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On the fractional triple Elzaki transform and its properties *

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Abstract In this work we introduce and prove the different properties and theorems of the fractional triple Elzaki transform like the linearity property, the first and the second shifting properties, the convolution theorem, the periodic function property and the operational formula. We also give an application of this new concept to solve a factional partial differential equation in three dimensions satisfying given initial and boundary value conditions.

Key words Fractional triple Elzaki transform, Inverse triple Elzaki transform, partial differential equations, Upadhyaya transform, Elzaki transform.

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1 Introduction

In the past two centuries, the integral transforms have been widely applied as a tool to solve various problems in pure and applied mathematics. Several integral transforms such as the most famous one introduced by P.S. Laplace (1749–1827) in 1782, called the Laplace transform [3,8] is defined by,

$$\mathcal{L}\left[f(t)\right] = F(u) = \int_0^\infty e^{-ut} f(t)dt \tag{1.1}$$

In the early 2011, Tarig M. Elzaki [15] introduced the modified Laplace transform, called the Elzaki transform (see also [10,12]), which is defined for a function of exponential order. Consider a function in the set S defined as

$$S = \{f(t) : \exists\, M, k_1, k_2 > 0, |f(t)| < M\, e^{\frac{|t|}{k_j}}, if\, t \in (-1)^j \times [0, \infty), j = 1, 2\}$$

For a given function $f(t)$ in the set S, the constant M must be finite, the numbers k_1, k_2may be finite or infinite. The modified Laplace transform, i.e.,the Elzaki transform denoted by the operator $\Im$ is defined by

$$\Im\left[f(t)\right] = T(\rho) \;= \rho \int_0^\infty e^{-\frac{t}{\rho}} f(t)dt \tag{1.2}$$

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The variable ρ in this transform is used to factorize the variable t.
The triple Elzaki transform of a function $f(x,y,t)$ of three variables x,y and t, that can be expressed as a convergent infinite series, and for $(x,y,t)\in\mathbb{R}_3^+$ defined in the first octant of the xyt- plane is defined by the triple modified Laplace transform in the form [4]:

$$\Im_{xyt}f(x,y,t)=T(\sigma,\rho,\delta)=\sigma\rho\delta\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})}f(x,y,t)dxdydt \tag{1.3}$$

We mention here that the Elzaki transform defined by (1.2) follows as the special case of the very recently introduced and the most powerful and versatile generalization of the Laplace transform, called the Upadhyaya transform (see, Upadhyaya [14, (2.2), (2.3), p.473]). We point out below the connection between the Upadhyaya transform and the Elzaki transform in terms of the notation of Upadhyaya [14, subsection 4.5, pp.476–477] as

$$\mathcal{U}\left\{f(t),v,\frac{1}{v}\right\}=\mathfrak{u}\left(v,\frac{1}{v},1\right)=\Im\left[f(t),v\right]=T(\rho) \tag{1.4}$$

It is also to be noted here that the triple Elzaki transform (1.3) introduced early this year by Elzaki and Mousa [4], is also a particular case of the Triple Upadhyaya Transform (TUT) (see, Upadhyaya [14, subsection 6.14, p.501]) and the relation between the two is given by:

$$\begin{aligned}\mathcal{U}_3\left\{f(x,y,t);\sigma,\frac{1}{\sigma},1,\rho,\frac{1}{\rho},1,\delta,\frac{1}{\delta},1\right\}&=\mathfrak{u}_3\left(\sigma,\frac{1}{\sigma},1,\rho,\frac{1}{\rho},1,\delta,\frac{1}{\delta},1\right)\\&=\Im\left[f(x,y,t);\sigma,\rho,\delta\right]=T(\sigma,\rho,\delta)\end{aligned} \tag{1.5}$$

As the above work of Upadhyaya [14] opens up many new future directions of work and applications of the Upadhyaya transform, we propose to take up the further study and applications of the Upadhyaya transform in our future works. For our present considerations the structure of this paper is organized as follows: first, we begin with some basic definitions of Fractional Calculus in section 2, then define the fractional triple Elzaki transform in the Definition 3.1 in section 3 and then prove the linearity property, the convolution theorem, the first and the second shitting properties, the periodic function property and the operational formula (differential property) of this new transform in the same section. In the section 4 we obtain the exact solution of a fractional partial differential equation in three dimensions satisfying some initial value and boundary conditions as an application of the results developed in section 3 and finally the conclusions are stated in section 5.

