Re(a)>-1.\end{aligned} \tag{5.12}$$

which also renders the companion result for the inverse UT of the above function as

$$\mathcal{U}^{-1}\left\{\frac{1}{(\lambda_2-b\lambda_3)^{a+1}};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{t^a e^{bt}}{\Gamma(a+1)\lambda_1\lambda_3^a},\Re(\lambda_2-b\lambda_3)>0,\Re(a)>-1. \tag{5.13}$$

A special case of (5.12) deserves special mention. When $a=0$, (5.12) reduces to

$$\mathcal{U}\{e^{bt};\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{(\lambda_2-b\lambda_3)},\Re(\lambda_2-b\lambda_3)>0. \tag{5.14}$$

and the inverse UT formula for recovering the original function is

$$\mathcal{U}^{-1}\left\{\frac{1}{(\lambda_2-b\lambda_3)};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{e^{bt}}{\lambda_1},\Re(\lambda_2-b\lambda_3)>0. \tag{5.15}$$

If we set $b=ik\left(i=\sqrt{-1}\right)$ in (5.14) we see that

$$\mathcal{U}\{e^{ikt};\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{(\lambda_2-ik\lambda_3)},\Re(\lambda_2-ik\lambda_3)>0.$$

Noting that $e^{ikt}=\cos kt+i\sin kt$ and the linearity property of UT (Theorem 5.3) in the above relation yields

$$\mathcal{U}\{\cos kt;\lambda_1,\lambda_2,\lambda_3\}+i\mathcal{U}\{\sin kt;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1(\lambda_2+ik\lambda_3)}{(\lambda_2-ik\lambda_3)(\lambda_2+ik\lambda_3)}=\frac{\lambda_1(\lambda_2+ik\lambda_3)}{\lambda_2^2+k^2\lambda_3^2}.$$

Now equating the real and imaginary parts this last relation at once admits

$$\mathcal{U}\{\cos kt;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1\lambda_2}{\lambda_2^2+k^2\lambda_3^2},\Re(\lambda_2-ik\lambda_3)>0 \tag{5.16}$$

and

$$\mathcal{U}\{\sin kt;\lambda_1,\lambda_2,\lambda_3\}=\frac{k\lambda_1\lambda_3}{\lambda_2^2+k^2\lambda_3^2},\Re(\lambda_2-ik\lambda_3)>0. \tag{5.17}$$

for which the corresponding formulae of inverse UT for recovering the original functions are respectively

$$\mathcal{U}^{-1}\left\{\frac{\lambda_2}{\lambda_2^2+k^2\lambda_3^2};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\cos kt}{\lambda_1},\Re(\lambda_2-ik\lambda_3)>0 \tag{5.18}$$

and

$$\mathcal{U}^{-1}\left\{\frac{1}{\lambda_2^2+k^2\lambda_3^2};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\sin kt}{k\lambda_1\lambda_3},\Re(\lambda_2-ik\lambda_3)>0. \tag{5.19}$$

For finding the UT of the hyperbolic functions we proceed as below:

$$\mathcal{U}\{\sinh at;\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\int_0^\infty e^{-\lambda_2 t}\sinh(a\lambda_3 t)\,dt=\frac{\lambda_1}{2}\int_0^\infty e^{-\lambda_2 t}\left(e^{a\lambda_3 t}-e^{-a\lambda_3 t}\right)dt$$

$$=\frac{\lambda_1}{2}\int_0^\infty\left(e^{-(\lambda_2-a\lambda_3)t}-e^{-(\lambda_2+a\lambda_3)t}\right)dt=\frac{\lambda_1}{2}\left[\frac{1}{\lambda_2-a\lambda_3}-\frac{1}{\lambda_2+a\lambda_3}\right],$$

for $\Re(\lambda_2-a\lambda_3)>0$ and $\Re(\lambda_2+a\lambda_3)>0$, which on simplification yields,

$$\mathcal{U}\{\sinh at;\lambda_1,\lambda_2,\lambda_3\}=\frac{a\lambda_1\lambda_3}{\lambda_2^2-a^2\lambda_3^2},\Re(\lambda_2\pm a\lambda_3)>0. \tag{5.20}$$

The inverse UT formula corresponding to the last result is

$$\mathcal{U}^{-1}\left\{\frac{1}{\lambda_2^2-a^2\lambda_3^2};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\sinh at}{a\lambda_1\lambda_3},\Re(\lambda_2\pm a\lambda_3)>0. \tag{5.21}$$

Similarly, we can show that

$$\mathcal{U}\{\cosh at;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1\lambda_2}{\lambda_2^2-a^2\lambda_3^2},\Re(\lambda_2\pm a\lambda_3)>0. \tag{5.22}$$

and

$$\mathcal{U}^{-1}\left\{\frac{\lambda_2}{\lambda_2^2-a^2\lambda_3^2};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{\cosh at}{\lambda_1},\Re(\lambda_2\pm a\lambda_3)>0. \tag{5.23}$$

Following Spiegel [6, p.3] we provide yet another example of the first shifting of UT below:

Example 5.6 From (5.16) we can write $\mathcal{U}\{\cos 2t;\lambda_1,\lambda_2,\lambda_3\}=\dfrac{\lambda_1\lambda_2}{\lambda_2^2+4\lambda_3^2}$ which in view of (5.11) at once yields that $\mathcal{U}\{e^{-t}\cos 2t;\lambda_1,\lambda_2,\lambda_3\}=\mathcal{U}\{\cos 2t;\lambda_1,\lambda_2+\lambda_3,\lambda_3\}=\dfrac{\lambda_1(\lambda_2+\lambda_3)}{(\lambda_2+\lambda_3)^2+4\lambda_3^2}$. It can be seen that in the special case when $\lambda_1=1,\lambda_2=s,\lambda_3=1$, when the UT reduces to the Laplace transform, this expression gives $\mathcal{U}\{e^{-t}\cos 2t;1,s,1\}=\mathcal{U}\{\cos 2t;1,s+1,1\}=\dfrac{(s+1)}{(s+1)^2+4}=\mathcal{L}\{e^{-t}\cos 2t;s\}$ as given in [6, p.3].

It can also be seen here that, as an illustration of the Upadhyaya-Laplace Duality relation (5.3), we can also obtain the UT of the function $e^{-t}\cos 2t$ with respect to the parameters $\lambda_1',\lambda_2',\lambda_3'$ given that its Laplace transform with respect to the parameter s is $\dfrac{(s+1)}{(s+1)^2+4}$. To that end we start from the left hand side of (5.3) as follows

$$\mathcal{L}\left\{e^{-t}\cos 2t;\frac{\lambda_2'}{\lambda_3'}\right\}=\frac{\frac{\lambda_2'}{\lambda_3'}+1}{\left(\frac{\lambda_2'}{\lambda_3'}+1\right)^2+4}=\frac{(\lambda_2'+\lambda_3')\lambda_3'}{(\lambda_2'+\lambda_3')^2+4\lambda_3'^2}$$

$$=\frac{\lambda_3'}{\lambda_1'}\left[\frac{(\lambda_2'+\lambda_3')\lambda_1'}{(\lambda_2'+\lambda_3')^2+4\lambda_3'^2}\right]=\frac{\lambda_3'}{\lambda_1'}\mathcal{U}\{e^{-t}\cos 2t;\lambda_1',\lambda_2',\lambda_3'\}.$$

5.6 Second Shifting (Second Translation) of UT

Theorem 5.7 Let $F(t)$ be a function with UT $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$. Let

$$G(t)=\begin{cases}F(t-a), t>a\\ 0, t<a\end{cases}=F(t-a)H(t-a)$$

where, $H(t-a)$ is the Heaviside step function defined by (3.3). Then

$$\mathcal{U}\{F(t-a)H(t-a);\lambda_1,\lambda_2,\lambda_3\}=e^{-\lambda_2 a/\lambda_3}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=e^{-\lambda_2 a/\lambda_3}\mathrm{u}(\lambda_1,\lambda_2,\lambda_3). \tag{5.24}$$

Proof From the definition of UT (2.3) leads us to

$$\mathcal{U}\{G(u);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\int_0^\infty e^{-\lambda_2 u}G(\lambda_3 u)\,du.$$

Noting that,

$$G(\lambda_3 u)=\begin{cases}F(\lambda_3 u-a),\ \lambda_3 u>a\\ 0,\ \lambda_3 u<a\end{cases}$$

the last expression gives

$$\mathcal{U}\{G(u);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\int_{a/\lambda_3}^\infty e^{-\lambda_2 u}F(\lambda_3 u-a)\,du$$

which on putting $v=\lambda_3 u-a$ changes into

$$\mathcal{U}\{G(u);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{\lambda_3}e^{-\lambda_2 a/\lambda_3}\int_0^\infty e^{-\lambda_2 v/\lambda_3}F(v)\,dv.$$

Now substituting $v=\lambda_3 w$, the above expression finally yields

$$\mathcal{U}\{G(u);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1 e^{-\lambda_2 a/\lambda_3}\int_0^\infty e^{-\lambda_2 w}F(\lambda_3 w)\,dw=e^{-\lambda_2 a/\lambda_3}\mathcal{U}\{G(u);\lambda_1,\lambda_2,\lambda_3\}.$$

■

Alternative Proof of the Second Shifting Theorem of UT by using the ULD
A shorter proof of (5.24) can be given by invoking the Upadhyaya-Laplace Duality (ULD) (5.3) which gives that for any function $F(t)$ its Laplace transform and UT are connected by the relation

$$\mathcal{L}\left\{F(t);\frac{\lambda_2}{\lambda_3}\right\}=\frac{\lambda_3}{\lambda_1}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}. \tag{5.25}$$

From Speigel [6, (4), p. 3] we have $\mathcal{L}\{G(t);s\}=e^{-as}\mathcal{L}\{F(t);s\}$, which for $s\to\frac{\lambda_2}{\lambda_3}$ gives

$$\mathcal{L}\left\{G(t);\frac{\lambda_2}{\lambda_3}\right\}=e^{-a\lambda_2/\lambda_3}\mathcal{L}\left\{F(t);\frac{\lambda_2}{\lambda_3}\right\}. \tag{5.26}$$

Now on using (5.25) on both sides of (5.26) yields

$$\frac{\lambda_3}{\lambda_1}\mathcal{U}\{G(t);\lambda_1,\lambda_2,\lambda_3\}=e^{-a\lambda_2/\lambda_3}\frac{\lambda_3}{\lambda_1}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}$$

from where the sought result follows immediately. ■

An illustration of the Second Shifting Theorem of UT is given below:

Example 5.8 Following Spiegel [6, p.4] we consider $F(t)=t^3$, then $\mathcal{U}\{t^3;\lambda_1,\lambda_2,\lambda_3\}=\frac{3!\lambda_1\lambda_3^3}{\lambda_2^4}$ (from (5.6)), thus (5.24) gives

$$\mathcal{U}\left\{(t-2)^3 H(t-2);\lambda_1,\lambda_2,\lambda_3\right\}=\frac{3!\lambda_1\lambda_3^3}{\lambda_2^4}e^{-2\lambda_2/\lambda_3}.$$

This result reduces in the limiting case (when $\lambda_1=1,\lambda_2=s,\lambda_3=1$) to the known result $\mathcal{L}\left\{(t-2)^3 H(t-2)\right\}=\frac{6e^{-2s}}{s^4}$ (see Spiegel [6, p.4]).

