Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

Bull. Pure Appl. Sci. Sect. E Math. Stat.
38E(2), 592–602 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2019.00060.2

Isometries of differentiation and composition operators on Zygmund type space *

A. Kamal[1] and M. Hamza. Eissa[2]

1,2. Department of Mathematics, Faculty of Science,
Port Said University, Port Said, Egypt.
1. Department of Mathematics, College of Sciences and Arts in Muthnib,
Qassim University, Kingdom of Saudi Arabia.
1. E-mail: alaa_mohamed1@yahoo.com, alaa_kamal@sci.psu.edu.eg

Abstract The goal of this paper is to characterize the isometries of the products of differentiation and composition on Zygmund type space.

Key words Composition operator, Bloch type space, Products of Differentiation and Composition Operators, Isometries.

2010 Mathematics Subject Classification 47B38, 31C25.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the **open unit disk** in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ is the class of all analytic functions in $\mathbb{D}$.
Now, we give some definitions which we will use in this paper.

Let f be an analytic self-map of the unit disk $\mathbb{D}$. The Bloch space is defined as (see [9–11]).

Definition 1.1. Let $0 < \alpha < \infty$. The **α-Bloch space** ,$\mathcal{B}_\alpha$, is defined by

$$\mathcal{B}_\alpha := \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha|f'(z)| < \infty\}.$$

The **little α-Bloch space** ,$\mathcal{B}_{\alpha,0}$, is given by the following

$$\mathcal{B}_{\alpha,0} := \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_{\alpha,\prime}} = \lim_{|z|\to1^-}(1 - |z|^2)^\alpha|f'(z)| = 0\}.$$

Definition 1.2. Let $\mathcal{Z}$ denote the space of all f $\in (H(\mathbb{D})) \cap (\mathbb{C}(\bar{\mathbb{D}}))$ f $\in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}}|(1 - |z|^2)|f''(z)| < \infty.$$

Hence, the following relation holds:

$$||f||_{\mathcal{Z}} \asymp \sup_{z\in\mathbb{D}}|(1 - |z|^2)|f''(z)| < \infty.$$

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received February 18, 2019 / Revised September 19, 2019 / Accepted October 24, 2019. Online First Published on December 24, 2019 at https://www.bpasjournals.com/.
Corresponding author A. Kamal, E-mail: alaa_mohamed1@yahoo.com , alaa_kamal@sci.psu.edu.eg

For more information on Zygmund type spaces and some operators on them can be found in (see [2,6–8]).
The little Zygmund space $\mathcal{Z}_0$ was introduced by Li and Stević in (see [5]) as the following :

$$f \in \mathcal{Z}_0 \Leftrightarrow \lim_{|z|\to 1} |(1-|z|^2)|f''(z)| = 0.$$

Definition 1.3. (see [3]) For any analytic self-mapping ϕ of $\mathbb{D}$ the symbol ϕ induces a linear composition operator $C_\phi(f) := f \circ \phi$ from $H(\mathbb{D})$ into itself.

Now, we will show the definition of the products of composition operators followed by differentiation operator which defined in (see [4]) as follows:

Definition 1.4. The differentiation operator D is defined by $Df = f'$, while the operators DC_ϕ are defined by $DC_\phi(f) = (f \circ \phi)' = f'(\phi)\phi'$. And operators CD_ϕ are defined by $C_\phi D(f) = f' \circ \phi = f'(\phi)$.

The Hardly space can be defined as follows see ([12]).

Definition 1.5. The space H^∞ denotes the space of all bounded analytic functions f on the unit disk $\mathbb{D}$ such that

$$||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty. \tag{1.1}$$

Definition 1.6. Let X and Y be two Banach spaces, recall that a linear isometry is a linear operator T from X to Y such that

$$||Tf||_Y = ||f||_X.$$

2 Auxiliary results

In this section, we will introduce some notation and state a couple of lemmas which are used to prove the main results.
For $a \in \mathbb{D}$ and $z \in \mathbb{D}$, the definition of involution ϕ_a which interchanges the origin and point a, is defined as

$$\varphi_a(z) := \frac{a-z}{1-\bar{a}z}, \quad \text{for } z \in \mathbb{D}.$$

The definition of pseudo-hyperbolic distance between z and w where z, w in $\in \mathbb{D}$ is given by

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\bar{z}w}\right|,$$

and the hyperbolic metric is given by

$$\beta(z,w) = \inf_\gamma \int_\gamma \frac{d\xi}{1-|\xi|^2} = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where γ is any piecewise smooth curve in $\mathbb{D}$ from z to w.
Now, we list up following two lemmas which are needed to prove our main results.
The first lemma is introduced in (see [13]).