2 Fundamental concepts of fractional calculus

Definition 2.1. [9, 10] Let $g(x)$ be a continuous function and not necessarily differentiable function, where $\lambda>0$ denote a constant discretization span, the fractional difference of $g(x)$ is known as

$$\Delta^\alpha g(x)=\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\k\end{array}\right)g[x+k\lambda]\quad\text{for}\quad 0<\alpha<1 \tag{2.1}$$

where $\left(\begin{array}{c}\alpha\\k\end{array}\right)=\frac{\alpha!}{k!(\alpha-k)!}$ and the α-derivative of $g(x)$ is known as

$$g^{(\alpha)}(x)=\lim_{\lambda\to 0}\frac{\Delta_\lambda^\alpha g(x)}{\lambda^\alpha}$$

See the details in [9, 10].

Definition 2.2. [13] Let $g(x)$ be a continuous function, but not necessarily differentiable, then
(i). Let us assume that $g(x)=\lambda$ where λ is a constant, thus α- derivative of the function $g(x)$ is

$$D_x^\alpha\lambda=\begin{cases}\lambda\frac{x^\alpha}{\Gamma(1+\alpha)}, & \alpha>0,\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, when $g(x) \neq \lambda$ then

$$g(x) = g(0) + (g(x) - g(0)),$$

and the fractional derivative of the function $g(x)$ is given as

$$D^{\alpha} g(x) = D_x^{\alpha} g(0) + D_x^{\alpha} (g(x) - g(0)),$$

(ii). For any $(\alpha > 0)$ one has

$$D^{-\alpha} g(x) = J^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} g(\tau) d\tau, \ \alpha > 0. \tag{2.2}$$

Definition 2.3. [11] The Caputo fractional derivative of the left sided $g \in C_{-1}^{n}, n \in \mathbb{N} \bigcup \{0\}$ is defined by

$$D^{\alpha} g(\tau) = \frac{\partial^{\alpha} g(\tau)}{\partial \tau^{\alpha}} = J^{m-\alpha} \left[\frac{\partial^{m} g(\tau)}{\partial \tau^{m}} \right], \ m-1 < \alpha \leq m, \ m \in N. \tag{2.3}$$

We record properties of the operator J^{α} (see [11])

(i). $J^{\alpha} J^{\beta} g(\tau) = J^{\alpha+\beta} g(\tau)$, $\alpha, \beta \geq 0$

(ii). $J^{\alpha} \tau^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} \tau^{\alpha+\mu}$, $\alpha > 0, \ \mu > -1, \tau > 0$

(iii). $J^{\alpha}(D_{*}^{\alpha} g(\tau)) = g(\tau) - \sum_{k=0}^{n-1} g^{k}(0^{+}) \frac{\tau^{k}}{k!}$,

Definition 2.4. [5] Let $g(x)$ be a continuous function, so the solution of the fractional differential equation

$$dy = g(x)(dx)^{\alpha}, \ x \geq 0, \ y(0) = 0, \ \alpha > 0,$$

by integration with respect to $(dx)^{\alpha}$ is the following

$$y(x) = \int_0^x g(\tau)(d\tau)^{\alpha}, \ y(0) = 0,$$

i.e.,

$$y(x) = \alpha \int_0^x (x-\tau)^{\alpha-1} g(\tau) d\tau, \ 0 < \alpha < 1 \tag{2.4}$$

For example, if $g(x) = x^{\beta}$ one obtains:

$$\int_0^x \tau^{\beta} (d\tau)^{\alpha} = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \ 0 < \alpha < 1.$$

Definition 2.5. [11] If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, then the fractional double Elzaki transform of the fractional derivative is,

$$\Im_{xt} \left[D_{*}^{\alpha} g(x,t) \right] = \frac{T_{\alpha}^{2}(x,\rho)}{\rho^{\alpha}} - \sum_{k=0}^{m-1} \rho^{2-\alpha+k} g^{(k)}(x,0), \ m-1 < \alpha \leq m, \tag{2.5}$$

3 Theorems and properties of the fractional triple Elzaki transform

In this section we define the fractional triple Elzaki transform of the functions dependent on three variables and give some properties for the same as pointed out earlier in the abstract of the paper and also in the section 1 above.