5.7 Change of Scale Property of UT

Theorem 5.9 Let $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$ then

$$\mathcal{U}\{F(at);\lambda_1,\lambda_2,\lambda_3\}=\mathcal{U}\left\{F(t);\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right\}=\mathrm{u}\left(\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right). \tag{5.27}$$

Proof Let $G(t)=F(at)$. Then

$$\mathcal{U}\{G(u);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\int_0^\infty e^{-\lambda_2 u}G(\lambda_3 u)\,du=\lambda_1\int_0^\infty e^{-\lambda_2 u}F(a\lambda_3 u)\,du,$$

which upon putting $au=v$ gives

$$\mathcal{U}\{F(au);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{a}\int_0^\infty e^{-\lambda_2 v/a}F(\lambda_3 v)\,dv=\mathcal{U}\left\{F(v);\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right\}.$$

■

An illustration of the Change of Scale Property of UT now follows:

Example 5.10 Following Spiegel [6, p.3] we consider $F(t)=\sin t$ which gives

$$\mathcal{U}\{\sin t;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2},$$

thus follows

$$\mathcal{U}\{\sin 3t;\lambda_1,\lambda_2,\lambda_3\}=\frac{\left(\frac{\lambda_1}{3}\right)\lambda_3}{\left(\frac{\lambda_2}{3}\right)^2+\lambda_3^2}=\frac{3\lambda_1\lambda_3}{\lambda_2^2+9\lambda_3^2}.$$

This expression in the limiting case $\lambda_1=1,\lambda_2=s,\lambda_3=1$ gives the known result (see [6, p.3])

$$\mathcal{U}\{\sin 3t;1,s,1\}=\frac{3}{s^2+9}=\mathcal{L}\{\sin 3t;s\}.$$

5.8 UT of Derivatives

Theorem 5.11 If $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$ and if $F(t)$ is continuous for $0\le t\le N$ and is of exponential order for $t>N$, while $F'(t)$ is sectionally continuous for $0\le t\le N$ then

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_2}{\lambda_3}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}-\frac{\lambda_1}{\lambda_3}F(0), \tag{5.28}$$

and

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{\lambda_3}\mathcal{U}\{F(t);\lambda_2,\lambda_2,\lambda_3\}-\frac{\lambda_1}{\lambda_3}F(0). \tag{5.29}$$

Proof From (2.3) we have

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\int_0^\infty e^{-\lambda_2 t}F'(\lambda_3 t)\,dt.$$

Put $u=\lambda_3 t$ to get

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{\lambda_3}\int_0^\infty e^{-\lambda_2 u/\lambda_3}F'(u)\,du=-\frac{\lambda_1}{\lambda_3}F(0)+\frac{\lambda_1\lambda_2}{\lambda_3^2}\int_0^\infty e^{-\lambda_2 u/\lambda_3}F(u)\,du$$

which on putting $u=\lambda_3 t$ in the last integral yields

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=-\frac{\lambda_1}{\lambda_3}F(0)+\frac{\lambda_1\lambda_2}{\lambda_3}\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt \tag{5.30}$$

Now (5.28) follows from (5.30) if we take out $\frac{\lambda_2}{\lambda_3}$ outside the second integral on the right hand side of (5.30) and interpret the remaining expression as $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}$. Similarly, by taking out $\frac{\lambda_1}{\lambda_3}$ outside the second integral on the right hand side of (5.30) and interpreting the remaining expression as $\mathcal{U}\{F(t);\lambda_2,\lambda_2,\lambda_3\}$ produces (5.29). ■

Example 5.12 Following Spiegel [6, p.4] we take $F(t)=\cos 3t$, which gives $F'(t)=-3\sin 3t$ and $F(0)=1$ then

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\mathcal{U}\{-3\sin 3t;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_2}{\lambda_3}\mathcal{U}\{\cos 3t;\lambda_1,\lambda_2,\lambda_3\}-\frac{\lambda_1}{\lambda_3}\times 1$$

$$=\frac{\lambda_1\lambda_2^2}{\lambda_3\left(\lambda_2^2+9\lambda_3^2\right)}-\frac{\lambda_1}{\lambda_3}.$$

This result in the limiting case when $\lambda_1=1,\lambda_2=s,\lambda_3=1$ gives the known result (see [6, p.4])

$$\mathcal{U}\{-3\sin 3t;1,s,1\} = -1 + \frac{s^2}{s^2+9} = \frac{-9}{s^2+9} = \mathcal{L}\{-3\sin 3t;s\}.$$

For finding the *UT of the second derivative* of $F(t)$ we assume that $F(t)$ and $F'(t)$ are continuous in $0 \le t \le N$ and of exponential order for $t > N$ while $F''(t)$ is sectionally continuous for $0 \le t \le N$ then

$$\mathcal{U}\{F''(t);\lambda_1,\lambda_2,\lambda_3\} = \left(\frac{\lambda_2}{\lambda_3}\right)^2 \mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} - \frac{\lambda_1\lambda_2}{\lambda_3^2}F(0) - \frac{\lambda_1}{\lambda_3}F'(0) \tag{5.31}$$

To prove the result of (5.31) we let $G(t) = F'(t)$ then $G'(t) = F''(t)$ thus

$$\mathcal{U}\{F''(t);\lambda_1,\lambda_2,\lambda_3\} = \mathcal{U}\{G'(t);\lambda_1,\lambda_2,\lambda_3\} = \frac{\lambda_2}{\lambda_3}\mathcal{U}\{G(t);\lambda_1,\lambda_2,\lambda_3\} - \frac{\lambda_1}{\lambda_3}G(0)$$

on using (5.28), which may be rewritten as

$$\mathcal{U}\{F''(t);\lambda_1,\lambda_2,\lambda_3\} = \frac{\lambda_2}{\lambda_3}\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\} - \frac{\lambda_1}{\lambda_3}F'(0).$$

A further application of (5.28) on the right hand side of this equation leads us to (5.31). By induction we may similarly show that the *UT of the* n^{th} *- derivative of* $F(t)$ is given by

$$\begin{aligned}\mathcal{U}\{F^{(n)}(t);\lambda_1,\lambda_2,\lambda_3\} = {} & \left(\frac{\lambda_2}{\lambda_3}\right)^n \mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} - \frac{\lambda_1\lambda_2^{n-1}}{\lambda_3^n}F(0) - \frac{\lambda_1\lambda_2^{n-2}}{\lambda_3^{n-1}}F'(0) \\ & - \frac{\lambda_1\lambda_2^{n-3}}{\lambda_3^{n-2}}F''(0) - \ldots - \frac{\lambda_1}{\lambda_3}F^{(n-1)}(0).\end{aligned} \tag{5.32}$$

5.9 UT of Integrals

Theorem 5.13 Let $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$ then

$$\mathcal{U}\left\{\int_0^t F(u)\,du;\lambda_1,\lambda_2,\lambda_3\right\} = \frac{\lambda_3}{\lambda_2}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \frac{\lambda_3}{\lambda_2}\mathrm{u}(\lambda_1,\lambda_2,\lambda_3). \tag{5.33}$$

Proof Let $G(t) = \int_0^t F(u)\,du$, then $G'(t) = F(t)$ and $G(0) = \int_0^0 F(u)\,du = 0$. Now,

$$\mathcal{U}\{G'(t);\lambda_1,\lambda_2,\lambda_3\} = \mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}$$

Or,

$$\frac{\lambda_2}{\lambda_3}\mathcal{U}\{G(t);\lambda_1,\lambda_2,\lambda_3\} - \frac{\lambda_1}{\lambda_3}G(0) = \mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}$$

which at once yields (5.33) after a little simplification. ■

Example 5.14 Following Spiegel [6, p.4] we consider $F(t) = \sin 2t$ which yields

$$\mathcal{U}\{\sin 2t;\lambda_1,\lambda_2,\lambda_3\} = \frac{2\lambda_1\lambda_3}{\lambda_2^2 + 4\lambda_3^2}.$$

Thus,

$$\mathcal{U}\left\{\int_0^t \sin 2u\,du;\lambda_1,\lambda_2,\lambda_3\right\} = \frac{\lambda_3}{\lambda_2}\cdot\frac{2\lambda_1\lambda_3}{\lambda_2^2+4\lambda_3^2} = \frac{2\lambda_1\lambda_3^2}{\lambda_2\left(\lambda_2^2+4\lambda_3^2\right)}.$$

This result for the limiting case $\lambda_1 = 1, \lambda_2 = s, \lambda_3 = 1$ gives the known result (see [6, p.4])

$$\mathcal{U}\left\{\int_0^t \sin 2u\,du;1,s,1\right\} = \frac{2}{s\left(s^2+4\right)} = \mathcal{L}\left\{\int_0^t \sin 2u\,du;s\right\}.$$

5.10 UT for Multiplication by Powers of t

Theorem 5.15 For any natural number n, if the UT of any function $F(t)$, i.e. $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}$ $=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$ is differentiable $n-$ times with respect to the parameter λ_2, then we have

$$\mathcal{U}\{t^nF(t);\lambda_1,\lambda_2,\lambda_3\}=(-\lambda_3)^n\frac{d^n}{d\lambda_2^n}\{\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}\}=(-\lambda_3)^n\frac{d^n}{d\lambda_2^n}\{\mathrm{u}(\lambda_1,\lambda_2,\lambda_3).\} \tag{5.34}$$

Proof We observe that for $n=1$,

$$\frac{d}{d\lambda_2}\{\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}\}=\frac{d}{d\lambda_2}\left[\lambda_1\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt\right]=\lambda_1\int_0^\infty\frac{\partial}{\partial\lambda_2}\{e^{-\lambda_2 t}F(\lambda_3 t)\}\,dt$$

$$=\lambda_1\int_0^\infty(-t)e^{-\lambda_2 t}F(\lambda_3 t)\,dt=-\frac{\lambda_1}{\lambda_3}\int_0^\infty e^{-\lambda_2 t}(\lambda_3 t)F(\lambda_3 t)\,dt=-\frac{1}{\lambda_3}\mathcal{U}\{tF(t);\lambda_1,\lambda_2,\lambda_3\}$$

which gives that

$$\mathcal{U}\{tF(t);\lambda_1,\lambda_2,\lambda_3\}=(-\lambda_3)\frac{d}{d\lambda_2}\left[\lambda_1\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt\right]$$

showing that (5.34) holds for $n=1$. Suppose it holds for the case $n=r$, r being a positive integer greater than 1 thus, we can write

$$\frac{d^r}{d\lambda_2^r}\{\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}\}=\left(\frac{-1}{\lambda_3}\right)^r\mathcal{U}\{t^rF(t);\lambda_1,\lambda_2,\lambda_3\}.$$

Differentiating both sides of the above equation with respect to λ_2 gives

$$\frac{d^{r+1}}{d\lambda_2^{r+1}}\{\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}\}=\left(\frac{-1}{\lambda_3}\right)^r\frac{d}{d\lambda_2}\{\mathcal{U}\{t^rF(t);\lambda_1,\lambda_2,\lambda_3\}\}=\left(\frac{-1}{\lambda_3}\right)^r\lambda_1\int_0^\infty\frac{\partial}{\partial\lambda_2}(e^{-\lambda_2 t})(\lambda_3 t)^rF(\lambda_3 t)\,dt$$

$$=\left(\frac{-1}{\lambda_3}\right)^r\lambda_1\int_0^\infty e^{-\lambda_2 t}(-t)(\lambda_3 t)^rF(\lambda_3 t)\,dt=\left(\frac{-1}{\lambda_3}\right)^{r+1}\lambda_1\int_0^\infty e^{-\lambda_2 t}(\lambda_3 t)^{r+1}F(\lambda_3 t)\,dt=\left(\frac{-1}{\lambda_3}\right)^{r+1}\mathcal{U}\{t^{r+1}F(t);\lambda_1,\lambda_2,\lambda_3\}.$$

(5.34) now follows immediately from this equation by the principle of induction. ■

Example 5.16 Following Spiegel [6, p.5] we consider $F(t)=e^{2t}$, then $\mathcal{U}\{e^{2t};\lambda_1,\lambda_2,\lambda_3\}=\dfrac{\lambda_1}{\lambda_2-2\lambda_3}$ which gives

$$c(-\lambda_3)\frac{d}{d\lambda_2}\left\{\frac{\lambda_1}{\lambda_2-2\lambda_3}\right\}=\frac{\lambda_1\lambda_3}{(\lambda_2-2\lambda_3)^2}$$

and

$$\mathcal{U}\{t^2e^{2t};\lambda_1,\lambda_2,\lambda_3\}=(-\lambda_3)^2\frac{d^2}{d\lambda_2^2}\left\{\frac{\lambda_1}{\lambda_2-2\lambda_3}\right\}=\frac{2\lambda_1\lambda_3^2}{(\lambda_2-2\lambda_3)^3}.$$

