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On Curvature Inheritance of the Fourth Cartan Tensor in Generalized Fifth-Order Recurrent Finsler Spaces

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ABSTRACT

This paper builds upon a new curvature inheritance called $K-$curvature inheritance in generalized $\mathcal{BK}$-fifth recurrent Finsler space. we establish the commutative relationship between the Lie -derivative and the Berwald covariant derivative of the fifth order for K^i_{jk} . We obtain an equality relation between the Lie-derivative of the Cartan's fourth curvature tensor K^i_{jk} and H^i_{jk} in such space that admits $K-$curvature inheritance and $H-$curvature inheritance. Furthermore, we prove the force acting on the flow of covariant vector field of fifth order a_{sqlnm} is opposite to the direction of that vector field's flow and we establish the scalar value of the smooth vector field $v^j(x)$ under certain conditions.

Subject Classification: 53B40, 53C60.

Keyword: $G\mathcal{B}K-5RF_n$, Lie-derivative L_v , curvature inheritance.

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1. INTRODUCTION AND PRELIMINARIES

In differential geometry, the Lie derivative provides a measure of how a tensor field evolves as it is transported along a smooth vector field. It is named after Sophus Lie, who introduced this concept in the 19th century. Singh [19] studied the Lie-derivative in ageneralized structure manifold. Pandy [15] established various identities on Lie recurrent in Finsler space. Some new remarks on Lie – derivative have been introduced by [8, 20].

The generalized $\mathcal{BK}$-fifth recurrent Finsler space was introduced by Al-Qashbari and Baleedi [3] this space whose K^i_{jkh} satisfies a fifth recurrence relation. Opondo and Singh [14] introduced $W-$ curvature inheritance in bi-recurrent Finsler space. While $H-$ curvature inheritance in bi-recurrent Finsler space introduced by Singh [18].

The $D-$recurrent Finsler metrics introduced by Atashafrouz and Najafi [6]. The generalized recurrent in Finsler space of higher orders have been discussed by [1, 2, 5, 9, 10]. Bidabad and Sepasi [7] studied the complete Finsler spaces of constant negative Ricci curvature. Various identities for curvature tensors in Finsler space was established by authors [11-13].

Let us explore generalized $\mathcal{B}K$-fifth recurrent Finsler space ($G\mathcal{B}K-5RF_n$) for which K^i_{jkh} satisfying the following relation [3, 4]

(1.1) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm}\, K^i_{jkh} + b_{sqlnm}\big(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\big) - c_{sqlnm}\big(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\big) - d_{sqlnm}\big(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\big) - e_{sqlnm}\big(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\big) - 2b_{qlnm} y^r \mathcal{B}_r\big(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\big),$$

where $K^i_{jk} \neq 0$ and a_{sqlnm} , b_{sqlnm}, c_{sqlnm}, d_{sqlnm} and e_{sqlnm} are non-zero covariant vector fields of fifth order.

(1.2) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm}\, H^i_{kh}$.

The metric tensor g_{ij}, non-zero vector y^i and Kronecker delta δ^k_j are satisfying the following relation [16]

(1.3) a) $\dot{\partial}_i y^i = 1$ b) $\delta^i_h = \dot{\partial}_h y^i$ and c) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j = k\ , \\ 0 & if\ \ j \neq k\ . \end{cases}$

The tensors H^i_{jkh}, K^i_{jkh}, H_{jk} and K_{jk} are satisfying the relations [16]

(1.4) a) $K^i_{jki} = K_{jk}$ b) $H_{jk} = H^i_{jki}$ and c) $H^i_{jkh} = K^i_{jkh} + \dot{\partial}_j K^i_{mkh} y^m$.

The Lie-derivative of a general mixed tensor field T^i_{jkh} is given by following formula [17]

(1.5) $$L_v\, T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh}\mathcal{B}_m\, v^i + T^i_{mkh}\mathcal{B}_j\, v^m + T^i_{jmh}\mathcal{B}_k\, v^m + T^i_{jkm}\mathcal{B}_h\, v^m + \dot{\partial}_m T^i_{mkh}\, \mathcal{B}_r v^m y^r,$$

where $v^i(x) \neq 0$.