Definition 3.1. The fractional triple Elzaki transform of the function $f(x,y,t)$ of three variables x, y, t is defined as follows:

$$\Im_{xyt} f(x,y,t) = T_{\alpha}^{3}(\sigma,\rho,\delta) = \sigma^{\alpha}\rho^{\alpha}\delta^{\alpha} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^{\alpha}] f(x,y,t)(dx)^{\alpha}(dy)^{\alpha}(dt)^{\alpha}$$

$$= (\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}) \lim_{\substack{N\to\infty \\ M\to\infty \\ K\to\infty}} \int_0^{K}\int_0^{M}\int_0^{N} E_{\alpha}[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^{\alpha}]\, f(x,y,t)(dx)^{\alpha}(dy)^{\alpha}(dt)^{\alpha} \tag{3.1}$$

where $\sigma, \rho, \delta \in C$, $x, y, t > 0$, and $E_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{x^{m}}{\Gamma(\alpha m+1)}$ is the Mittag-Leffler function.

Definition 3.2. [13] Let $f(x,y,t)$ denote a function which vanishes for negative values of x,y,t. Its triple Laplace's transform of order α (or its α^{th} fractional Laplace transform) is defined by the following expression:

$$L_{xyt}f(x,y,t) = F_\alpha^3(\sigma,\rho,\delta) = \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\sigma x+\rho y+\delta t)^\alpha]\, f(x,y,t)(dx)^\alpha(dy)^\alpha(dt)^\alpha \tag{3.2}$$

$$= \lim_{\substack{N\to\infty\\ M\to\infty\\ K\to\infty}} \int_0^K \int_0^M \int_0^N E_\alpha[-(\sigma x+\rho y+\delta t)^\alpha]\, f(x,y,t)(dx)^\alpha(dy)^\alpha(dt)^\alpha$$

provided that integral exists.

Theorem 3.3. The Linearity of the triple fractional Elzaki transform: *Let $f(x,y,t)$ and $g(x,y,t)$ be functions whose triple fractional Elzaki transforms exist, then*

$$\Im_{xyt}\left[\theta f(x,y,t)+\beta g(x,y,t)\right] = \theta\, \Im_{xyt}\left[f(x,y,t)\right] + \beta\, \Im_{xyt}\left[g(x,y,t)\right]$$

where θ and β are constants.

Proof.

$$\begin{aligned}
&\Im_{xyt}\left[\theta f(x,y,t)+\beta g(x,y,t)\right]\\
&= \sigma^\alpha\rho^\alpha\delta^\alpha \int_0^\infty\int_0^\infty\int_0^\infty \left[\theta f(x,y,t)+\beta g(x,y,t)\right] E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha](dx)^\alpha(dy)^\alpha(dt)^\alpha\\
&= \sigma^\alpha\rho^\alpha\delta^\alpha \int_0^\infty\int_0^\infty\int_0^\infty \left[\theta f(x,y,t)\right] E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha](dx)^\alpha(dy)^\alpha(dt)^\alpha +\\
&\qquad \sigma^\alpha\rho^\alpha\delta^\alpha \int_0^\infty\int_0^\infty\int_0^\infty \left[\beta g(x,y,t)\right] E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha](dx)^\alpha(dy)^\alpha(dt)^\alpha\\
&= \sigma^\alpha\rho^\alpha\delta^\alpha\, \theta \int_0^\infty\int_0^\infty\int_0^\infty \left[f(x,y,t)\right] E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha](dx)^\alpha(dy)^\alpha(dt)^\alpha +\\
&\qquad \sigma^\alpha\rho^\alpha\delta^\alpha\, \beta \int_0^\infty\int_0^\infty\int_0^\infty \left[g(x,y,t)\right] E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha](dx)^\alpha(dy)^\alpha(dt)^\alpha\\
&= \theta\, \Im_{xyt}\left[f(x,y,t)\right] + \beta\, \Im_{xyt}\left[g(x,y,t)\right]
\end{aligned}$$

□

Theorem 3.4. The First Shifting Property: *If $\Im_{xyt}\left[f(x,y,t)\right] = T_\alpha^3(\sigma,\rho,\delta)$, then*

$$\Im_{xyt}\left[E_\alpha[-(\frac{\theta x}{\sigma}+\frac{\beta y}{\rho}+\frac{\kappa t}{\delta})^\alpha]f(x,y,t)\right] = T_\alpha^3(1+\theta,1+\beta,1+\kappa)$$