Both these results reduce in the limiting case $\lambda_1=1,\lambda_2=s,\lambda_3=1$ to the known results of [6, p.5] viz.,

$$\mathcal{U}\{te^{2t};1,s,1\}=\frac{1}{(s-2)^2}=\mathcal{L}\{te^{2t};s\}\text{ and }\mathcal{U}\{t^2e^{2t};1,s,1\}=\frac{2}{(s-2)^3}=\mathcal{L}\{t^2e^{2t};s\}.$$

5.11 UT for Division by t

Theorem 5.17 Let $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)$, if $\lim_{t\to 0}\dfrac{F(t)}{t}$ exists then

$$\mathcal{U}\left\{\frac{F(t)}{t};\lambda_1,\lambda_2,\lambda_3\right\}=\frac{1}{\lambda_3}\int_{\lambda_2}^\infty\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)\,d\lambda_2. \tag{5.35}$$

Proof By the definition of UT of $F(t)$ we have

$$\mathrm{u}(\lambda_1,\lambda_2,\lambda_3)=\lambda_1\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt\,.$$

If we integrate with respect to λ_2 from $\lambda_2=\lambda_2$ to $\lambda_2=\infty$ on both sides of this relation then

$$\int_{\lambda_2}^\infty \mathrm{u}(\lambda_1,\lambda_2,\lambda_3)\,d\lambda_2=\lambda_1\int_{\lambda_2}^\infty d\lambda_2\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt.$$

Since the variables λ_2 and t are independent therefore the order of integration on the right hand side of the last relation can be interchanged to achieve

$$\int_{\lambda_2}^\infty \mathrm{u}(\lambda_1,\lambda_2,\lambda_3)\,d\lambda_2=\lambda_1\int_0^\infty dt\int_{\lambda_2}^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,d\lambda_2=\lambda_1\int_0^\infty F(\lambda_3 t)\,dt\int_{\lambda_2}^\infty e^{-\lambda_2 t}\,d\lambda_2$$

$$=\lambda_1\int_0^\infty F(\lambda_3 t)\,dt\left(\frac{e^{-\lambda_2 t}}{-t}\right)_{\lambda_2=\lambda_2}^\infty=\lambda_1\int_0^\infty \frac{e^{-\lambda_2 t}}{t}F(\lambda_3 t)\,dt=\lambda_1\lambda_3\int_0^\infty \frac{e^{-\lambda_2 t}}{\lambda_3 t}F(\lambda_3 t)\,dt=\mathcal{U}\left\{\frac{F(t)}{t};\lambda_1,\lambda_2,\lambda_3\right\}.$$

■

5.12 UT of Periodic Functions

Theorem 5.18 Let $F(t)$ be a periodic function with period T, i.e. $F(t+nT)=F(t)$ for $n=1,2,3,\ldots$, then

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\sum_{n=0}^\infty\int_0^T e^{-\lambda_2(x+nT)}F\{\lambda_3(x+nT)\}\,dx. \tag{5.36}$$

Proof We have from the definition of UT,

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\int_0^\infty e^{-\lambda_2 t}F(\lambda_3 t)\,dt=\lambda_1\sum_{n=0}^\infty\int_{nT}^{(n+1)T}e^{-\lambda_2 t}F(\lambda_3 t)\,dt$$

Put $t=x+nT$ so that, $dt=dx$ in the above equation to get the desired result. ■

Corollary 5.19 If λ_3 is an integer then (5.36) assumes the form

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_1}{1-e^{-\lambda_2 T}}\int_0^T e^{-\lambda_2 x}F(\lambda_3 x)\,dx. \tag{5.37}$$

Proof When λ_3 is an integer then $F\{\lambda_3(x+nT)\}=F(\lambda_3 x)$ which renders (5.36) as

$$\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\lambda_1\sum_{n=0}^\infty e^{-\lambda_2 nT}\int_0^T e^{-\lambda_2 x}F(\lambda_3 x)\,dx=\lambda_1\int_0^T e^{-\lambda_2 x}F(\lambda_3 x)\,dx\left(\sum_{n=0}^\infty e^{-\lambda_2 nT}\right)$$

$$=\lambda_1\int_0^T e^{-\lambda_2 x}F(\lambda_3 x)\,dx\left(1+e^{-\lambda_2 T}+e^{-2\lambda_2 T}+\ldots\right)=\frac{\lambda_1}{1-e^{-\lambda_2 T}}\int_0^T e^{-\lambda_2 x}F(\lambda_3 x)\,dx.$$

■

5.13 Initial value Theorem of UT

Theorem 5.20

$$\lambda_1\lim_{t\to0}F(t)=\lambda_1\lim_{t\to0}F(\lambda_3 t)=\lim_{\lambda_2\to\infty}\lambda_2\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}. \tag{5.38}$$

Proof From (5.28) we have

$$\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_2}{\lambda_3}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}-\frac{\lambda_1}{\lambda_3}F(0).$$

Taking limits of the above expression as $\lambda_2\to\infty$ gives

$$\lim_{\lambda_2\to\infty}\mathcal{U}\{F'(t);\lambda_1,\lambda_2,\lambda_3\}=\lim_{\lambda_2\to\infty}\lambda_1\int_0^\infty e^{-\lambda_2 t}F'(\lambda_3 t)\,dt=\lim_{\lambda_2\to\infty}\frac{\lambda_2}{\lambda_3}\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}-\frac{\lambda_1}{\lambda_3}F(0).$$

Or,

$$0 = \lim_{\lambda_2 \to \infty} \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0).$$

From where (5.38) at once follows keeping in mind that $F(0) = \lim_{t \to 0} F(t) = \lim_{t \to 0} F(\lambda_3 t)$.

■

5.14 Final Value Theorem of UT
Theorem 5.21

$$\lambda_1 \lim_{t \to \infty} F(\lambda_3 t) = \lim_{\lambda_2 \to 0} \lambda_2 \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\}. \tag{5.39}$$

Proof We recall (5.28)

$$\mathcal{U}\{F'(t); \lambda_1, \lambda_2, \lambda_3\} = \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0).$$

On proceeding to the limits of the above equation as $\lambda_2 \to 0$ we get

$$\lim_{\lambda_2 \to 0} \mathcal{U}\{F'(t); \lambda_1, \lambda_2, \lambda_3\} = \lim_{\lambda_2 \to 0} \lambda_1 \int_0^\infty e^{-\lambda_2 t} F'(\lambda_3 t)\, dt = \lim_{\lambda_2 \to 0} \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0).$$

Or

$$\lambda_1 \int_0^\infty \left(\lim_{\lambda_2 \to 0} e^{-\lambda_2 t} \right) F'(\lambda_3 t)\, dt = \lim_{\lambda_2 \to 0} \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0)$$

$$\Rightarrow \lambda_1 \int_0^\infty F'(\lambda_3 t)\, dt = \lim_{\lambda_2 \to 0} \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0).$$

Or

$$\lambda_1 \left(\frac{F(\lambda_3 t)}{\lambda_3} \right)_{t=0}^{\infty} = \lim_{\lambda_2 \to 0} \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0).$$

Or

$$\frac{\lambda_1}{\lambda_3} \lim_{t \to \infty} F(\lambda_3 t) - \frac{\lambda_1}{\lambda_3} F(0) = \lim_{\lambda_2 \to 0} \frac{\lambda_2}{\lambda_3} \mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} - \frac{\lambda_1}{\lambda_3} F(0).$$

The required result is now evident. ■

5.15 UT of UT of a Function
Theorem 5.22

$$\mathcal{U}\{\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\}; \lambda_1', \lambda_2', \lambda_3'\} = \lambda_1 \lambda_1' \int_0^\infty \frac{F(\lambda_3 t)}{\lambda_2' + \lambda_3' t}\, dt. \tag{5.40}$$

Proof

The definition (2.3) of UT of a function $F(t)$ with respect to the parameters $\lambda_1, \lambda_2 \lambda_3', \lambda_3$ gives

$$\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\} = \mathrm{u}(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \int_0^\infty e^{-\lambda_2 t} F(\lambda_3 t)\, dt$$

On taking the UT of the above equation again with respect to the parameters $\lambda_1', \lambda_2', \lambda_3'$ we obtain

$$\mathcal{U}\{\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\}; \lambda_1', \lambda_2', \lambda_3'\} = \lambda_1' \int_0^\infty e^{-\lambda_2' \lambda_2} \mathrm{u}(\lambda_1, \lambda_2 \lambda_3', \lambda_3)\, d\lambda_2$$

$$= \lambda_1' \int_0^\infty e^{-\lambda_2' \lambda_2} \left[\lambda_1 \int_0^\infty e^{-\lambda_3' \lambda_2 t} F(\lambda_3 t)\, dt \right] d\lambda_2 = \lambda_1 \lambda_1' \int_0^\infty e^{-\lambda_2' \lambda_2} \left[\int_0^\infty e^{-\lambda_3' \lambda_2 t} F(\lambda_3 t)\, dt \right] d\lambda_2.$$

As the area (region) of integration in this double integral is the whole of the positive quadrant of the $\lambda_2 - t$ plane, on interchanging the order of integration of these independent variables we have

$$\mathcal{U}\{\mathcal{U}\{F(t); \lambda_1, \lambda_2, \lambda_3\}; \lambda_1', \lambda_2', \lambda_3'\} = \lambda_1 \lambda_1' \int_0^\infty F(\lambda_3 t)\, dt \left[\int_0^\infty e^{-(\lambda_2' + \lambda_3' t)\lambda_2}\, d\lambda_2 \right].$$

The desired result now follows at once by evaluating the integral inside the square brackets on the right hand side of the last equation. ■

5.16 Convolution Theorem of UT

Definition 5.23 The convolution or faltung of two functions $F(t)$ and $G(t)$ is defined by (see, for example, [6, (11), p.45]

$$H(\mu)=\int_0^{\mu} F(a)G(\mu-a)\,da=\int_0^{\mu} F(\mu-a)G(a)\,da=F*G \tag{5.41}$$

Theorem 5.24 Let $F(t)$ and $G(t)$ be two functions such that $\mathcal{U}\{F(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)$ and $\mathcal{U}\{G(t);\lambda_1,\lambda_2,\lambda_3\}=\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)$ then the UT of their convolution $H=F*G$ is given by

$$\mathcal{U}\{H;\lambda_1,\lambda_2,\lambda_3\}=\mathcal{U}\{F*G;\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_3}{\lambda_1}\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)\cdot\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3). \tag{5.42}$$

Proof Consider the right hand side of (5.42)

$$\begin{aligned}\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)\cdot\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)&=\lambda_1\int_0^{\infty}e^{-\lambda_2 t}F(\lambda_3 t)\,dt\cdot\lambda_1\int_0^{\infty}e^{-\lambda_2 p}G(\lambda_3 p)\,dp\\&=\lambda_1^2\int_0^{\infty}\int_0^{\infty}e^{-\lambda_2(t+p)}F(\lambda_3 t)G(\lambda_3 p)\,dtdp.\end{aligned} \tag{5.43}$$

where the double integral is taken over the entire first quadrant of the $t-p$ plane bounded by the axes $t=0$ and $p=0$. On making the change of variables $p=q$ and $t=r-p=r-q$ which transforms the original axes $t=0$ and $p=0$ of the $t-p$ plane into the lines $q=0$ and $q=r$ respectively in the new $q-r$ plane (see also, [7, (3.5.5), p. 158 and Fig. (3.3), p. 159]). Thus the new limits of integration in the above double integral (5.43) are $q=0$ to $q=r$ for the $q-$ variable and $r=0$ to $r=\infty$ for the $r-$ variable and the Jacobian for the change of variables $(p,t)\to(q,r)$ is

$$dtdp=\left|\frac{\partial(t,p)}{\partial(r,q)}\right|drdq=\begin{vmatrix}\dfrac{\partial t}{\partial r} & \dfrac{\partial t}{\partial q}\\ \dfrac{\partial p}{\partial r} & \dfrac{\partial p}{\partial q}\end{vmatrix}drdq=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}drdq=drdq.$$