The Berwald covariant derivative of the vector field v^i vanish identically, i.e. [17]

(1.6) $\mathcal{B}_m\, v^i = 0.$

The tensor T whose components are related to the components of a vector field $v^i(x)$ during the application of a force, and which inherits it to a new tensor $\alpha(x)T$ is called the inheritance tensor.

The $H-$curvature inheritance and $K-$curvature inheritance are satisfying the relations [17]

(1.7) a) $L_v H^i_{jkh} = \alpha(x) H^i_{jkh}$ b) $L_v H^i_{kh} = \alpha(x) H^i_{kh}$ and c) $L_v K^i_{jkh} = \alpha(x) K^i_{jkh}$,

where $\alpha(x)$is a non-zero scalar function.

The scalar function $\alpha(x)$of the torsion tensor T^i_{kh} is given in [3]

(1.8) $\alpha(x) T^i_{kh} = \alpha(x) T^i_{ki} = \alpha(x) T_k = \sum_{k=1}^{n} T_k y^k$.

This paper is concerned with the investigation of $K-$curvature inheritance in generalized $\mathcal{B}K-$ fifth recurrent Finsler space (). The main objective is to establish conditions under which Cartan's fourth curvature tensor admits inheritance with respect to the Lie derivative.

2. MAIN RESULTS

Theorem 2.1. *In $G\mathcal{B}K-5RF_n$, the Lie- derivative of K-curvature inheritance and the Berwald covariant derivative of the fifth order are commutative if and only if the non-zero covariant vector field of fifth order a_{sqlnm}anishes simultaneously with the vanishing of the scalar function $\alpha(x)$ by Berwald covariant derivative of the fifth order.*

Proof. (I): $L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) = 0$

$\rightarrow \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big).$

Taking the Berwald covariant derivative of [(1.7)c] with x^m, x^n, x^l, x^q and x^s respectively, we get

(2.1) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = \alpha(x)\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}.$

Since $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0$, then above equation can be written as

(2.2) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = K^i_{jkh}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) + \alpha(x)\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\,.$

Using (1.1) in above equation, we get

(2.3) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = K^i_{jkh}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x)$
$+\alpha(x)\big[a_{sqlnm}K^i_{jkh} + b_{sqlnm}\big(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\big) - c_{sqlnm}\big(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\big) - d_{sqlnm}\big(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\big)$
$- e_{sqlnm}\big(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\big) - 2b_{qlnm}y^r\mathcal{B}_r\big(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\big)\big].$

Taking the Lie - derivative of (1.1) and since the components of the Kronecker delta δ^i_h are constant functions (one or zero), so $L_v\delta^i_h = 0$, thus

(2.4) $L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big) = \big(L_v a_{sqlnm}\big)K^i_{jkh} + a_{sqlnm}L_v\big(K^i_{jkh}\big) +$
$L_v b_{sqlnm}\big(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\big) + b_{sqlnm}\big(\delta^i_h L_v g_{jk} - \delta^i_k\, L_v g_{jh}\big) - L_v c_{sqlnm}\big(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\big)$
$- c_{sqlnm}\big(\delta^i_h L_v C_{jkn} - \delta^i_k\, L_v C_{jhn}\big) - L_v d_{sqlnm}\big(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\big) - d_{sqlnm}\big(\delta^i_h L_v C_{jkl} - \delta^i_k\, L_v C_{jhl}\big)$
$- L_v e_{sqlnm}\big(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\big) - e_{sqlnm}\big(\delta^i_h L_v C_{jkq} - \delta^i_k\, L_v C_{jhq}\big)$
$- 2\big(L_v b_{qlnm}\big)y^r\mathcal{B}_r\big(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\big) - 2b_{qlnm}L_v\big[y^r\mathcal{B}_r\big(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\big)\big].$

Using [(1.3)c] when $j \neq k$ and [(1.7)c] in above equation, we get

(2.5) $\quad L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big) = \big(L_v a_{sqlnm} + \alpha(x)a_{sqlnm}\big)K^i_{jk}\;\;.$

Using [(1.3)c] in (3.3), we get

(2.6) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = \big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) + \alpha(x)a_{sqlnm}\big)K^i_{jkh}\,.$