Proof . Let

$$\begin{aligned}
&\Im_{xyt}\left[f(x,y,t)\right] = T_\alpha^3(\sigma,\rho,\delta)\\
&= \sigma^\alpha\rho^\alpha\delta^\alpha \int_0^\infty\int_0^\infty\int_0^\infty E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha]f(x,y,t)(dx)^\alpha(dy)^\alpha(dt)^\alpha
\end{aligned}$$

Then

$$\begin{aligned}
&\Im_{xyt}\left[E_\alpha[-(\tfrac{\theta x}{\sigma}+\tfrac{\beta y}{\rho}+\tfrac{\kappa t}{\delta})^\alpha]f(x,y,t)\right]\\
&= \sigma^\alpha\rho^\alpha\delta^\alpha \int_0^\infty\int_0^\infty\int_0^\infty E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha]\left[E_\alpha[-(\tfrac{\theta x}{\sigma}+\tfrac{\beta y}{\rho}+\tfrac{\kappa t}{\delta})^\alpha]\right] f(x,y,t)(dx)^\alpha(dy)^\alpha(dt)^\alpha
\end{aligned}$$

by using the equality $E_\alpha[\mu(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] = E_\alpha\mu(\frac{x}{\sigma})^\alpha E_\alpha\mu(\frac{y}{\rho})^\alpha E_\alpha\mu(\frac{t}{\delta})^\alpha$ which implies that,

$$\begin{aligned}
&\Im_{xyt}\left[E_\alpha[-(\tfrac{\theta x}{\sigma}+\tfrac{\beta y}{\rho}+\tfrac{\kappa t}{\delta})^\alpha]f(x,y,t)\right]\\
&= \sigma^\alpha\rho^\alpha\delta^\alpha \int_0^\infty\int_0^\infty\int_0^\infty E_\alpha[-(\tfrac{(1+\theta)x}{\sigma}+\tfrac{(1+\beta)y}{\rho}+\tfrac{(1+\kappa)t}{\delta})^\alpha]\, f(x,y,t)(dx)^\alpha(dy)^\alpha(dt)^\alpha\\
&= \sigma^\alpha \int_0^\infty E_\alpha[-(\tfrac{(1+\theta)x}{\sigma})^\alpha]\left\{\rho^\alpha\delta^\alpha \int_0^\infty E_\alpha[-(\tfrac{(1+\beta)y}{\rho}+\tfrac{(1+\kappa)t}{\delta})^\alpha]\, f(x,y,t)(dy)^\alpha(dt)^\alpha\right\}(dx)^\alpha\\
&= \sigma^\alpha \int_0^\infty E_\alpha[-(\tfrac{(1+\theta)x}{\sigma})^\alpha]f(x,1+\beta,1+\kappa)\,dx\\
&= T_\alpha^3(1+\theta,1+\beta,1+\kappa).
\end{aligned}$$

□

Theorem 3.5. The Periodic Property: *If $f(x,y,t)$ is a periodic function of periods θ, β and κ respectively, in the variables x, y and t , i.e., $f(x+\theta, y+\beta, t+\kappa) = f(x,y,t)$ and if $\Im_{xyt}[f(x,y,t)]$ exits then*

$$\Im_{xyt}[f(x,y,t)] = T_\alpha^3(\sigma,\rho,\delta)$$

$$= \frac{\sigma^\alpha \rho^\alpha \delta^\alpha}{1-\left[e_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha]\right]} \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha.$$

Proof . Let

$$\begin{aligned}
&\Im_{xyt}[f(x,y,t)] = T_\alpha^3(\sigma,\rho,\delta)\\
&= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha\\
&= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha +\\
&\sigma^\alpha \rho^\alpha \delta^\alpha \int_\alpha^\infty \int_\beta^\infty \int_\kappa^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha
\end{aligned}$$

Putting $x = u+\theta, y = v+\beta, t = w+\kappa$ in the second triple integral we get

$$\begin{aligned}
&\Im_{xyt}[f(x,y,t)] = T_\alpha^3(\sigma,\rho,\delta)\\
&= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha +\\
&\sigma^\alpha \rho^\alpha \delta^\alpha \int_\alpha^\infty \int_\beta^\infty \int_\kappa^\infty E_\alpha[-(\frac{(u+\theta)}{\sigma}+\frac{(v+\beta)}{\rho}+\frac{(w+\kappa)}{\delta})^\alpha] f(u+\alpha, v+\beta, w+\kappa)(du)^\alpha (dv)^\alpha (dw)^\alpha
\end{aligned}$$