Lemma 2.1. *For all $z, w \in \mathbb{D}$ we have*

$$1-\rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}.$$

For, $\phi \in S(\mathbb{D})$ the Schwarz-Pick lemma shows that $\rho(\phi(z), \phi(w)) \leq \rho(z,w)$ w), and if equality holds for some $z \neq w$, then ϕ is an automorphism of the disk. It is also well known that for$\phi \in S(\mathbb{D}), C_\phi$ is always bounded on $\mathcal{B}$.
From (see [1]) and by making a little modification of Lemma 2.1 we get the following lemma.

Lemma 2.2. *There exists a constant $C >$ such that*

$$\left|(1-|z|^2)f'(z)-(1-|w|^2)f'(w)\right| \leq C||f||_{\mathcal{B}}.\rho(z,w),$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}$.

$$\left|(1-|z|^2)f''(z)-(1-|w|^2)f''(w)\right| \leq C||f||_{\mathcal{Z}}.\rho(z,w),$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{Z}$.

Lemma 2.3. *(see([9])) Assume that $f \in H^\infty$. Then for each $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z\in\mathbb{D}}(1-\mid z\mid)^n \mid f^{(n)}(z)\mid \leq C \parallel f \parallel_\infty .$$

In this paper, we will denote to the letter C as a positive constant.

3 The isometries of $(DC_\phi f)(z) : \mathcal{Z} \to \mathcal{Z}$

In this section, we characterize the operators $(DC_\phi f)(z) : \mathcal{Z} \to \mathcal{Z}$. Moreover, we give the conditions which prove the isometries of the operators $(DC_\phi f)(z)$.

Theorem 3.1. *Let ϕ be analytic self-maps of the unit disk then, the operator $(DC_\phi f)(z)$ is an isometry in the seminorm if and only if the following conditions hold.*
(A)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^3(z)|}{(1-|\phi(z)|^2)^3} \leq 1,$$
$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi''(z)||\phi'(z)|}{(1-|\phi(z)|^2)} \leq 1$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'''(z)|}{(1-|\phi(z)|^2)} \leq 1.$$

(B) For every $a \in \mathbb{D}$, there exists at least a sequence $\{z_n\}$ in $\mathbb{D}$ such that

$$\lim_{n\to\infty} \rho(\phi(z_n), a) = 0.$$

And

$$\lim_{n\to\infty} \frac{(1-|z|^2)|\phi'^3(z_n)|}{(1-|\phi(z_n)|^2)^3} = 1,$$
$$\lim_{n\to\infty} \frac{(1-|z|^2)|\phi''(z_n)||\phi'(z_n)|}{(1-|\phi(z_n)|^2)} = 1,$$

and

$$\lim_{n\to\infty} \frac{(1-|z|^2)|\phi'''(z_n)|}{(1-|\phi(z_n)|^2)} = 1.$$

Proof. First, We will prove the sufficiency.By condition (A), for every $f \in \mathcal{Z}$, we have

$$\begin{aligned}
||(DC_\phi f)(z)||_{\mathcal{Z}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big((f\circ\phi)'(z)\Big)'' \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big(f'(\phi(z))\phi'(z)\Big)'' \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big[\phi'^2(z)f''(\phi(z))+\phi''(z)f'(\phi(z))\Big]' \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big[\phi'^3(z)f'''(\phi(z))+2\phi'(z)\phi''(z)f''(\phi(z)) \\
&+ \phi'''(z)f'(\phi(z))+\phi''(z)\phi'(z)f''(\phi(z))\Big] \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big[\phi'^3(z)f'''(\phi(z))+3\phi'(z)\phi''(z)f''(\phi(z)) \\
&+ \phi'''(z)f'(\phi(z))\Big] \\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'^3(z)||f'''(\phi(z)| \\
&+ \sup_{z\in\mathbb{D}}3(1-|z|^2)|\phi''(z)||\phi'(z)||f''(\phi(z)| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'''(z)||f'(\phi(z)|
\end{aligned}$$