Subtracting (2.5) from (2.6), we get

(2.7) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) - L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big)$
$= \big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) - L_v a_{sqlnm}\big)\,K^i_{jkh}\,.$

Which can be written as

(2.8) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big)$

if

(2.9) $\quad L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0.$

And now, we prove the converse

$$(II):\; \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big)$$
$$\rightarrow L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0.$$

Using [(1.3)c] in (1.1), we get

(2.10) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jk} \;= a_{sqlnm}\, K^i_{jkh}\,.$

Assume that the Lie-derivative of K –curvature inheritance and the Berwald covariant derivative of fifth order are commutative, meaning that the equation (2.8) is satisfied without any conditions. Using above equation in (2.8), we get

(2.11) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = L_v\big(a_{sqlnm}\, K^i_{jk}\;\big).$

Let as assume Lie-derivative of K^i_{jkh} behaves as fifth recurrent, i.e.

(2.12) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = a_{sqlnm}\,\big(L_v K^i_{jkh}\big).$

Using above equation in (3.11), we get

(2.13) $\quad a_{sqlnm}\,\big(L_v K^i_{jkh}\big) = L_v\big(a_{sqlnm}\, K^i_{jkh}\big).$

Which can be written as

(2.14) $\quad K^i_{jkh}\, L_v a_{sqlnm} = 0.$

Since $K^i_{jkh} \neq 0$ and $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0$, then above equation can be written as

$$L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0.$$

Hence from (I) and (II), we get

(2.15) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(L_v K^i_{jkh}\big) = L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\big)$
$\leftrightarrow L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0.$

Exactly as required.

Theorem 2.2. *In $\mathcal{GBK}-5RF_n$ that admits K −curvature inheritance, the scalar function of K^i_{jkh} is vanishing.*
Proof. Taking the Lie - derivative of (3.10), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = K^i_{jkh} L_v a_{sqlnm} + a_{sqlnm} L_v K^i_{jkh} .$$

Adding the tensor $[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh})]$ of both sides of above equation, we get

$$2L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = K^i_{jk}\ L_v a_{sqlnm} + a_{sqlnm} L_v K^i_{jkh} + L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}).$$

Using the [Theorem 2.1] in above equation, we get

$$2\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L_v K^i_{jk} = a_{sqlnm} L_v K^i_{jk} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L_v K^i_{jk} .$$

Using [(1.7) c] in above equation, we get

$$2\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) K^i_{jk} = a_{sqlnm}\, \alpha(x) K^i_{jk} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) K^i_{jk} .$$

Since $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) K^i_{jkh} = 0$, so above equation can be written as

$$a_{sqlnm}\, \alpha(x) K^i_{jk} = 0.$$

Since the covariant vector field of fifth order $a_{sqlnm} \neq 0$, so above equation can be written as
(2.16) $\quad \alpha(x) K^i_{jkh} = 0.$
Exactly as required.

Now, we have corollary related to the above theorem. Contracting the indices i and h in (2.16) and using [(1.4)a], we get
(2.17) $\quad \alpha(x) K_{jk} = 0.$
From [Lemma (1)], we get

$$\alpha(x) K^i_{jk} = \alpha(x) H^i_{jkh} .$$

Using (2.16) in above equation, we get
(2.18) $\quad \alpha(x) H^i_{jk} = 0.$
Contracting the indices i and h in above equation and using [(1.4)b], we get
(2.19) $\quad \alpha(x) H_{jk} = 0.$
Thus,
Corollary 2.1. *In $\mathcal{GBK}-5RF_n$ that admits K-curvature inheritance, the scalar function of K −Ricci tensor K_{jk}, the Berwald's curvature tensor H^i_{jkh} and the H-Ricci tensor H_{jk} are vanishing [provided (Theorem 3.1) holds].*

Theorem 2.3. *In $\mathcal{GBK}-5RF_n$ that admits K −curvature inheritance* and *H −curvature inheritance*, the Lie - derivative of K^i_{jkh} and H^i_{jkh} are equal if the Berwald's covariant derivative of fifth order for these curvature tensors are equal.
Proof. Let as assume the Berwald's covariant derivative of fifth order for K^i_{jkh} and H^i_{jkh} are equal, i.e.
(2.20) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jk} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{jkh}.$
Using [(1.4)c] in (2.10), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (H^i_{jk} - \dot{\partial}_j K^i_{mk}\ y^m) = a_{sqlnm} (H^i_{jkh} - \dot{\partial}_j K^i_{mkh} y^m).$$