Thus the double integral (5.43) takes the form

$$\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)\cdot\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)=\lambda_1^2\int_{r=0}^{\infty}e^{-\lambda_2 r}\int_{q=0}^{r}F(\lambda_3(r-q))G(\lambda_3 q)\,drdq$$

Or, $$\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)\cdot\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)=\frac{\lambda_1^2}{\lambda_3}\int_{r=0}^{\infty}e^{-\lambda_2 r}dr\int_{q=0}^{r}F(\lambda_3 r-\lambda_3 q)G(\lambda_3 q)\cdot\lambda_3 dq \tag{5.44}$$

Note that as $q\in(0,r)$ the variable, $\lambda_3 q\in(0,\lambda_3 r)$ thus setting $v=\lambda_3 q$ renders (5.44) with the help of (5.41) as

$$\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)\cdot\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)=\frac{\lambda_1^2}{\lambda_3}\int_{r=0}^{\infty}e^{-\lambda_2 r}dr\left[\int_{v=0}^{\lambda_3 r}F(\lambda_3 r-v)G(v)\,dv\right]$$

i.e.,

$$\mathrm{u}_1(\lambda_1,\lambda_2,\lambda_3)\cdot\mathrm{u}_2(\lambda_1,\lambda_2,\lambda_3)=\frac{\lambda_1^2}{\lambda_3}\int_{r=0}^{\infty}e^{-\lambda_2 r}F*G\,dr=\frac{\lambda_1}{\lambda_3}\mathcal{U}\{F*G;\lambda_1,\lambda_2,\lambda_3\}$$

■

Corollary 5.25 If $H_3=F_1*F_2*F_3$ then (5.42) gives

$$\mathcal{U}\{H_3;\lambda_1,\lambda_2,\lambda_3\}=\mathcal{U}\{F_1*(F_2*F_3);\lambda_1,\lambda_2,\lambda_3\}=\frac{\lambda_3}{\lambda_1}\mathcal{U}\{F_1;\lambda_1,\lambda_2,\lambda_3\}\mathcal{U}\{(F_2*F_3);\lambda_1,\lambda_2,\lambda_3\}$$

$$=\left(\frac{\lambda_3}{\lambda_1}\right)^2 \mathcal{U}\{F_1;\lambda_1,\lambda_2,\lambda_3\}\mathcal{U}\{F_2;\lambda_1,\lambda_2,\lambda_3\}\mathcal{U}\{F_3;\lambda_1,\lambda_2,\lambda_3\}.$$

If $H_k = F_1 * F_2 * \ldots * F_k$ then (5.42) on repeated use yields

$$\mathcal{U}\{H_k;\lambda_1,\lambda_2,\lambda_3\} = \left(\frac{\lambda_3}{\lambda_1}\right)^{k-1} \prod_{i=1}^{k} \mathcal{U}\{F_i;\lambda_1,\lambda_2,\lambda_3\}. \quad (5.45)$$

5.17 UT of Partial Derivatives of Functions

Theorem 5.26 If $Y = Y(x,t)$ and $\mathcal{U}\{Y(x,t);\lambda_1,\lambda_2,\lambda_3\} = \bar{Y}(x;\lambda_1,\lambda_2,\lambda_3)$ and $Y_t(x,0) = \left[\frac{\partial}{\partial t}(Y(x,t))\right]_{t=0}$ then

(i) $$\mathcal{U}\left\{\frac{\partial}{\partial t}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \frac{\lambda_2}{\lambda_3}\bar{Y}(x;\lambda_1,\lambda_2,\lambda_3) - \frac{\lambda_1}{\lambda_3}Y(x,0). \quad (5.46)$$

(ii) $$\mathcal{U}\left\{\frac{\partial^2}{\partial t^2}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \left(\frac{\lambda_2}{\lambda_3}\right)^2 \bar{Y}(x;\lambda_1,\lambda_2,\lambda_3) - \frac{\lambda_1\lambda_2}{\lambda_3^2}Y(x,0) - \frac{\lambda_1}{\lambda_3}Y_t(x,0). \quad (5.47)$$

(iii) $$\mathcal{U}\left\{\frac{\partial}{\partial x}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \frac{d}{dx}\left(\bar{Y}(x;\lambda_1,\lambda_2,\lambda_3)\right). \quad (5.48)$$

(iv) $$\mathcal{U}\left\{\frac{\partial^2}{\partial t^2}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \frac{d^2}{dx^2}\left(\bar{Y}(x;\lambda_1,\lambda_2,\lambda_3)\right). \quad (5.49)$$

Proof (i) Using the definition of UT, we get

$$\mathcal{U}\left\{\frac{\partial}{\partial t}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \lambda_1\int_0^\infty e^{-\lambda_2 t}\frac{\partial}{\partial t}\{Y(x,\lambda_3 t)\}dt$$

$$= \lambda_1 \lim_{a\to\infty}\left[\left\{e^{-\lambda_2 t}\frac{Y(x,\lambda_3 t)}{\lambda_3}\right\}_{t=0}^{a} + \int_0^a \frac{\lambda_2}{\lambda_3}Y(x,\lambda_3 t)e^{-\lambda_2 t}dt\right].$$

which on simplification gives (5.46).

(ii) Let $V(x,t) = \frac{\partial}{\partial t}\{Y(x,t)\} = Y_t(x,t)$. Now,

$$\mathcal{U}\left\{\frac{\partial^2}{\partial t^2}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \mathcal{U}\left\{\frac{\partial}{\partial t}\{V(x,t)\};\lambda_1,\lambda_2,\lambda_3\right\} = \frac{\lambda_2}{\lambda_3}\bar{V}(x;\lambda_1,\lambda_2,\lambda_3) - \frac{\lambda_1}{\lambda_3}V(x,0).$$

The second equality of the above expression follows from the use of (5.46). Proceeding further we have,

$$\mathcal{U}\left\{\frac{\partial^2}{\partial t^2}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} = \frac{\lambda_2}{\lambda_3}\mathcal{U}\{V(x,t);\lambda_1,\lambda_2,\lambda_3\} - \frac{\lambda_1}{\lambda_3}Y_t(x,0)$$

$$= \frac{\lambda_2}{\lambda_3}\mathcal{U}\left\{\frac{\partial}{\partial t}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\} - \frac{\lambda_1}{\lambda_3}Y_t(x,0).$$

Using (5.46) again for the value of the expression $\mathcal{U}\left\{\frac{\partial}{\partial t}(Y(x,t));\lambda_1,\lambda_2,\lambda_3\right\}$ in the above equation leads to (5.47) after a little simplification.

(iii)
$$\mathcal{U}\left\{\frac{\partial}{\partial x}\left(Y\left(x,t\right)\right);\lambda_1,\lambda_2,\lambda_3\right\}=\lambda_1\int_0^{\infty}e^{-\lambda_2 t}\frac{\partial}{\partial x}\left(Y\left(x,\lambda_3 t\right)\right)dt=\frac{d}{dx}\left[\lambda_1\int_0^{\infty}e^{-\lambda_2 t}Y\left(x,\lambda_3 t\right)dt\right]$$
$$=\frac{d}{dx}\left[\mathcal{U}\left\{Y\left(x,t\right);\lambda_1,\lambda_2,\lambda_3\right\}\right]=\frac{d}{dx}\left(\overline{Y}\left(x;\lambda_1,\lambda_2,\lambda_3\right)\right).$$

(iv) We note that $\mathcal{U}\left\{\frac{\partial^2}{\partial x^2}\left(Y\left(x,t\right)\right)\right\}=\mathcal{U}\left\{\frac{\partial}{\partial x}\left(Z\left(x,t\right)\right)\right\}$, where, $Z\left(x,t\right)=\frac{\partial}{\partial x}\left(Y\left(x,t\right)\right)$. Then form (5.48) we obtain
$$\mathcal{U}\left\{\frac{\partial^2}{\partial x^2}\left(Y\left(x,t\right)\right)\right\}=\frac{d}{dx}\left[\overline{Z}\left(x,t\right)\right]=\frac{d}{dx}\left[\mathcal{U}\left\{Z\left(x,t\right);\lambda_1,\lambda_2,\lambda_3\right\}\right]$$
$$=\frac{d}{dx}\left[\mathcal{U}\left\{\frac{\partial}{\partial x}\left(Y\left(x,t\right)\right);\lambda_1,\lambda_2,\lambda_3\right\}\right]=\frac{d}{dx}\left[\frac{d}{dx}\left(\overline{Y}\left(x;\lambda_1,\lambda_2,\lambda_3\right)\right)\right]=\frac{d^2}{dx^2}\left(\overline{Y}\left(x;\lambda_1,\lambda_2,\lambda_3\right)\right).$$
■

6. THE $n-$ DIMENSIONAL GENERALIZATION OF THE UPADHYAYA TRANSFORM

Sometimes the problems arising in physical sciences and engineering are also solved by the two dimensional analogue of the Laplace transform, often called the *double Laplace transform* (see, for instance. Patra [8, section 4.6, pp.161-165], or, Debnath and Bhatta [7, section 4.11, p. 274-280 and Exercises 59-62, pp.294-295]). Therefore, to cover such cases and many other possible fields of the future applications of the Upadhyaya transform we now present a generalization of the one- dimensional Upadhyaya transform defined by (2.3).

Analogous to (2.1) we define the complex parameters $\lambda_1^{(j)},\lambda_2^{(j)},\lambda_3^{(j)}$ by the relations

$$\lambda_1^{(j)}=\frac{\left(\sum_{i=0}^{r_1^{(j)}}a_i^{(j)}\left(z_1^{(j)}\right)^i\right)^{m_1^{(j)}}}{\left(\sum_{i=0}^{r_2^{(j)}}b_i^{(j)}\left(z_2^{(j)}\right)^i\right)^{m_2^{(j)}}},\lambda_2^{(j)}=\frac{\left(\sum_{i=0}^{r_3^{(j)}}c_i^{(j)}\left(z_3^{(j)}\right)^i\right)^{m_3^{(j)}}}{\left(\sum_{i=0}^{r_4^{(j)}}d_i^{(j)}\left(z_4^{(j)}\right)^i\right)^{m_4^{(j)}}},\lambda_3^{(j)}=\frac{\left(\sum_{i=0}^{r_5^{(j)}}k_i^{(j)}\left(z_5^{(j)}\right)^i\right)^{m_5^{(j)}}}{\left(\sum_{i=0}^{r_6^{(j)}}l_i^{(j)}\left(z_6^{(j)}\right)^i\right)^{m_6^{(j)}}} \quad (6.1)$$

where, $j=1,\ldots,n,$ $m_i^{(j)}$ and $r_i^{(j)}$ are nonnegative integers $\left(i=1,2,\ldots,6\right)$ and $a_i^{(j)},b_i^{(j)},c_i^{(j)},d_i^{(j)},k_i^{(j)},l_i^{(j)}$ are complex constants. To tackle the situation at hand, we can also choose the parameters $\lambda_1^{(j)},\lambda_2^{(j)},\lambda_3^{(j)}$ and the constants $a_i^{(j)},b_i^{(j)},c_i^{(j)},d_i^{(j)},k_i^{(j)},l_i^{(j)}$ to be real numbers also.

Now we define the $n-$dimensional Upadhyaya transform of a function $F\left(t_1,\ldots,t_n\right)$ of $n-$variables $t_1,\ldots,t_n$ as below. We assume that the function $F\left(t_1,\ldots,t_n\right)$ is of exponential order $a\left(a>0\right)$ for each of the $n-$variables $t_1,\ldots,t_n$.