$$\begin{aligned}
&= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\phi'^3(z)|}{(1-|\phi(z)|^2)^3}|f'''(\phi(z)|(1-|\phi(z)|^2)^3 \\
&+ \sup_{z\in\mathbb{D}}\frac{3(1-|z|^2)|\phi''(z)||\phi'(z)|}{(1-|\phi(z)|^2)}|f''(\phi(z)|(1-|\phi(z)|^2) \\
&+ \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\phi'''(z)|}{(1-|\phi(z)|^2)}|f'(\phi(z)|(1-|\phi(z)|^2) \\
&= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\phi'^3(z)|}{(1-|\phi(z)|^2)^3}||f||_\infty \\
&+ \sup_{z\in\mathbb{D}}\frac{3(1-|z|^2)|\phi''(z)||\phi'(z)|}{(1-|\phi(z)|^2)}||f||_{\mathcal{Z}} \\
&+ \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\phi'''(z)|}{(1-|\phi(z)|^2)}||f||_{\mathcal{B}} \\
&\le ||f||_\infty+||f||_{\mathcal{Z}}+||f||_{\mathcal{B}}. \qquad (3.1)
\end{aligned}$$

Now, we will show that property (B) implies $||(DC_\phi f)(z)||_{\mathcal{Z}} \ge ||f||_{\mathcal{Z}}$.
In fact, given any $f \in \mathcal{Z}$, then

$$||f||_{\mathcal{Z}} = \lim_{m\to\infty}(1-|a_m|^2)|f''(a_m)|,$$

$$||f||_\infty = \lim_{m\to\infty}(1-|a_m|^2)|f'''(a_m)|,$$

and

$$||f||_{\mathcal{B}} = \lim_{m\to\infty}(1-|a_m|^2)|f'(a_m)|,$$

for some sequence $\{a_m \subset \mathbb{D}\}$.
For any fixed m, it follows from (B) that there is a sequence $\{z_k^m \subset \mathbb{D}\}$ such that

$$\rho(\phi(z_k^m),a_m)\to 0, \frac{(1-|z_k^m|^2)|\phi'^3(z_k^m)|}{(1-|\phi(z_k^m)|^2)^3}\to 1,$$

$$\frac{3(1-|z_k^m|^2)|\phi''(z_k^m)||\phi'(z_k^m)|}{(1-|\phi(z_k^m)|^2)}\to 1 \; and \; \frac{(1-|z_k^m|^2)|\phi'''(z_k^m)|}{(1-|\phi(z_k^m)|^2)}\to 1, \qquad (3.2)$$

as $k \to \infty$ By Lemma 2.2, for all m and k,

$$\begin{aligned}
&\left|(1-|\phi(z_k^m)|^2)^3 f'''(z_k^m)-(1-|a_m|^2)f'''(a_m)\right| \leq C||f||_\infty.\rho(z_k^m,a_m),\\
&\left|(1-|\phi(z_k^m)|^2)f''(z_k^m)-(1-|a_m|^2)f''(a_m)\right| \leq C||f||_{\mathcal{Z}}.\rho(z_k^m,a_m),\\
&\left|(1-|\phi(z_k^m)|^2)f'(z_k^m)-(1-|a_m|^2)f'(a_m)\right| \leq C||f||_{\mathcal{B}}.\rho(z_k^m,a_m).
\end{aligned} \tag{3.3}$$

Hence,

$$\begin{aligned}
&|(1-|\phi(z_k^m)|^2)^3 f'''(z_k^m)| \geq |(1-|a_m|^2)f'''(a_m)| - C||f||_\infty.\rho(z_k^m,a_m),\\
&|(1-|\phi(z_k^m)|^2)f''(z_k^m)| \geq |(1-|a_m|^2)f''(a_m)| - C||f||_{\mathcal{Z}}.\rho(z_k^m,a_m),\\
&|(1-|\phi(z_k^m)|^2)f'(z_k^m)| \geq |(1-|a_m|^2)f'(a_m)| - C||f||_{\mathcal{B}}.\rho(z_k^m,a_m).
\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}
||(DC_\phi f)(z)||_{\mathcal{Z}} &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^3(z)|}{(1-|\phi(z)|^2)^3}|f'''(\phi(z)|(1-|\phi(z)|^2)^3\\
&+ \sup_{z\in\mathbb{D}} \frac{3(1-|z|^2)|\phi''(z)||\phi'(z)|}{(1-|\phi(z)|^2)}|f''(\phi(z)|(1-|\phi(z)|^2)\\
&+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'''(z)|}{(1-|\phi(z)|^2)}|f'(\phi(z)|(1-|\phi(z)|^2)\\
&\geq \lim_{k\to\infty}\sup \frac{(1-|z_k^m|^2)|\phi'^3(z_k^m)|}{(1-|\phi(z_k^m)|^2)^3}|f'''(\phi(z_k^m)|(1-|\phi(z_k^m)|^2)^3\\
&+ \lim_{k\to\infty}\sup \frac{3(1-|z_k^m|^2)|\phi''(z_k^m)||\phi'(z_k^m)|}{(1-|\phi(z_k^m)|^2)}|f''(\phi(z_k^m)|(1-|\phi(z_k^m)|^2)\\
&+ \lim_{k\to\infty}\sup \frac{(1-|z_k^m|^2)|\phi'''(z_k^m)|}{(1-|\phi(z_k^m)|^2)}|f'(\phi(z_k^m)|(1-|\phi(z_k^m)|^2)\\
&= (1-|a_m|^2)|f'''(a_m)| + 3(1-|a_m|^2)|f''(a_m)| + (1-|a_m|^2)|f'(a_m)|.
\end{aligned} \tag{3.5}$$