Or,

$$\begin{aligned}
&T_\alpha^3(\sigma,\rho,\delta) = \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha +\\
&\sigma^\alpha \rho^\alpha \delta^\alpha (E_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha]) \int_\alpha^\infty \int_\beta^\infty \int_\kappa^\infty E_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha] f(u+\theta, v+\beta, w+\kappa)(du)^\alpha (dv)^\alpha (dw)^\alpha\\
&= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha +\\
&\sigma^\alpha \rho^\alpha \delta^\alpha \left[E_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha]\right] \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{u}{\sigma}+\frac{v}{\rho}+\frac{w}{\delta})^\alpha] f(u,v,w)(du)^\alpha (dv)^\alpha (dw)^\alpha\\
&= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha + \left[E_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha]\right] T_\alpha^3(\sigma,\rho,\delta)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\theta \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha\\
&= T_\alpha^3(\sigma,\rho,\delta) - \left[e_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha]\right] T_\alpha^3(\sigma,\rho,\delta).
\end{aligned}$$

Hence,

$$T_\alpha^3(\sigma,\rho,\delta) = \frac{\sigma^\alpha \rho^\alpha \delta^\alpha}{1-\left[E_\alpha[-(\frac{\theta}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha]\right]} \int_0^\alpha \int_0^\beta \int_0^\kappa E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha$$

□

Theorem 3.6. The Second Shifting Property: *If $\Im_{xyt}[f(x,y,t)] = T_\alpha^3(\sigma,\rho,\delta)$ then,*

$$\Im_{xyt}[f(x-\theta, y-\beta, t-\kappa)H(x-\theta, y-\beta, t-\kappa)] = E_\alpha[-(\frac{\alpha}{\sigma}+\frac{\beta}{\rho}+\frac{\kappa}{\delta})^\alpha] T_\alpha^3(\sigma,\rho,\delta)$$

where $H(x,y,t)$ is the Heaviside unit step function defined by

$$H(x-\theta, y-\beta, t-\kappa) = \begin{cases} 1, & when,\ x > \theta\,, y > \beta, t > \kappa \\ 0, & when,\ x < \theta, y < \beta, t < \kappa. \end{cases}$$

Proof. Let $\Im_{xyt}\left[f(x,y,t)\right] = T_\alpha^3(\sigma,\rho,\delta) = \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f(x,y,t)(dx)^\alpha (dy)^\alpha (dt)^\alpha$. Then

$$\Im_{xyt}\left[f(x-\theta,y-\beta,t-\kappa)H(x-\theta,y-\beta,t-\kappa)\right] =$$
$$\sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha] f(x-\theta,y-\beta,t-\kappa)H(x-\theta,y-\beta,t-\kappa)(dx)^\alpha (dy)^\alpha (dt)^\alpha$$
$$= \sigma^\alpha \rho^\alpha \delta^\alpha \int_\theta^\infty \int_\beta^\infty \int_\kappa^\infty E_\alpha[-(\tfrac{x}{\sigma}+\tfrac{y}{\rho}+\tfrac{t}{\delta})^\alpha] f(x-\theta,y-\beta,t-\kappa)(dx)^\alpha (dy)^\alpha (dt)^\alpha$$

which, on putting $x-\theta = u, y-\beta = v, t-\kappa = w$ gives

$$\Im_{xyt}\left[f(x-\theta,y-\beta,t-\kappa)H(x-\theta,y-\beta,t-\kappa)\right]$$
$$= \left[E_\alpha[-(\tfrac{\alpha}{\sigma}+\tfrac{\beta}{\rho}+\tfrac{\kappa}{\delta})^\alpha]\right] \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\tfrac{u}{\sigma}+\tfrac{v}{\rho}+\tfrac{w}{\delta})^\alpha] f(u,v,w)(du)^\alpha (dv)^\alpha (dw)^\alpha$$
$$= \left[E_\alpha[-(\tfrac{\alpha}{\sigma}+\tfrac{\beta}{\rho}+\tfrac{\kappa}{\delta})^\alpha]\right] T_\alpha^3(\sigma,\rho,\delta).$$