Definition 6.1 The $n-$*dimensional Upadhyaya transform* of a function $F\left(t_1,\ldots,t_n\right)$ of $n-$variables $t_1,\ldots,t_n$, which is of exponential order $a\left(a>0\right)$ for each of the $n-$variables $t_1,\ldots,t_n$ is denoted by $\mathcal{U}_n\left\{F\left(t_1,\ldots,t_n\right);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\ldots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\}$ and defined by

$$\mathcal{U}_n\left\{F\left(t_1,\ldots,t_n\right);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\ldots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\}=\mathrm{u}_n\left\{\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\ldots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\}$$
$$=\prod_{i=1}^{n}\lambda_1^{(i)}\int_0^{\infty}\ldots(n)\ldots\int_0^{\infty}e^{-\left(\sum_{i=1}^{n}\lambda_2^{(i)}t_i\right)}F\left(\lambda_3^{(1)}t_1,\ldots,\lambda_3^{(n)}t_n\right)dt_1\ldots dt_n. \quad (6.2)$$

provided the integral exists. The symbol $\int_0^\infty \ldots(n)\ldots\int_0^\infty$ in the above equation means that the integral sign $\int_0^\infty$ appears n times in this equation corresponding to the n variables of integration $t_1,\ldots,t_n$. It is also to be noted that the subscript n attached with $\mathcal{U}$, (i.e., the symbol $\mathcal{U}_n$) signifies the $n-$dimensional UT of the function $F(t_1,\ldots,t_n)$.

We remark that (6.2) generalizes or will generalize a number of multidimensional variants of the Laplace transform which either currently exist in the literature or are yet unknown and may appear in the future. We mention below some special cases of (6.2) with which the author is aware of at present.

6.1 The Multivariable Laplace Transform

If $F(t_1,\ldots,t_n)$ be a function of n positive variables then the multivariable Laplace transform (i.e. the $n-$dimensional Laplace transform) of $F(t_1,\ldots,t_n)$ is defined by (see [54, (2.5.2), p.47])

$$\begin{aligned}&\mathcal{L}_n\{F(t_1,\ldots,t_n);p_1,\ldots,p_n\}=f_n(p_1,\ldots,p_n)\\&=\int_0^\infty\ldots(n)\ldots\int_0^\infty e^{-t_1p_1-\ldots-t_np_n}F(t_1,\ldots,t_n)\,dt_1\ldots dt_n.\end{aligned}\tag{6.3}$$

If in (6.2) we choose $\lambda_1^{(j)}=1=\lambda_3^{(j)}$ and $\lambda_2^{(j)}=p_j$ for $j=1,\ldots,n$ then it reduces to (6.3) and we can easily observe that

$$\begin{aligned}&\mathcal{U}_n\{F(t_1,\ldots,t_n);1,p_1,1;\ldots;1,p_n,1\}=\mathrm{u}_n\{1,p_1,1;\ldots;1,p_n,1\}\\&=\mathcal{L}_n\{F(t_1,\ldots,t_n);p_1,\ldots,p_n\}=f_n(p_1,\ldots,p_n).\end{aligned}$$

As far as this author knows till date the special case of (6.3) when $n=2$ has been studied so far by some authors and it is called the *Double Laplace Transform*. For the interested reader we refer to the instances as are pointed out at the beginning of this section of the paper. Still the double Laplace transform does not seem to be studied as extensively as its counterpart the classical Laplace transform and the various variants thereof all corresponding to the one-dimensional case. One more relevant reference in this direction is the very recent work of Aylikçi and Dernek [53]. These authors also highlight this fact by pointing out that: "But there is a very little work available for the double Laplace transform of $f(x,y)$ of two positive real variables x and y and their properties" (see, [53, p. 2 of 21]). Given these facts, now we proceed to study the special case $n=2$ of (6.2) in some detail. Consistent with the modish terminology in the literature, we call the $n=2$ case of (6.2) the *Double Upadhyaya Transform* (DUT) and define it as below:

Definition 6.2 The *Double Upadhyaya Transform* (DUT) of a function $F(t_1,t_2)$ of two variables t_1,t_2, which is of exponential order $a\,(a>0)$ for each of the variables t_1,t_2 is denoted by

$$\mathcal{U}_2\{F(t_1,t_2);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)}\}$$

and is defined by

$$\begin{aligned}&\mathcal{U}_2\{F(t_1,t_2);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)}\}=\mathrm{u}_2\{\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)}\}\\&\qquad=\lambda_1^{(1)}\lambda_1^{(2)}\int_0^\infty\int_0^\infty e^{-\left(\lambda_2^{(1)}t_1+\lambda_2^{(2)}t_2\right)}F\left(\lambda_3^{(1)}t_1,\lambda_3^{(2)}t_2\right)dt_1dt_2.\end{aligned}\tag{6.4}$$

whenever this double integral exists.

6.2 The Double Laplace Transform

Now we observe the double Laplace transform, which is the special case of (6.3) when $n=2$. Therefore, the following equation defines the double Laplace transform of a function $F(t_1,t_2)$ of two variables t_1,t_2, which is of exponential order $a\,(a>0)$ for each of the variables t_1,t_2

$$\mathcal{L}_2\left\{F\left(t_1,t_2\right);p_1,p_2\right\}=f_2\left(p_1,p_2\right)=\int_0^\infty\int_0^\infty e^{-t_1p_1-t_2p_2}F\left(t_1,t_2\right)dt_1dt_2. \qquad (6.5)$$

provided, of course, the double integral in question converges. In this context we refer the reader to Patra [8, (4.25), p.161] or, Debnath and Bhatta [7, (4.11.1), p.274] or, Aylikçi and Dernek [53, (1.5), p.2 of 21]. It can be very easily discerned that (6.4) reduces to (6.5) when we put $\lambda_1^{(1)}=1=\lambda_1^{(2)}=\lambda_3^{(1)}=\lambda_3^{(2)}$ and $\lambda_2^{(1)}=p_1$, $\lambda_2^{(2)}=p_2$ thereby at once showing that

$$\mathcal{U}_2\left\{F\left(t_1,t_2\right);1,p_1,1;1,p_2,1\right\}=\mathrm{u}_2\left\{1,p_1,1;1,p_2,1\right\}=\mathcal{L}_2\left\{F\left(t_1,t_2\right);p_1,p_2\right\}=f_2\left(p_1,p_2\right).$$

6.3 The Iterated Laplace Transform

This Laplace transform is a particular case of the double Laplace transform of a function of two variables in which the transformation variables are equal (see Patra [8, section 4.7, p.166]). The iterated Laplace transform is defined by (see Patra [8, (4.46), p.166])

$$I_{\mathcal{L}}\left\{F\left(t_1,t_2\right);p\right\}=\mathcal{L}_2\left\{F\left(t_1,t_2\right);p,p\right\}=f_2\left(p,p\right)=\int_0^\infty\int_0^\infty e^{-p\left(t_1+t_2\right)}F\left(t_1,t_2\right)dt_1dt_2 \qquad (6.6)$$

where the symbol $I_{\mathcal{L}}$ denotes the iterated Laplace transform. For the setting of the parameters $\lambda_1^{(1)}=1=\lambda_1^{(2)}=\lambda_3^{(1)}=\lambda_3^{(2)}$ and $\lambda_2^{(1)}=p=\lambda_2^{(2)}$ in (6.4) we obtain (6.6) from it to see that

$$\mathcal{U}_2\left\{F\left(t_1,t_2\right);1,p,1;1,p,1\right\}=\mathrm{u}_2\left\{1,p,1;1,p,1\right\}=I_{\mathcal{L}}\left\{F\left(t_1,t_2\right);p\right\}=\mathcal{L}_2\left\{F\left(t_1,t_2\right);p,p\right\}=f_2\left(p,p\right).$$

6.4 The Double $\mathcal{L}_{22}-$ Integral Transform of Aylikçi and Dernek

Very recently in 2018 Aylikçi and Dernek [53] have defined the $\mathcal{L}_{22}-$ integral transform which extends the $\mathcal{L}_2-$ integral transform of Yürekli and Sadek [52]. For a function $f\left(x,y\right)$ of two variables x and y the double Lapalce – type transform $\mathcal{L}_{22}-$is defined in the positive quadrant of the $xy-$plane by the double integral (see [53, (2.1), p. 3 of 21])

$$\mathcal{L}_{22}\left\{f\left(x,y\right);p,q\right\}=\tilde{\tilde{F}}\left(p,q\right)=\int_0^\infty\int_0^\infty xye^{-x^2p^2-y^2q^2}f\left(x,y\right)dxdy, \qquad (6.7)$$

whenever the integral exists. A comparison between (6.4) and (6.7) shows that the choice of the parameters $\lambda_1^{(1)}=\lambda_1^{(2)}=\frac{1}{2},\lambda_3^{(1)}=\lambda_3^{(2)}=1,\lambda_2^{(1)}=p^2,\lambda_2^{(2)}=q^2$ and the choice of the variables $t_1=x^2,t_2=y^2$ with the following definition of the function $F\left(t_1,t_2\right)$,

$$F\left(t_1,t_2\right)=F\left(x^2,y^2\right)=\begin{cases}f\left(x,y\right), & \text{whenever } x>0 \text{ and } y>0\\ 0, & \text{otherwise}\end{cases}$$

leads us to (6.7). Thus we see the following relation

$$\mathcal{U}_2\left\{F\left(x^2,y^2\right);\frac{1}{2},p^2,1;\frac{1}{2},q^2,1\right\}=\mathrm{u}_2\left\{\frac{1}{2},p^2,1;\frac{1}{2},q^2,1\right\}=\mathcal{L}_{22}\left\{f\left(x,y\right);p,q\right\}=\tilde{\tilde{F}}\left(p,q\right).$$

6.5 The Double Integral Transform of Atangana and Alkaltani

A new double integral transform was introduced recently by Atangana and Alkaltani [55] as below:

For a function $f\left(x,t\right)$ which is continuous and its Laplace transform is n times and m times partially differentiable, the new integral transform of Atangana and Alkaltani is defined by (see [55, (2.1), p. 425])

$$K_{n,m}\left\{f\left(x,t\right);s,p\right\}=\mathcal{F}_{AA}\left(s,p\right)=\int_0^\infty\int_0^\infty x^nt^me^{-\left(sx+pt\right)}f\left(x,t\right)dxdt. \qquad (6.8)$$

Here the subscript AA attached to the symbol $\mathcal{F}$ signifies the words Atangana and Alkaltani. On setting the variables $t_1=x,t_2=t$ and by observing that from (6.5) the double Laplace transform of the function $F\left(t_1,t_2\right)=F\left(x,t\right)=f\left(x,t\right)$ is given by

$$\mathcal{L}_2\{f(x,t);s,p\}=f_2(s,p)=\int_0^\infty\int_0^\infty e^{-sx-pt}f(x,t)\,dxdt \tag{6.9}$$

which, on differentiating partially n times with respect to s and m times with respect to p yields

$$\frac{\partial^{n+m}}{\partial s^n\partial p^m}\left(\mathcal{L}_2\{f(x,t);s,p\}\right)=\frac{\partial^{n+m}}{\partial s^n\partial p^m}\left(f_2(s,p)\right)=\frac{\partial^{n+m}}{\partial s^n\partial p^m}\left(\int_0^\infty\int_0^\infty e^{-sx-pt}f(x,t)\,dxdt\right)$$
$$=\int_0^\infty\int_0^\infty x^n t^m e^{-sx-pt}f(x,t)\,dxdt=K_{n,m}\{f(x,t);s,p\}=\mathcal{F}_{AA}(s,p) \tag{6.10}$$

and now utilizing the relation between the double Upadhyaya transform and the double Laplace transform from the subsection 6.2 above we can rewrite (6.10) as

$$\frac{\partial^{n+m}}{\partial s^n\partial p^m}\left(\mathcal{U}_2\{f(x,t);1,s,1;1,p,1\}\right)=\frac{\partial^{n+m}}{\partial s^n\partial p^m}\left(\mathrm{u}_2\{1,s,1;1,p,1\}\right)$$
$$=K_{n,m}\{f(x,t);s,p\}=\mathcal{F}_{AA}(s,p). \tag{6.11}$$

which shows the relation between a special case of the double Upadhyaya transform (6.4) and the double integral transform of Atangana and Alkaltani [55] given by (6.8).