The inequality $||(DC_\phi f)(z)||_{\mathcal{Z}} \geq ||f||_{\mathcal{Z}}$ follows by letting $m \to 0$.

From the above, we have $||(DC_\phi f)(z)||_{\mathcal{Z}} = ||f||_{\mathcal{Z}}$,which means that $(DC_\phi f)(z)$ is an isometry operator on the Zygmund type space.
Now, we prove will the necessity. For any $a \in \mathbb{D}$, we will begin by taking the following test function

$$f_a(z) = \frac{(1-|a|^2)}{a(1-|\bar{a}z|)}. \tag{3.6}$$

It is clear that

$$f_a'(z) = \frac{1-|a|^2}{(1-|\bar{a}z|)^2},$$

$$f_a''(z) = \frac{2a(1-|a|^2)}{(1-|\bar{a}z|)^3} = \frac{(1-|a|^2)}{(1-|\bar{a}z|)^2}.\frac{2a}{(1-|\bar{a}z|)} = C\frac{(1-|a|^2)}{(1-|\bar{a}z|)^2},$$

and

$$f_a'''(z) = \frac{6a^2(1-|a|^2)}{(1-|\bar{a}z|)^4} = \frac{6a^2}{(1-|\bar{a}z|)^2}.\frac{(1-|a|^2)}{(1-|\bar{a}z|)^2} = C\frac{(1-|a|^2)}{(1-|\bar{a}z|)^2}.$$

Using Lemma 2.1, we have

$$(1-|z|^2)|f_a'(z)| = \frac{(1-|z|^2)(1-|a|^2)}{(1-\bar{a}z)^2} = (1-\rho^2(a,z)),$$

$$(1-|z|^2)|f_a''(z)| = C\frac{(1-|z|^2)(1-|a|^2)}{(1-\bar{a}z)^2} = C(1-\rho^2(a,z)),$$

and

$$(1-|z|^2)|f_a'''(z)| = C\frac{(1-|z|^2)(1-|a|^2)}{(1-\bar{a}z)^2} = C(1-\rho^2(a,z)). \tag{3.7}$$

So,

$$||f||_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f_a'(z)| \le 1,$$
$$||f||_{\mathcal{Z}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f_a''(z)| \le 1,$$

and

$$||f||_{\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f_a'''(z)| \le 1. \tag{3.8}$$

Moreover, since

$$(1-|a|^2)|f_a'(a) = \frac{(1-|a|^2)(1-|a|^2)}{(1-|a|^2)^2} = 1,$$

$$(1-|a|^2)|f_a''(a) = \frac{(1-|a|^2)(1-|a|^2)}{(1-|a|^2)^2} = 1$$

and

$$(1-|a|^2)|f_a'''(a) = \frac{(1-|a|^2)(1-|a|^2)}{(1-|a|^2)^2} = 1,$$

we have $||f||_{\mathcal{Z}} = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we obtain