□

Theorem 3.7. The Convolution Theorem: *If $\Im_{xyt}\left[F(x,y,t)\right] = f_\alpha^3(\sigma,\rho,\delta), \Im_{xyt}\left[G(x,y,t)\right] = g_\alpha^3(\sigma,\rho,\delta)$, then the convolution of the functions $F(x,y,t)$ and $G(x,y,t)$ is denoted by $F***G$ and is defined by*

$$\Im_{xyt}\left[(F***G)\,(x,y,t)\right] = \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^x \int_0^y \int_0^t F(x-\theta,y-\beta,t-\kappa)\, G(\theta,\beta,\kappa)\,(dx)^\alpha (dy)^\alpha (dt)^\alpha$$

and we have

$$\Im_{xyt}\left[(F***G)\,(x,y,t)\right] = \Im_{xyt}\left[F(x,y,t)\right] \cdot \Im_{xyt}\left[G(x,y,t)\right] = f_\alpha^3(\sigma,\rho,\delta) \cdot g_\alpha^3(\sigma,\rho,\delta).$$

Proof. From the definition of the convolution we have

$$\Im_{xyt}\left[(F***G)\,(x,y,t)\right]$$
$$= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha]\,(F***G)\,(x,y,t)\,(dx)^\alpha (dy)^\alpha (dt)^\alpha$$
$$= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha]\times$$
$$\left[\sigma^\alpha \rho^\alpha \delta^\alpha \int_0^x \int_0^y \int_0^t F(x-\theta,y-\beta,t-\kappa)\, G(\alpha,\beta,\kappa)(d\theta)^\alpha (d\beta)^\alpha (d\kappa)^\alpha\right] (dx)^\alpha (dy)^\alpha (dt)^\alpha$$

which on using the Heaviside unit step function yields

$$\Im_{xyt}\left[(F***G)\,(x,y,t)\right]$$
$$= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty G(\theta,\beta,\kappa)(d\theta)^\alpha (d\beta)^\alpha (d\kappa)^\alpha \times$$
$$\left[\sigma^\alpha \rho^\alpha \delta^\alpha \int_0^x \int_0^y \int_0^t E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] F(x-\theta,y-\beta,t-\kappa)\, H(x-\theta,y-\beta,t-\kappa)\right] (dx)^\alpha (dy)^\alpha (dt)^\alpha.$$

The above expression may be simplified by using the result of the Theorem 3.6

$$\Im_{xyt}\left[(F***G)\,(x,y,t)\right]$$
$$= \sigma^\alpha \rho^\alpha \delta^\alpha \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] f_\alpha^3(\theta,\rho,\delta) G(\theta,\beta,\kappa)(d\theta)^\alpha (d\beta)^\alpha (d\kappa)^\alpha$$
$$= \sigma^\alpha \rho^\alpha \delta^\alpha f_\alpha^3(\sigma,\rho,\delta) \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^\alpha] G(\theta,\beta,\kappa)(d\theta)^\alpha (d\beta)^\alpha (d\kappa)^\alpha$$
$$= f_\alpha^3(\sigma,\rho,\delta) \cdot g_\alpha^3(\sigma,\rho,\delta).$$

□

Theorem 3.8. *[2] Let $\alpha, \beta, \chi > 0$, $n-1 < \alpha \leq n, m-1 < \beta \leq m, h-1 < \chi \leq h$, $n, m, h \in N$ be such that $f \in C^{\lambda}(\mathbb{R}^{+} \times \mathbb{R}^{+} \times \mathbb{R}^{+}), \lambda = \max\{\alpha, \beta, \chi\}, f^{1} \in L_1[(0,a) \times (0,b) \times (0,c)]$ for any $a, b, c > 0, |f(x,y,t)| \leq ke^{\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta}}, x > a > 0, y > b > 0, t > c > 0$ the triple Elzaki transform of $f(x,y,t)$ and $\frac{\partial^{i+j+r} f(x,y,t)}{\partial x^i \partial y^j \partial t^r}, i = 0, 1, \dots, n,\ j = 0, 1, \dots, m,\ r = 0, 1, \dots, h$ exist. Then,*

$$\bullet \Im_{xyt}\left\{{}^{c}_{x}D^{\alpha}_{0^+} f(x,y,t)\right\} = \sigma^{-\alpha}\left[E_{xyt}\{f(x,y,t)\} - \sum_{i=0}^{n-1} \sigma^{2+i} E_{yt}\left\{\frac{\partial^i f(0,y,t)}{\partial x^i}\right\}\right],$$

$$\bullet \Im_{xyt}\left\{{}^{c}_{y}D^{\beta}_{0^+} f(x,y,t)\right\} = \rho^{-\beta}\left[E_{xyt}\{f(x,y,t)\} - \sum_{j=0}^{m-1} \rho^{2+j} E_{xt}\left\{\frac{\partial^j f(x,0,t)}{\partial y^j}\right\}\right],$$

$$\bullet \Im_{xyt}\left\{{}^{c}_{t}D^{\chi}_{0^+} f(x,y,t)\right\} = \delta^{-\chi}\left[E_{xyt}\{f(x,y,t)\} - \sum_{r=0}^{h-1} \rho^{2+r} E_{xy}\left\{\frac{\partial^r f(x,y,0)}{\partial t^r}\right\}\right].$$

Proof. We refer the reader to [2] for the proof of this theorem.