6.6 The Double Aboodh Transform

Aboodh et al. [56,57] have introduced the double Aboodh transform in the literature defined as (see, [56, section 1.1, p.48213]) below:

For a function $F(t_1,t_2)$ of positive real variables t_1,t_2 which is capable of being expressible as a convergent infinite series, the double Aboodh transform is given by

$$A_2\{F(t_1,t_2);u,v\}=K_2(u,v)=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(ut_1+vt_2)}F(t_1,t_2)\,dt_1dt_2. \tag{6.12}$$

Comparing (6.4) and (6.12) shows that for the choice of the parameters $\lambda_1^{(1)}=\frac{1}{u},\lambda_1^{(2)}=\frac{1}{v},\lambda_3^{(1)}=1=\lambda_3^{(2)}$ and $\lambda_2^{(1)}=u,\lambda_2^{(2)}=v$ in (6.4) it reduces to (6.12), thereby yielding the relation

$$\mathcal{U}_2\left\{F(t_1,t_2);\frac{1}{u},u,1;\frac{1}{v},v,1\right\}=\mathrm{u}_2\left\{\frac{1}{u},u,1;\frac{1}{v},v,1\right\}=A_2\{F(t_1,t_2);u,v\}=K_2(u,v).$$

6.7 The Double Mahgoub Transform (in fact, the Double Laplace-Carson Transform)

Thangavelu et al. [58] have used the double Mahgoub transform (*which, in fact, is nothing but the double Lapalce-Carson transform (see, subsections 4.2 and 4.10 above)*) which is defined by them in Definition 1.2 p. 16 (see [58]) as under:

For a function $F(t_1,t_2)$ of positive real variables t_1,t_2 which can be written as a convergent infinite series, the double Mahgoub transform (i.e., *truly speaking, the double Laplace-Carson transform*) is given by

$$M_2\{F(t_1,t_2);u,v\}=H_2(u,v)=uv\int_0^\infty\int_0^\infty e^{-(ut_1+vt_2)}F(t_1,t_2)\,dt_1dt_2, \tag{6.13}$$

u,v being complex variables.

A straightforward comparison between (6.4) and (6.13) with the choice of parameters $\lambda_1^{(1)}=u,\lambda_1^{(2)}=v$, $\lambda_2^{(1)}=u,\lambda_2^{(2)}=v,\lambda_3^{(1)}=1=\lambda_3^{(2)}$ in (6.4) at once lends (6.13) to show that

$$\mathcal{U}_2\{F(t_1,t_2);u,u,1;v,v,1\}=\mathrm{u}_2\{u,u,1;v,v,1\}=M_2\{F(t_1,t_2);u,v\}=H_2(u,v).$$

6.8 The Double Elzaki Transform

Elzaki and Hilal [59] used the double Elzaki transform to solve the telegraph equation. A few other works using the double Elzaki transform are [60, 61]. In [59] the double Elzaki transform is defined by the relation

$$E_2\{F(t_1,t_2);u,v\}=T_2(u,v)=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{t_1}{u}+\frac{t_2}{v}\right)}F(t_1,t_2)\,dt_1dt_2 \tag{6.14}$$

for complex values of u,v, where $F(t_1,t_2)$ is a function of the positive variables t_1,t_2 which possesses convergent infinite series expansion (see [59, (1-4), p. 96]). From (6.4) we observe that for the choice of the parameters $\lambda_1^{(1)} = u, \lambda_1^{(2)} = v, \lambda_2^{(1)} = \frac{1}{u}, \lambda_2^{(2)} = \frac{1}{v}, \lambda_3^{(1)} = 1 = \lambda_3^{(2)}$ in it, we obtain (6.14) from it to have the relation

$$\mathcal{U}_2\left\{F(t_1,t_2);u,\frac{1}{u},1;v,\frac{1}{v},1\right\} = \mathrm{u}_2\left\{u,\frac{1}{u},1;v,\frac{1}{v},1\right\} = E_2\{F(t_1,t_2);u,v\} = T_2(u,v).$$

6.9 The Double Kashuri and Fundo Transform

Kashuri and Fundo [62] defined this double transform as below:

For a function $F(t_1,t_2)$ is a function of the positive variables t_1,t_2 which has a convergent infinite series expansion

$$\mathcal{K}_2\left[F(t_1,t_2);u,v\right] = \mathfrak{k}_2(u,v) = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{t_1}{u^2}+\frac{t_2}{v^2}\right)}F(t_1,t_2)\,dt_1dt_2. \qquad (6.15)$$

We can at once deduce from (6.4) that for the setting of the parameters $\lambda_1^{(1)} = \frac{1}{u}, \lambda_1^{(2)} = \frac{1}{v}, \lambda_2^{(1)} = \frac{1}{u^2}$, $\lambda_2^{(2)} = \frac{1}{v^2}, \lambda_3^{(1)} = 1 = \lambda_3^{(2)}$ in it we arrive at (6.15) thus showing that

$$\mathcal{U}_2\left\{F(t_1,t_2);\frac{1}{u},\frac{1}{u^2},1;\frac{1}{v},\frac{1}{v^2},1\right\} = \mathrm{u}_2\left\{\frac{1}{u},\frac{1}{u^2},1;\frac{1}{v},\frac{1}{v^2},1\right\} = \mathcal{K}_2\left[F(t_1,t_2);u,v\right] = \mathfrak{k}_2(u,v).$$

6.10 The Double Sumudu Transform

Watugala [63] introduced the double Sumudu transform in 2002. Some other studies involving the double Sumudu transform are [64, 65, 66]. The double Sumudu transform is defined in Tchuenche and Mbare [64] for a function $F(t_1,t_2)$ of the positive variables t_1,t_2, which is capable of being expanded in a convergent infinite series, by the relation (see [64, (2.1), p.33])

$$\mathcal{S}_2\{F(t_1,t_2);u,v\} = \mathbb{S}_2\{u,v\} = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{t_1}{u}+\frac{t_2}{v}\right)}F(t_1,t_2)\,dt_1dt_2. \qquad (6.16)$$

We can see that if in (6.4) we set parameters $\lambda_1^{(1)} = \frac{1}{u}, \lambda_1^{(2)} = \frac{1}{v}, \lambda_2^{(1)} = \frac{1}{u}, \lambda_2^{(2)} = \frac{1}{v}, \lambda_3^{(1)} = 1 = \lambda_3^{(2)}$ we get (6.16) to see the relation

$$\mathcal{U}_2\left\{F(t_1,t_2);\frac{1}{u},\frac{1}{u},1;\frac{1}{v},\frac{1}{v},1\right\} = \mathrm{u}_2\left\{\frac{1}{u},\frac{1}{u},1;\frac{1}{v},\frac{1}{v},1\right\} = \mathcal{S}_2\{F(t_1,t_2);u,v\} = \mathbb{S}_2\{u,v\}.$$

6.11 The Double Natural Transform

Kiliçman and Omran [67] have introduced the double Natural transform and studied its relation with the double Laplace and the double Sumudu transforms and applied it to solve partial integro-differential equations. They define the double Natural transform for a $F(t_1,t_2)$ of the positive variables t_1,t_2 as (see, [67, (3.1), p. 1746])

$$\mathbb{N}_+^2\{F(t_1,t_2);(s,p);(u,v)\} = R_+^2\left[(s,p);(u,v)\right] = \int_0^\infty\int_0^\infty e^{-(st_1+pt_2)}F(ut_1,vt_2)\,dt_1dt_2. \quad (6.17)$$

From (6.4) we can conclude that with the choice of the parameters $\lambda_1^{(1)} = 1, \lambda_1^{(2)} = 1, \lambda_2^{(1)} = s, \lambda_2^{(2)} = p$, $\lambda_3^{(1)} = u, \lambda_3^{(2)} = v$ in it we at once get (6.17) to see the relation

$$\mathcal{U}_2\{F(t_1,t_2);1,s,u;1,p,v\} = \mathrm{u}_2\{1,s,u;1,p,v\} = \mathbb{N}_+^2\{F(t_1,t_2);(s,p);(u,v)\} = R_+^2\left[(s,p);(u,v)\right].$$

6.12 The Double Ramadan Group Integral Transform

Mohamed A. Ramadan and Adel R. Hadhoud [76] introduced the Double Ramadan Group Integral Transform in 2018 as an extension to their earlier work [36]. According to them (see [76, p. 389]), for the function $F(x,t)$ belonging to the set A defined by

$$A=\left\{F(x,t):\exists\ M,\tau_1,\tau_2>0,\left|F(x,t)\right|\le Me^{\frac{x+t}{\tau_i^2}},i=1,2\ \text{ if }(x,t)\in\mathbb{R}_+^2\right\}$$

the Double Ramadan Group Integral Transform of $F(x,t)$ in the positive quadrant of the $x-t$ plane is defined by

$$RG_2\left[F(x,t):(s,p,u,v)\right]=K(s,p,u,v)=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)}F(x,t)\,dxdt \tag{6.18}$$

(see, [76, (1), p. 390]) where, s,p,u and v are complex variables, s and p are the transform variables for x and t respectively, and $u,v\in(-\tau_1,\tau_2)$ where, $\tau_1,\tau_2>0$ and $\mathrm{Re}(s)>0,\mathrm{Re}(p)>0$.

We can now at once infer from (6.4) that for the choice of the parameters $\lambda_1^{(1)}=\frac{1}{u},\lambda_1^{(2)}=\frac{1}{v},\lambda_2^{(1)}=\frac{s}{u}$, $\lambda_2^{(2)}=\frac{p}{v},\lambda_3^{(1)}=1,\lambda_3^{(2)}=1$ in it and by setting the variables $t_1=x,t_2=t$ in it we obtain (6.18) thereby establishing the relation

$$\mathcal{U}_2\left\{F(x,t);\frac{1}{u},\frac{s}{u},1;\frac{1}{v},\frac{p}{v},1\right\}=\mathrm{u}_2\left\{\frac{1}{u},\frac{s}{u},1;\frac{1}{v},\frac{p}{v},1\right\}=RG_2\left[F(x,t):(s,p,u,v)\right]=K(s,p,u,v).$$

We also remark that for the choice of variables $t_1=\frac{x}{u},t_2=\frac{t}{v}$ *in (6.17), it reduces immediately to (6.18) showing thereby that the Double Ramadan Group Integral Transform of Ramadan and Hadhoud [76] which was introduced in 2018 is exactly identical with the Double Natural Transform of Kiliçman and Omran [67] which appeared one year earlier in the literature i.e., in 2017.*

We also find it important to point out to the interested reader that some triple integral transforms of the Laplace type as far as known till date to this author are also available in the literature. With this thing in mind we are therefore inclined to explicitly define the special case of (6.2) corresponding to the case $n=3$ and in conformity with the ongoing pattern of the literature we call it the *Triple Upadhyaya Transform* (TUT).

Definition 6.3 The *Triple Upadhyaya Transform* (TUT) of a function $F(t_1,t_2,t_3)$ of three variables t_1,t_2,t_3, which is of exponential order $a\,(a>0)$ for each of the variables t_1,t_2,t_3 is denoted by

$$\mathcal{U}_3\left\{F(t_1,t_2,t_3);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)};\lambda_1^{(3)},\lambda_2^{(3)},\lambda_3^{(3)}\right\}$$

and is defined by

$$\begin{aligned}&\mathcal{U}_3\left\{F(t_1,t_2,t_3);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)};\lambda_1^{(3)},\lambda_2^{(3)},\lambda_3^{(3)}\right\}\\&\quad=\mathrm{u}_3\left\{\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)};\lambda_1^{(3)},\lambda_2^{(3)},\lambda_3^{(3)}\right\}\\&\quad=\lambda_1^{(1)}\lambda_1^{(2)}\lambda_1^{(3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\lambda_2^{(1)}t_1+\lambda_2^{(2)}t_2+\lambda_2^{(3)}t_3\right)}F\left(\lambda_3^{(1)}t_1,\lambda_3^{(2)}t_2,\lambda_3^{(3)}t_3\right)dt_1dt_2dt_3.\end{aligned}\tag{6.19}$$

whenever this triple integral exists.