$$\begin{aligned}
5 &= ||f_{\phi(a)}||_{\mathcal{Z}} = ||(DC_\phi f_{\phi(a)})||_{\mathcal{Z}} \\
&= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\phi'^3(z)|}{(1-|\phi(z)|^2)^3}(1-|\phi(z)|^2)^3|f_{\phi(a)}'''| \\
&+ \sup_{z\in\mathbb{D}}\frac{3(1-|z|^2)|\phi''(z)||\phi'(z)|}{(1-|\phi(z)|^2)}(1-|\phi(z)|^2)|f_{\phi(a)}''| \\
&+ \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\phi'''(z)|}{(1-|\phi(z)|^2)}(1-|\phi(z)|^2)|f_{\phi(a)}'| \\
&\ge \frac{(1-|a|^2)|\phi'^3(a)|}{(1-|\phi(a)|^2)^3}(1-|\phi(a)|^2)^3|f_{\phi(a)}'''| \\
&+ \frac{3(1-|a|^2)|\phi''(a)||\phi'(a)|}{(1-|\phi(a)|^2)}(1-|\phi(a)|^2)|f_{\phi(a)}''| \\
&+ \frac{(1-|a|^2)|\phi'''(a)|}{(1-|\phi(a)|^2)}(1-|\phi(a)|^2)|f_{\phi(a)}'| \\
&\ge \frac{(1-|a|^2)|\phi'^3(a)|}{(1-|\phi(a)|^2)^3} \\
&+ \frac{3(1-|a|^2)|\phi''(a)||\phi'(a)|}{(1-|\phi(a)|^2)} \\
&+ \frac{(1-|a|^2)|\phi'''(a)|}{(1-|\phi(a)|^2)}.
\end{aligned} \tag{3.9}$$

Hence A follows by noticing that a is arbitrary.
Since $||DC_\phi f_a||_{\mathcal{Z}} = ||f_a||_{\mathcal{Z}} = 1$ there exists a sequence $z_m \subset D$ such that

$$\begin{aligned}
(1-|z_m|^2)\left|\frac{d(DC_\phi f_{\phi(a)})}{dz_m}(z_m)\right| &= (1-|z_m|^2)|\phi'^3(z_m)||f_a'''(\phi(z_m))| \\
&+ 3(1-|z_m|^2)|\phi''(z_m)||\phi'(z_m)||f_a''(\phi(z_m))| \\
&+ (1-|z_m|^2)|\phi'''(z_m)||f_a'(\phi(z_m))| \\
&\to 5,
\end{aligned} \tag{3.10}$$

as $m \to \infty$.

$$\begin{aligned}(1-|z_m|^2) &\quad |\phi'^3(z_m)||f_a'''(\phi(z_m))| \\ &= \frac{(1-|z_m|^2)|\phi'^3(z_m)|}{(1-|\phi(z_m)|^2)^3}(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m))| \\ &\leq (1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)|. \end{aligned} \tag{3.11}$$

$$\begin{aligned}(1-|z_m|^2) &\quad |\phi''(z_m)||\phi'(z_m)||f_a''(\phi(z_m))| \\ &= \frac{(1-|z_m|^2)|\phi''(z_m)||\phi'(z_m)|}{(1-|\phi(z_m)|^2)}(1-|\phi(z_m)|^2)|f_a''(\phi(z_m))| \\ &\leq (1-|\phi(z_m)|^2)|f_a''(\phi(z_m))|. \end{aligned} \tag{3.12}$$

$$\begin{aligned}(1-|z_m|^2) &\quad |\phi'''(z_m)||f_a'(\phi(z_m))| \\ &= \frac{(1-|z_m|^2)|\phi'''(z_m)|}{(1-|\phi(z_m)|^2)}(1-|\phi(z_m)|^2)|f_a'(\phi(z_m))| \\ &\leq (1-|\phi(z_m)|^2)|f_a'(\phi(z_m))|. \end{aligned} \tag{3.13}$$

By combining (3.11),(3.12),(3.13) and (3.10), it follows that

$$\begin{aligned} 1 &\leq \lim_{m\to\infty} \inf(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)| \\ &\leq \lim_{m\to\infty} \sup(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)|. \end{aligned} \tag{3.14}$$

$$\begin{aligned} 1 &\leq \lim_{m\to\infty} \inf(1-|\phi(z_m)|^2)|f_a''(\phi(z_m)| \\ &\leq \lim_{m\to\infty} \sup(1-|\phi(z_m)|^2)|f_a''(\phi(z_m)|. \end{aligned} \tag{3.15}$$

$$\begin{aligned} 1 &\leq \lim_{m\to\infty} \inf(1-|\phi(z_m)|^2)|f_a'(\phi(z_m)| \\ &\leq \lim_{m\to\infty} \sup(1-|\phi(z_m)|^2)|f_a'(\phi(z_m)|. \end{aligned} \tag{3.16}$$