□

Theorem 3.9. The Operational Formula: *Let $f(x,y,t) \in C^{\lambda}(\mathbb{R}^{+} \times \mathbb{R}^{+} \times \mathbb{R}^{+})$, then the operational formula for the triple fractional Elzaki transform is given by*

$$\Im\left[D^{\alpha}_{x} f(x,y,t) : (\sigma, \rho, \delta)\right] = \frac{T^{3}_{\alpha}(\sigma, \rho, \delta)}{\sigma^{\alpha}} - \sigma^{\alpha}\Gamma(\alpha+1) T^{3}_{\alpha}(0, \rho, \delta) \tag{3.3}$$

Proof. Let

$$\Im_{xyt} f(x,y,t) = T^{3}_{\alpha}(\sigma, \rho, \delta) = \sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^{\alpha}] f(x,y,t)(dx)^{\alpha}(dy)^{\alpha}(dt)^{\alpha}.$$

Then

$$\begin{aligned}
&\Im\left[D^{\alpha}_{x} f(x,y,t) : (\sigma, \rho, \delta)\right] \\
&= \sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma}+\frac{y}{\rho}+\frac{t}{\delta})^{\alpha}] f^{(\alpha)}(x,y,t)(dx)^{\alpha}(dy)^{\alpha}(dt)^{\alpha} \\
&= \rho^{\alpha}\delta^{\alpha}\int_0^{\infty} E_{\alpha}[-(\frac{t}{\delta})^{\alpha}\left[\int_0^{\infty} E_{\alpha}[-(\frac{y}{\rho})^{\alpha}\left[\sigma^{\alpha}\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma})^{\alpha} f^{\alpha}(x,y,t)(dx)^{\alpha}\right]\right](dy)^{\alpha}(dt)^{\alpha}.
\end{aligned}$$

Applying the integration by parts to the expressions inside the square brackets on the right hand side of the above equation we have

$$\begin{aligned}
&\Im\left[D^{\alpha}_{x} f(x,y,t) : (\sigma, \rho, \delta)\right] \\
&= \rho^{\alpha}\delta^{\alpha}\int_0^{\infty} E_{\alpha}[-(\frac{t}{\delta})^{\alpha}]\left[\int_0^{\infty} E_{\alpha}[-(\frac{y}{\rho})^{\alpha}]\left\{\sigma^{\alpha}\left[\Gamma(1+\alpha) f(x,y,t) E_{\alpha}[-(\frac{x}{\sigma})^{\alpha}]\right]_0^{\infty} + \right.\right. \\
&\left.\left.\frac{1}{\sigma^{\alpha}}\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma})^{\alpha}] f(x,y,t)(dx)^{\alpha}\right\}\right](dy)^{\alpha}(dt)^{\alpha} \\
&= \rho^{\alpha}\delta^{\alpha}\int_0^{\infty} E_{\alpha}[-(\frac{t}{\delta})^{\alpha}]\left[\int_0^{\infty} E_{\alpha}[-(\frac{y}{\rho})^{\alpha}]\left\{-\sigma^{\alpha}\left[\Gamma(1+\alpha) f(0,y,t) + \right.\right.\right. \\
&\left.\left.\left.\frac{1}{\sigma^{\alpha}}\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma})^{\alpha}] f(x,y,t)(dx)^{\alpha}\right]\right\}\right](dy)^{\alpha}(dt)^{\alpha} \\
&= \rho^{\alpha}\delta^{\alpha}\int_0^{\infty} E_{\alpha}[-(\frac{t}{\delta})^{\alpha}]\left[\int_0^{\infty} E_{\alpha}[-(\frac{y}{\rho})^{\alpha}]\left[\int_0^{\infty} E_{\alpha}[-(\frac{x}{\sigma})^{\alpha} f(x,y,t)(dx)^{\alpha} - \right.\right. \\
&\left.\left.\sigma^{\alpha}\Gamma(1+\alpha) f(0,y,t)\right]\right](dy)^{\alpha}(dt)^{\alpha} \\
&= \frac{T^{3}_{\alpha}(\sigma, \rho, \delta)}{\sigma^{\alpha}} - \sigma^{\alpha}\Gamma(1+\alpha)\ T^{3}_{\alpha}(0, \rho, \delta)
\end{aligned}$$