6.13 The Triple Laplace Transform

Atangana [68] introduced the triple Laplace transform in 2013 and solved the Mboctara equations which are third order differential equations, using it. If $F(t_1,t_2,t_3)$ be a continuous function of three variables then its triple Laplace transform is defined as (see, [68, (1), p. 1 of 10]

$$\mathcal{L}_3\left\{F(t_1,t_2,t_3);p,s,k\right\}=f_3(p,s,k)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-(pt_1+st_2+kt_3)}F(t_1,t_2,t_3)\,dt_1dt_2dt_3 \tag{6.20}$$

One more relevant reference using the triple Laplace transform is the work of Khan et al. [69] in which the two dimensional fractional order homogeneous heat equation is solved by them. We see that (6.20) is a special case of the triple Upadhyaya transform (TUT) (6.19) for the choice of parameters

$$\lambda_1^{(1)} = 1 = \lambda_1^{(2)} = \lambda_1^{(3)} = \lambda_3^{(1)} = \lambda_3^{(2)} = \lambda_3^{(3)}, \lambda_2^{(1)} = p, \lambda_2^{(2)} = s, \lambda_2^{(3)} = k$$

in it which shows the expected relation between these two transforms

$$\mathcal{U}_3\left\{F(t_1,t_2,t_3);1,p,1;1,s,1;1,k,1\right\} = \mathrm{u}_3\left\{1,p,1;1,s,1;1,k,1\right\} = \mathcal{L}_3\left\{F(t_1,t_2,t_3);p,s,k\right\} = f_3(p,s,k).$$

6.14 The Triple Elzaki Transform

Elzaki and Mousa [70] very recently introduced the triple Elzaki transform and studied its convergence properties and applied it to solve the Volterra integro-partial differential equation. For a function $F(t_1,t_2,t_3)$ of three positive variables t_1,t_2,t_3 which is expressible as a convergent infinite series, the triple Elzaki transform is defined by

$$E_3\left\{F(t_1,t_2,t_3);(\rho,s,\delta)\right\} = T_3(\rho,s,\delta) = \rho s\delta \int_0^\infty \int_0^\infty \int_0^\infty e^{-\left(\frac{t_1}{\rho}+\frac{t_2}{s}+\frac{t_3}{\delta}\right)} F(t_1,t_2,t_3)\, dt_1 dt_2 dt_3 \quad (6.21)$$

provided the integral is convergent. A comparison between (6.19) and (6.21) shows that for the choice of parameters $\lambda_1^{(1)} = \rho, \lambda_1^{(2)} = s, \lambda_1^{(3)} = \delta, \lambda_2^{(1)} = \frac{1}{\rho}, \lambda_2^{(2)} = \frac{1}{s}, \lambda_2^{(3)} = \frac{1}{\delta}, \lambda_3^{(1)} = \lambda_3^{(2)} = \lambda_3^{(3)} = 1$ in (6.19) reduces it to (6.21) to show that

$$\mathcal{U}_3\left\{F(t_1,t_2,t_3);\rho,\frac{1}{\rho},1;s,\frac{1}{s},1;\delta,\frac{1}{\delta},1\right\} = \mathrm{u}_3\left\{\rho,\frac{1}{\rho},1;s,\frac{1}{s},1;\delta,\frac{1}{\delta},1\right\}$$
$$= E_3\left\{F(t_1,t_2,t_3);(\rho,s,\delta)\right\} = T_3(\rho,s,\delta).$$

6.15 The Quadruple Laplace Transform

The quadruple Laplace transform was recently introduced by Rehman et al. [74]. In their work [74] they prove some properties of this transform and solve some homogeneous and non-homogeneous partial differential equations involving four variables. They define the quadruple Laplace transform for a continuous function $F(w,x,y,z)$ of four positive w,x,y,z by the relation (see, [74, (2.1), p. 3374])

$$\begin{aligned}&\mathcal{L}_4\left\{F(w,x,y,z);p,q,r,s\right\} = f_4(p,q,r,s)\\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(pw+qx+ry+sz)} F(w,x,y,z)\, dwdxdydz\end{aligned} \quad (6.22)$$

We mention that (6.22) is a special case of (6.2) when $n = 4$ in it and for the choice of parameters

$$\lambda_1^{(1)} = 1 = \lambda_1^{(2)} = \lambda_1^{(3)} = \lambda_1^{(4)} = \lambda_3^{(1)} = \lambda_3^{(2)} = \lambda_3^{(3)} = \lambda_3^{(4)}, \lambda_2^{(1)} = p, \lambda_2^{(2)} = q, \lambda_2^{(3)} = r, \lambda_2^{(4)} = s$$

and the variables $t_1 = w, t_2 = x, t_3 = y, t_4 = z$ to show that (as expected)

$$\mathcal{U}_4\left\{F(w,x,y,z);1,p,1;1,q,1;1,r,1;1,s,1\right\} = \mathrm{u}_4\left\{1,p,1;1,q,1;1,r,1;1,s,1\right\}$$
$$= \mathcal{L}_4\left\{F(w,x,y,z);p,q,r,s\right\} = f_3(p,q,r,s).$$

7. THE DEGENERATE UPADHYAYA TRANSFORM

We record that the *degenerate Laplace transform* was introduced recently by Kim and Kim [9] in 2017. They defined the *degenerate exponential function* e_μ^t as a function of two variables μ and t, where, $\mu \in (0,\infty)$, $t \in \mathbb{R}$, which is defined by (see, [9, (1.3), p. 241])

$$e_\mu^t = (1+\mu t)^{\frac{1}{\mu}} \quad (7.1)$$

This degenerate exponential function generalizes the classical exponential function e^t defined by the infinite series expansion $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ and buy observing that

$$\lim_{\mu \to 0+} e_\mu^t = \lim_{\mu \to 0+} (1+\mu t)^{\frac{1}{\mu}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t \quad (7.2)$$

On the basis of this degenerate exponential function, Kim and Kim [9] defined the *degenerate Laplace transform* of a function $F(t)$ be a function defined for $t \geq 0$ and for $\mu \in (0,\infty)$ by the relation (see [9, (3.1), p.244])

$$\mathcal{L}_\mu\{F(t);s\} = f_\mu(s) = \int_0^\infty (1+\mu t)^{\frac{-s}{\mu}} F(t)\,dt \tag{7.3}$$

Kim and Kim [9] developed the theory of the degenerate Laplace transform to some extent in their work [9]. This work of Kim and Kim was extended by the present author in a series of four papers [10-13]. Corresponding to this work of Kim and Kim we also define a degenerate version of the Upadhyaya transform here and analogously call it the *degenerate Upadhyaya transform* (DUT) for a function $F(t)$ of a positive variable t and for $\mu \in (0,\infty)$ as follows:

$$\mathcal{U}_\mu^D\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathrm{u}_\mu^D(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\int_0^\infty (1+\mu t)^{\frac{-\lambda_2}{\mu}} F(\lambda_3 t)\,dt \tag{7.4}$$

provided the integral converges, where, the superscript D in the symbols $\mathcal{U}_\mu^D$ or u_μ^D signifies the degenerate nature of the UT in (7.4).

We can observe that in the limiting case when $\mu \to 0$ in (7.3) and (7.4) they respectively give the usual Laplace transform and the Upadhyaya transform (2.3) respectively. We also see that the degenerate Laplace transform (7.3) of Kim and Kim [9] is a special case of the degenerate Upadhyaya transform (7.4) because for the choice of parameters $\lambda_1 = 1, \lambda_2 = s, \lambda_3 = 1$ in (7.4) it reduces to (7.3) thus showing that

$$\mathcal{U}_\mu^D\{F(t);1,s,1\} = \mathrm{u}_\mu^D(1,s,1) = \mathcal{L}_\mu\{F(t);s\} = f_\mu(s).$$

Similarly corresponding to the $n-$dimensional Upadhyaya Transform of (6.2) we define the *n-Dimensional Degenerate Upadhyaya Transform* (n-DDUT) for a function $F(t_1,\ldots,t_n)$ of n positive variables $t_1,\ldots,t_n$ and for $\mu \in (0,\infty)$ by the relation

$$\begin{aligned}\mathcal{U}_{\mu(n)}^D\left\{F(t_1,\ldots,t_n);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\ldots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\} &= \mathrm{u}_{\mu(n)}^D\left\{\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\ldots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\}\\ &= \prod_{i=1}^n \lambda_1^{(i)} \int_0^\infty \ldots(n)\ldots\int_0^\infty \left(\prod_{i=1}^n (1+\mu t_i)^{\frac{-\lambda_2^{(i)}}{\mu}}\right) F\left(\lambda_3^{(1)}t_1,\ldots,\lambda_3^{(n)}t_n\right)dt_1\ldots dt_n.\end{aligned} \tag{7.5}$$

whenever the integral converges. We can at once see that in the limit as $\mu \to 0$ (7.5) reduces to (6.2).

8. THE MODIFIED DEGENERATE UPADHYAYA TRANSFORM

Very recently in the month of October 2018 Kim et al. [71] introduced the *modified degenerate Laplace transform* for a function $F(t)$ defined for $t \geq 0$ and for $\mu \in (0,\infty)$ defined as follows (see [71, (26), p. 4 of 8])

$$\mathcal{L}_\mu^*\{F(t);s\} = f_\mu^*(s) = \int_0^\infty (1+\mu s)^{\frac{-t}{\mu}} F(t)\,dt. \tag{8.1}$$

Corresponding to this we define the *Modified Degenerate Upadhyaya Transform* (MDUT) for a function $F(t)$ defined for $t \geq 0$ and for $\mu \in (0,\infty)$ by the relation

$$\mathcal{U}_\mu^{*D}\{F(t);\lambda_1,\lambda_2,\lambda_3\} = \mathrm{u}_\mu^{*D}(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\int_0^\infty (1+\mu\lambda_2)^{\frac{-t}{\mu}} F(\lambda_3 t)\,dt. \tag{8.2}$$

if the integral converges. We can at once notice that in the limit as $\mu \to 0$ then (8.1) and (8.2) reduce to the Laplace transform and the Upadhyaya transform (2.3) respectively. We also see it easily that for the choice of parameters $\lambda_1 = 1, \lambda_2 = s, \lambda_3 = 1$ in (8.2) it reduces to (8.1) thereby showing that the modified degenerate Laplace transform of Kim et al. [71] is a special case of the modified degenerate Upadhyaya transform and the relation between the two is

$$\mathcal{U}_\mu^{*D}\{F(t);1,s,1\} = \mathrm{u}_\mu^{*D}(1,s,1) = \mathcal{L}_\mu^*\{F(t);s\} = f_\mu^*(s).$$

In a similar fashion corresponding to the $n-$dimensional Upadhyaya Transform of (6.2) we define the *n-Dimensional Modified Degenerate Upadhyaya Transform* (n-DMDUT) for a function $F(t_1,\dots,t_n)$ of n positive variables $t_1,\dots,t_n$ and for $\mu\in(0,\infty)$ by the relation

$$\mathcal{U}^{*D}_{\mu(n)}\left\{F(t_1,\dots,t_n);\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\dots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\}=\mathrm{u}^{*D}_{\mu(n)}\left\{\lambda_1^{(1)},\lambda_2^{(1)},\lambda_3^{(1)};\dots;\lambda_1^{(n)},\lambda_2^{(n)},\lambda_3^{(n)}\right\}$$
$$=\prod_{i=1}^{n}\lambda_1^{(i)}\int_0^\infty\dots(n)\dots\int_0^\infty\left(\prod_{i=1}^{n}\left(1+\mu\lambda_2^{(i)}\right)^{\frac{-t_i}{\mu}}\right)F\left(\lambda_3^{(1)}t_1,\dots,\lambda_3^{(n)}t_n\right)dt_1\dots dt_n. \tag{8.3}$$

provided the integral converges. It is obvious that as we take the limit $\mu\to 0$ (8.3) reduces to (6.2).