The last inequality follows (3.7) since $\phi(z_m) \in D$. Consequently,

$$\lim_{m\to\infty}(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)| = (1-\rho^2(\phi(z_m),a)),$$
$$\lim_{m\to\infty}(1-|\phi(z_m)|^2)|f_a''(\phi(z_m)| = (1-\rho^2(\phi(z_m),a)),$$

and

$$\lim_{m\to\infty}(1-|\phi(z_m)|^2)|f_a'(\phi(z_m)| = (1-\rho^2(\phi(z_m),a)). \tag{3.17}$$

That is,

$$\lim_{m\to\infty}(\rho(\phi(z_m),a)) = 0.$$

By combining (3.10), (3.11), (3.12), (3.13) and (3.17), we know that

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)|\phi'^3(z_m)|}{(1-|\phi(z_m)|^2)^3} = 1,$$
$$\lim_{m\to\infty}\frac{(1-|z_m|^2)|\phi''(z_m)||\phi'(z_m)|}{(1-|\phi(z_m)|^2)} = 1,$$

and

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)|\phi'''(z_m)|}{(1-|\phi(z_m)|^2)} = 1. \tag{3.18}$$

This completes the proof of theorem. □

Theorem 3.2. *Let ϕ be analytic self-maps of the unit disk, such that ϕ fixes the origin, then the operator $C_\phi D : \mathcal{Z} \to \mathcal{Z}$ is an isometry in the seminorm if and only if the following conditions hold.*
(C)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^2(z)|}{(1-|\phi(z)|^2)^3} \le 1,$$

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi''(z)|}{(1-|\phi(z)|^2)} \le 1.$$

(D) For every $a \in \mathbb{D}$, there exists at least a sequence $\{z_n\}$ in $\mathbb{D}$ such that

$$\lim_{n\to\infty} \rho(\phi(z_n), a) = 0,$$

and

$$\lim_{n\to\infty} \frac{(1-|z_m|^2)|\phi'^2(z_n)|}{(1-|\phi(z_n)|^2)^3} = 1,$$

$$\lim_{n\to\infty} \frac{(1-|z_m|^2)|\phi''(z_n)|}{(1-|\phi(z_n)|^2)} = 1.$$

Proof. First, we will prove the sufficiency. By condition (C), for every $f \in \mathcal{Z}$, we have

$$\begin{aligned}
||(C_\phi Df)(z)||_{\mathcal{Z}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big(f'(\phi(z))\Big)'' \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big[\phi'(z)f''(\phi(z))\Big]' \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big[\phi'^2(z)f'''(\phi(z)) + \phi''(z)f''(\phi(z))\Big] \\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'^2(z)||f'''(\phi(z)| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi''(z)||f''(\phi(z)| \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^2(z)|}{(1-|\phi(z)|^2)^3}|f'''(\phi(z)|(1-|\phi(z)|^2)^3 \\
&+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi''(z)|}{(1-|\phi(z)|^2)}|f''(\phi(z)|(1-|\phi(z)|^2) \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^2(z)|}{(1-|\phi(z)|^2)^3}||f||_\infty \\
&+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi''(z)|}{(1-|\phi(z)|^2)}||f||_{\mathcal{Z}} \\
&\le ||f||_\infty + ||f||_{\mathcal{Z}}.
\end{aligned} \tag{3.19}$$

Now, we will prove that property (D) implies $||(C_\phi Df)(z)||_{\mathcal{Z}} \ge ||f||_{\mathcal{Z}}$.
In fact, given any $f \in \mathcal{Z}$, then

$$||f||_{\mathcal{Z}} = \lim_{m\to\infty}(1-|a_m|^2)|f''(a_m)|,$$

$$||f||_\infty = \lim_{m\to\infty}(1-|a_m|^2)|f'''(a_m)|,$$

for some sequence $\{a_m \subset \mathbb{D}\}$. For any fixed m, it follows from (B) that there is a sequence $\{z_k^m \subset \mathbb{D}\}$ such that

$$\rho(\phi(z_k^m), a_m) \to 0, \frac{(1-|z_k^m|^2)|\phi'^2(z_k^m)|}{(1-|\phi(z_k^m)|^2)^3} \to 1,$$

$$\frac{(1-|z_k^m|^2)|\phi''(z_k^m)|}{(1-|\phi(z_k^m)|^2)} \to 1, \tag{3.20}$$

as $k \to \infty$. By Lemma 2.2, for all m and k,

$$\left|(1-|\phi(z_k^m)|^2)^3 f'''(\phi(z)_k^m) - (1-|a_m|^2)^3 f'''(a_m)\right| \le C||f||_\infty.\rho(z_k^m, a_m),$$
$$\left|(1-|\phi(z_k^m)|^2) f''(\phi(z)_k^m) - (1-|a_m|^2) f''(a_m)\right| \le C||f||_{\mathcal{Z}}.\rho(z_k^m, a_m). \tag{3.21}$$