□

4 Applications

In this section on the assumption that the inverse fractional triple Elzaki transform exists, we use the inverse fractional triple Elzaki transform to obtain the exact solutions of the partial differential equations of fractional order in three dimensions with initial and boundary conditions.

Example 4.1. Consider the following partial differential equation of fractional order

$$D_t^{\alpha} f(x,y,t) = \frac{\partial^2 f(x,y,t)}{\partial x^2}, \ 0<\alpha\leq 1 \tag{4.1}$$

with the following initial and boundary conditions

$$\begin{aligned} &f(0,y,t)=0 \ \ f_x(0,y,t) = \cos y\, E_{\alpha}(-t^{\alpha}) \\ &f(x,y,0)=\cos x\, \cos y \end{aligned}$$

Solution. Taking the fractional triple Elzaki transform of (4.1) and the fractional double Elzaki transform of the initial and the boundary conditions gives

$$\Im_{xyt}\left[D_t^{\alpha} f(x,y,t)\right] = \Im_{xyt}\left[\frac{\partial^2 f(x,y,t)}{\partial x^2}\right]$$

$$\begin{aligned} &\frac{1}{\delta^{\alpha}}\Im_{xyt}\left[f(x,y,t)\right] - \delta^{\alpha}\Gamma(\alpha+1)\Im_{xy}\left[f(x,y,0)\right] \\ &= \frac{1}{\sigma^2}\Im_{xyt}\left[f(x,y,t)\right] - \Im_{xyt}\left[f(0,y,t)\right] - \sigma\Im_{xyt}\left[\frac{\partial f(0,y,t)}{\partial x}\right] \end{aligned} \tag{4.2}$$

$$T_{\alpha}^3(\sigma,\rho,0) = \frac{\sigma^3}{(\sigma^2+1)}\frac{\rho^2}{(\rho^2+1)}\,,\ T_{\alpha}^3(0,\rho,\delta)=0\,,\ \frac{\partial T_{\alpha}^3(0,\rho,\delta)}{\partial x} = \frac{\rho^2}{(\rho^2+1)}\frac{\delta^{\alpha}\Gamma(\alpha+1)}{(\delta^2+1)} \tag{4.3}$$

Then (4.2) becomes

$$\begin{aligned} \Im_{xyt}\left[f(x,y,t)\right]\left(\frac{1}{\delta^{\alpha}}-\frac{1}{\sigma^2}\right) &= \delta^{\alpha}\Gamma(\alpha+1)\frac{\sigma^3}{(\sigma^2+1)}\frac{\rho^2}{(\rho^2+1)} - \sigma\frac{\rho^2}{(\rho^2+1)}\frac{\delta^{2\alpha}\Gamma(\alpha+1)}{(\delta^{\alpha}+1)} \\ \Im_{xyt}\left[f(x,y,t)\right]\left(\frac{1}{\delta^{\alpha}}-\frac{1}{\sigma^2}\right) &= \frac{\delta^{\alpha}\sigma^3\rho^2\Gamma(\alpha+1) - \delta^{2\alpha}\sigma\rho^2\Gamma(\alpha+1)}{(\sigma^2+1)(\rho^2+1)(\delta^{\alpha}+1)} \\ \Im_{xyt}\left[f(x,y,t)\right] &= \frac{\sigma^3\rho^2\delta^{2\alpha}\Gamma(\alpha+1)}{(\sigma^2+1)(\rho^2+1)(\delta^{\alpha}+1)} \end{aligned}$$

Applying inverse fractional triple Elzaki transform, we get

$$f(x,y,t) = \sin x\, \cos y\, E_{\alpha}\left[-t^{\alpha}\right]$$

which is the required exact solution of (4.1).

5 Conclusion

This work introduces the definition of the fractional triple Elzaki transform and the various properties like the linearity property, the first and the second shifting properties, the periodic property, the convolution theorem and the operational formula are deduced and the results obtained are applied to find the exact solution of a fractional partial differential equation in three dimensions satisfying some initial and boundary value conditions.

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