9. THE UPADHYAYA TRANSFORM OF REAL MATRIX ARGUMENTS

The functions of matrix arguments find vital applications in statistics, probability, astrophysics, etc. Till date the monograph by Mathai [14] is the most comprehensive and consolidated reference work available on this subject. The author's doctoral dissertation [15] also presents a comprehensive study of multiple hypergeometric functions of matrix arguments. We consider, in this section, only the $(p\times p)$ real symmetric positive definite matrices, i.e. all the matrices appearing in this section of the paper are real symmetric positive definite matrices. Referring to Mathai [14, Definition 5.2, p.255] we consider a function $F(X)$ which is a scalar function of a real symmetric positive definite matrix X of order $(p\times p)$ and a matrix $T=\left[t_{ij}\right]_{p\times p}$ of parameters whose diagonal elements are $t_{jj},\ j=1,\dots,p$ and the non-diagonal elements are $\frac{1}{2}t_{jk},\ j,k=1,\dots,p,\ j\neq k$ which satisfy the condition $t_{jk}=t_{kj},\forall j,k$, then for such matrices X,T the trace of the product matrix TX or XT is given by (see, [14, (5.1.6), p. 255]

$$\mathrm{tr}(TX)=\mathrm{tr}(XT)=\sum_{j\geq k}t_{jk}x_{jk}. \tag{9.1}$$

Then we define the Laplace transform of the function $F(X)$ of real (symmetric positive definite) matrix argument by the relation (see Mathai [14, (5.1.7), p.255])

$$\mathcal{L}_F(T)=f_F(T)=\int_{X>0}e^{-\mathrm{tr}(TX)}F(X)\,dX \tag{9.2}$$

where, $\mathrm{tr}(A)$ denotes the trace of the matrix A, $X>0$ shows that the matrix X is positive definite and the symbol $\int_{X>0}$ shows that the integration is carried out over the set of all positive definite matrices X and $X>0$ means that the matrix X is positive definite. Whenever the integral in (9.2) exists it is called the matrix argument Laplace transform of the function $F(X)$. With the same choice of the matrices X,Λ_2 as above and the conditions imposed on the matrix Λ_2 are the same as those imposed above on the matrix T, we define the *Upadhyaya Transform of Real (Symmetric Positive Definite) Matrix Arguments* (UTRSPDMA) by the equation

$$\mathcal{U}_{\Lambda_2}\left\{F(X);\Lambda_1,\Lambda_2,\Lambda_3\right\}=\mathrm{u}_{\Lambda_2}(\Lambda_1,\Lambda_2,\Lambda_3)=|\Lambda_1|\int_{X>0}e^{-\mathrm{tr}(\Lambda_2X)}F\left(X^{1/2}\Lambda_3X^{1/2}\right)dX \tag{9.3}$$

provided the integral exists. It may be mentioned here that the function $F(\Lambda_3X)$ in (9.3) is symmetric in the sense $F(\Lambda_3X)=F\left(\Lambda_3^{1/2}X\Lambda_3^{1/2}\right)=F\left(X^{1/2}\Lambda_3X^{1/2}\right)=F(X\Lambda_3)$ (see Mathai [14, section 6.1.3, p. 367], for a more detailed discussion regarding the symmetry of the function of matrix argument we refer the reader to Mathai [72, p. 515 and (1.7), p. 516]), $\Lambda_1,\Lambda_2,\Lambda_3$ are all real symmetric positive definite matrices of order $(p\times p)$ and $X^{1/2}$ denotes the real symmetric positive definite square root of the matrix X and $|X|$ denotes the determinant of the matrix X.

We mention that (9.3) is the matrix generalization of (2.3) for the case of functions of real symmetric positive definite matrix arguments. When in (9.3) we choose $\Lambda_1 = I_p = \Lambda_3$, ($I_p$ being the identity matrix of order p) and $\Lambda_2 = T$, it reduces to (9.2) thus showing that

$$\mathcal{U}_T\left\{F(X);I_p,T,I_p\right\} = \mathfrak{u}_{\Lambda_2}\left(I_p,T,I_p\right) = \mathcal{L}_F(T) = f_F(T).$$

10. THE UPADHYAYA TRANSFORM OF COMPLEX MATRIX ARGUMENTS

For defining the *Upadhyaya Transform of Complex Hermitian Positive Definite Matrix Arguments* (UTCHPDMA) we assume that all the matrices appearing in this section of the paper are $(p\times p)$ complex Hermitian positive definite matrices. For the theory of the functions of complex Hermitian positive definite matrix arguments we refer the reader to the Chapter 6 of Mathai [14] and we use the same notations in this section of the paper to denote the complex matrices as are used by Mathai in the Chapter 6 of his book [14] (see also section 4, pp. 213-215 of [73]). Following Mathai [14, Chapter 6, pp.361-362] we consider a $(p\times p)$ complex Hermitian matrix $\tilde{X}$ of functionally independent variables. We assume that $\tilde{X} = X_1 + iX_2$, where X_1, X_2 are real matrices such that $X_1 = X_1^{'}$ and $X_2 = -X_2^{'}$, ($\tilde{X} = \tilde{X}^*$). Further we consider a $(p\times p)$ complex Hermitian matrix $\tilde{T}$ of parameters such that $\tilde{T} = T_1 + iT_2$, T_1, T_2 being real matrices such that $T_1 = T_1^{'}$ and $T_2 = -T_2^{'}$ ($\tilde{T} = \tilde{T}^*$) then it can be shown that (see Mathai [14, (6.1.4) p. 361])

$$\mathrm{tr}\left(\tilde{T}^*\tilde{X}\right) = \mathrm{tr}\left(T_1X_1\right) + \mathrm{tr}\left(T_2X_2\right) \tag{10.1}$$

Further if we assume that $\tilde{T}^* = \left[\eta_{jk}\tilde{t}_{jk}\right]_{p\times p} = \left(T_1 + iT_2\right)$ where, $\eta_{jk} = 1, j = k$ and $\eta_{jk} = \frac{1}{2}, j \neq k$ and $\tilde{T} = \tilde{T}^*, \tilde{X} = \tilde{X}^*$ it follows that (see Mathai [14, (6.1.5), p. 362])

$$\mathrm{tr}\left(\tilde{T}^*\tilde{X}\right) = \sum_{j=1}^{p} t_{jj1}x_{jj1} + \sum_{j>k} t_{jk1}x_{jk1} + \sum_{j>k} t_{jk2}x_{jk2} \tag{10.2}$$

where $T_1 = \left(t_{jk1}\right), T_2 = \left(t_{jk2}\right), X_1 = \left(x_{jk1}\right), X_2 = \left(x_{jk2}\right)$ (see, Matahi [14, p. 361]) . With this background, the Laplace transform of complex Hermitian positive definite matrix argument $\tilde{X}$ is defined as follows (see Mathai [14, Definition 6.7, (6.1.6), p. 362]) for a function $F\left(\tilde{X}\right)$, which is a real valued scalar function of $\tilde{X}$:
For positive definite Hermitain matrices $\tilde{T}$ and $\tilde{X}$ of order $(p\times p)$ and the restrictions on $\tilde{T}$ as stated above, the Laplace transform of a real valued scalar function $F\left(\tilde{X}\right)$ is defined by,

$$\mathcal{L}_F\left(\tilde{T}\right) = f_F\left(\tilde{T}\right) = \int_{\tilde{X}=\tilde{X}^*>0} e^{-\left[\mathrm{tr}\left(\tilde{T}^*\tilde{X}\right)\right]} F\left(\tilde{X}\right) d\tilde{X} \tag{10.3}$$

whenever the integral exists.
Corresponding to this, we define the *Upadhyaya Transform of Complex Hermitian Positive Definite Matrix Arguments* (UTCHPDMA) for three $(p\times p)$ complex Hermitian positive definite matrices $\tilde{\Lambda}_1, \tilde{\Lambda}_2, \tilde{\Lambda}_3$ of parameters, a function $F\left(\tilde{X}\right)$, which is a real valued scalar function of the complex Hermitian positive definite matrix $\tilde{X}$ of order $(p\times p)$ as below

$$\mathcal{U}_{\tilde{\Lambda}_2}\left\{F\left(\tilde{X}\right);\tilde{\Lambda}_1,\tilde{\Lambda}_2,\tilde{\Lambda}_3\right\} = \mathfrak{u}_{\tilde{\Lambda}_2}\left(\tilde{\Lambda}_1,\tilde{\Lambda}_2,\tilde{\Lambda}_3\right) = \left|\det\left(\tilde{\Lambda}_1\right)\right| \int_{\tilde{X}=\tilde{X}^*>0} e^{-\left[\mathrm{tr}\left(\tilde{\Lambda}_2^*\tilde{X}\right)\right]} F\left(\tilde{X}^{1/2}\tilde{\Lambda}_3\tilde{X}^{1/2}\right) d\tilde{X} \tag{10.4}$$

provided the integral exists and in which $\left|\det\left(\tilde{\Lambda}_1\right)\right|$ represents the absolute value of the determinant of the matrix $\tilde{\Lambda}_1$. It is pertinent to record here that (10.4) is the matrix generalization of (2.3) for the case of functions of Hermitain positive definite matrix arguments. When we choose $\tilde{\Lambda}_1 = I_p = \tilde{\Lambda}_3, \tilde{\Lambda}_2 = \tilde{T}$ in (10.4) it reduces to (10.3) hence establishing that

$$\mathcal{U}_{\tilde{T}}\left\{F\left(\tilde{X}\right);I_p,\tilde{T},I_p\right\}=\mathfrak{u}_{\tilde{\Lambda}_2}\left(I_p,\tilde{T},I_p\right)=\mathcal{L}_F\left(\tilde{T}\right)=f_F\left(\tilde{T}\right).$$

We also mention here that we can also define the *basic extensions ($q-$ analogues) of the one-dimensional and $n-$ dimensional Upadhyaya transforms*, which we may call the *Basic One- Dimensional Upadhyaya Transform* (BODUT) and the *Basic $n-$ Dimensional Upadhyaya Transform* (BnDUT) respectively, for (2.3) and (6.2) on similar lines by using the notations and methods of the $q-$calculus given in the classical work of Gasper and Rahman [75] by using the $q-$exponential functions $e_q\left(z\right)$ and $E_q\left(z\right)$ defined respectively in (1.3.15), p. 10 and (1.3.16), p. 11 of Gasper and Rahman [75] by the following equations in the usual notations of the theory of the basic hypergeometric series:

$$e_q\left(z\right)={}_1\phi_0\left(0;-;q,z\right)=\sum_{n=0}^{\infty}\frac{z^n}{\left(q;q\right)_n}=\frac{1}{\left(z;q\right)_\infty},\left|z\right|<1 \tag{10.5}$$

and

$$E_q\left(z\right)={}_0\phi_0\left(-;-;q,-z\right)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{\left(q;q\right)_n}z^n=\left(-z;q\right)_\infty \tag{10.6}$$

Similarly by using the concepts of fractional calculus the *Fractional Upadhyaya Transforms of one and $n-$dimensions* respectively can be defined for (2.3) and (6.2). It is to be noted that the Theorem 5.11, equations (5.31), (5.32) and the Theorem 5.26 open the gateway of applications of the Upadhyaya Transform (UT) for solving the initial value problems and the boundary value problems involving the ordinary and partial differential coefficients arising most frequently in applied mathematics, physics, engineering and other allied areas of study like biomathematics, biophysics, etc. We foresee that all the areas of research and possibly many more than we can think of at present, wherever the Laplace transform and its above mentioned extant variants are being currently applied for finding the solutions of the problems arising there, the Upadhyaya transform will be used most extensively in the coming years and the results developed in this paper and the various generalizations of the Upadhyaya transform pointed out here will be developed by us and other researchers around the world and all these proposed works will find the maximum possible applications in the coming years. With this highly optimistic vision about the future scope of this work we conclude the paper and also remark that our future communications will focus on these subjects and various other possible applications and the author most humbly invites all the respected readers and researchers worldwide to extend this work, generalize it and apply it as extensively as possible so that together we all the researchers of the worldwide mathematics community make the Upadhyaya Transform an everlasting tool in the mathematical arena, a powerful tool whose potential powers of applications are yet to be expounded by all of us!

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