Hence,

$$|(1-|\phi(z_k^m)|^2) f'''(\phi(z)_k^m)| \ge |(1-|a_m|^2) f'''(a_m)| - C||f||_\infty.\rho(z_k^m, a_m),$$
$$|(1-|\phi(z_k^m)|^2) f''(\phi(z)_k^m)| \ge |(1-|a_m|^2) f''(a_m)| - C||f||_{\mathcal{Z}}.\rho(z_k^m, a_m). \tag{3.22}$$

Therefore,

$$\begin{aligned} ||(C_\phi Df)(z)||_{\mathcal{Z}} &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^2(z)|}{(1-|\phi(z)|^2)^3}|f'''(\phi(z)|(1-|\phi(z)|^2)^3 \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi''(z)|}{(1-|\phi(z)|^2)}|f''(\phi(z)|(1-|\phi(z)|^2) \\ &\ge \lim_{k\to\infty} \sup \frac{(1-|z_k^m|^2)|\phi'^2(z_k^m)|}{(1-|\phi(z_k^m)|^2)^3}|f'''(\phi(z_k^m)|(1-|\phi(z_k^m)|^2)^3 \\ &+ \lim_{k\to\infty} \sup \frac{(1-|z_k^m|^2)|\phi''(z_k^m)|}{(1-|\phi(z_k^m)|^2)}|f''(\phi(z_k^m)|(1-|\phi(z_k^m)|^2) \\ &= (1-|a_m|^2)|f'''(a_m)| + (1-|a_m|^2)|f''(a_m)|. \end{aligned} \tag{3.23}$$

The inequality $||(C_\phi Df)(z)||_{\mathcal{Z}} \ge ||f||_{\mathcal{Z}}$ follows by letting $m \to \infty$.
For any $a \in \mathbb{D}$, and we will use the same test function f_a defined by (3.6) which satisfies $||f_a|| = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we have

$$\begin{aligned} 2 &= ||f_{\phi(a)}||_{\mathcal{Z}} = ||(C_\phi Df_{\phi(a)})||_{\mathcal{Z}} \\ &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'^2(z)|}{(1-|\phi(z)|^2)^3}(1-|\phi(z)|^2)^3|f'''_{\phi(a)}(\phi(z))| \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi''(z)|}{(1-|\phi(z)|^2)}(1-|\phi(z)|^2)|f''_{\phi(a)}(\phi(z))| \\ &\ge \frac{(1-|a|^2)|\phi'^2(a)|}{(1-|\phi(a)|^2)^3}(1-|\phi(a)|^2)^3|f'''_{\phi(a)}(\phi(z))| \\ &+ \frac{(1-|a|^2)|\phi''(a)|}{(1-|\phi(a)|^2)}(1-|\phi(a)|^2)|f''_{\phi(a)}(\phi(z))| \\ &\ge \frac{(1-|a|^2)|\phi'^2(a)|}{(1-|\phi(a)|^2)^3} + \frac{(1-|a|^2)|\phi''(a)|}{(1-|\phi(a)|^2)}. \end{aligned} \tag{3.24}$$

Hence, C follows by noticing a is arbitrary.
Since $||C_\phi Df_a||_{\mathcal{Z}} = ||f_a||_{\mathcal{Z}} = 2$ there exists a sequence $\{z_m\} \subset D$ such that

$$\begin{aligned} (1-|z_m|^2)\left|\frac{d(C_\phi Df_{\phi(a)})}{dz_m}(z_m)\right| &= (1-|z_m|^2)|\phi'^2(z_m)||f'''_a(\phi(z_m))| \\ &+ (1-|z_m|^2)|\phi''(z_m)||f''_a(\phi(z_m))| \\ &\to 2, \end{aligned} \tag{3.25}$$

as $m \to \infty$.

$$\begin{aligned} (1-|z_m|^2) &\quad |\phi'^2(z_m)||f'''_a(\phi(z_m))| \\ &= \frac{(1-|z_m|^2)|\phi'^2(z_m)|}{(1-|\phi(z_m)|^2)^3}(1-|\phi(z_m)|^2)^3|f'''_a(\phi(z_m))| \\ &\le (1-|\phi(z_m)|^2)^3|f''_a(\phi(z_m)|. \end{aligned} \tag{3.26}$$

$$\begin{aligned}
(1-|z_m|^2) & \quad |\phi''(z_m)||f_a''(\phi(z_m))| \\
= & \quad \frac{(1-|z_m|^2)|\phi''(z_m)|}{(1-|\phi(z_m)|^2)}(1-|\phi(z_m)|^2)|f_a''(\phi(z_m))| \\
\leq & \quad (1-|\phi(z_m)|^2)|f_a''(\phi(z_m))|. \qquad (3.27)
\end{aligned}$$

By combining (3.26),(3.27) and (3.25), it follows that

$$\begin{aligned}
1 \quad \leq & \quad \lim_{m\to\infty} \inf(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)| \\
\leq & \quad \lim_{m\to\infty} \sup(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)|. \qquad (3.28)
\end{aligned}$$

$$\begin{aligned}
1 \quad \leq & \quad \lim_{m\to\infty} \inf(1-|\phi(z_m)|^2)|f_a''(\phi(z_m)| \\
\leq & \quad \lim_{m\to\infty} \sup(1-|\phi(z_m)|^2)|f_a''(\phi(z_m)|. \qquad (3.29)
\end{aligned}$$

The last inequality follows (3.7) since $\phi(z_m) \in D$. Consequently,

$$\lim_{m\to\infty}(1-|\phi(z_m)|^2)^3|f_a'''(\phi(z_m)| = (1-\rho^2(\phi(z_m),a)),$$

and

$$\lim_{m\to\infty}(1-|\phi(z_m)|^2)|f_a''(\phi(z_m)| = (1-\rho^2(\phi(z_m),a)). \qquad (3.30)$$

That is,

$$\lim_{m\to\infty}(\rho(\phi(z_m),a)) = 0.$$

By combining (3.25), (3.26), (3.27) and (3.30), we know that

$$\lim_{m\to\infty} \frac{(1-|z_m|^2)|\phi'^2(z_m)|}{(1-|\phi(z_m)|^2)^3} = 1,$$

and

$$\lim_{m\to\infty} \frac{(1-|z_m|^2)|\phi''(z_m)|}{(1-|\phi(z_m)|^2)} = 1. \qquad (3.31)$$

From the above results follows the proof of theorem is completed. □

References

[1] José Bonet, Mikael Lindström, and Elke Wolf, *Isometric weighted composition operators on weighted banach spaces of type*, Proceedings of the American Mathematical Society (2008), 4267–4273.

[2] Boo Rim Choe, Hyungwoon Koo, and Wayne Smith, *Composition operators on small spaces*, Integral Equations and Operator Theory **56** (2006), no. 3, 357–380.

[3] A El-Sayed Ahmed and M.A Bakhit, *Composition operators acting between some weighted Möbius invariant spaces*, Annals of Functional Analysis **2** (2011), no. 2, 138–152.

[4] GL Li and ZH Zhou, *Isometries of composition and differentiation operators from bloch type space to H^∞*, J. Comput. Anal. Appl **15** (2013), no. 3, 526–533.

[5] Songxiao Li and Stevo Stević, *Volterra-type operators on Zygmund spaces*, Journal of Inequalities and Applications **2007**, no. 1.

[6] Songxiao Li and Stevo Stević, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, Journal of Mathematical Analysis and Applications **338** (2008), no. 2, 1282–1295.

[7] Songxiao Li and Stevo Stević, *Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces*, Journal of Mathematical Analysis and Applications **345** (2008), no. 1, 40–52.

[8] Songxiao Li and Stevo Stević, *Weighted composition operators from Zygmund spaces into Bloch spaces*, Applied Mathematics and Computation **206** (2008), no. 2, 825–831.

[9] Yongmin Liu and Yanyan Yu, *Composition followed by differentiation between H^∞ and Zygmund spaces*, Complex Analysis and Operator Theory **6** (2012), no. 1, 121–137.

[10] ZHAO Ru-han, *On α-Bloch functions and $VMOA$*, Acta Mathematica Scientia. Series B. English Edition **16** (1996), no. 3, 349–360.

[11] Wen Xu, *Superposition operators on Bloch-type spaces*, Computational Methods and Function Theory **7** (2007), no. 2, 501–507.

[12] Yanyan Yu and Yongmin Liu, *On stević type operator from H^∞ space to the logarithmic Bloch spaces*, Complex Analysis and Operator Theory **9** (2015), no. 8, 1759–1780.

[13] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, vol. 226, Springer Science & Business Media, 2005.