Using [(1.3)b,c] in above equation, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{jkh} = a_{sqlnm} H^i_{jk} .$$

Using above equation and (2.10) in (2.20), we get

$$a_{sqlnm} K^i_{jk} = a_{sqlnm} H^i_{jk} .$$

Since the covariant vector field of fifth order $a_{sqlnm} \neq 0$, so above equation can be written as

$$K^i_{jk} = H^i_{jk} .$$

Transvecting above equation by scalar function $\alpha(x)$ then using [(1.7)a,c] in the result equation equation, we get
(2.21) $\quad L_v K^i_{jkh} = L_v H^i_{jk} .$
Exactly as required.

Theorem 2.4. *In $GBK-5RF_n$ that admits H −curvature inheritance, the force acting on the flow of covariant vector field of fifth order a_{sqlnm} is opposite to the direction of that vector field's flow and given by (2.22) if the Lie-derivative and the Berwald's covariant derivative of fifth order of the $h(v)$ −torsion tensor H^i_{kh} are commutative.*

Proof. Taking the Lie - derivative of (1.2), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = H^i_{kh} L_v a_{sqlnm} + a_{sqlnm} L_v H^i_{kh}.$$

Let as assume the Lie-derivative and the Berwald's covariant derivative of fifth order of the $h(v)-$ torsion tensor H^i_{kh} are commutative, then above equation can be written as

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L_v H^i_{kh} = H^i_{kh} L_v a_{sqlnm} + a_{sqlnm} L_v H^i_{kh}.$$

Using [(1.7)b] in above equation, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) H^i_{kh} = H^i_{kh} L_v a_{sqlnm} + a_{sqlnm}\, \alpha(x) H^i_{kh}.$$

Since$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) H^i_{kh} = 0$, so above equation can be written as

$$L_v a_{sqlnm} = -\left(\frac{\alpha(x) H^i_{kh}}{H^i_{kh}}\right) a_{sqlnm}.$$

Using (1.8) in above equation, we get

(2.22) $\quad L_v a_{sqlnm} = -\left(\frac{\sum_{k=1}^n H_k y^k}{H^i_{kh}}\right) a_{sqlnm}.$

Exactly as required.

Theorem 2.5. *In $GBK-5RF_n$ that admits K −curvature inheritance and H −curvature inheritance, the Lie-derivative of K^i_{jk} in the direction of vector y^h is vanishing [provided (2.20) holds].*

Proof. Using [(1.3)a] in the right side of (2.21), we get

$$L_v K^i_{jk} = \dot{\partial}_i y^i L_v H^i_{jk}.$$

Transvecting above equation by y^h and using [(1.3)b,c], we get

(2.23) $\quad y^h L_v K^i_{jkh} = 0.$

Exactly as required.

Now, we have corollary related to the above theorem. Using (2.21) in (2.23), we get

$$y^h L_v H^i_{jk} = 0.$$

Thus, we conclude

Corollary 2.2. *In $GBK-5RF_n$, that admits K −curvature inheritance and H −curvature inheritance, the Lie-derivative of Berwald's curvature tensor H^i_{jk} in the direction of vector y^h is vanishing [provided (2.20) holds].*

Theorem 2.6. *In $GBK-5RF_n$ that admits H-curvature inheritance, the scalar value of the smooth vector field $v^j(x)$ is given by (2.24) if the Lie-derivative and the Berwald's covariant derivative of fifth order of H^i_{kh} are commutative [provided (2.25) holds].*

Proof. Using formula (1.5) in (2.22), we get

$$v^j\mathcal{B}_j a_{sqlnm} + a_{jqlnm}\,\mathcal{B}_s v^j + a_{sjlnm}\,\mathcal{B}_q v^j + a_{sqjnm}\,\mathcal{B}_l v^j + a_{sqljm}\,\mathcal{B}_n v^j + a_{sqlnj}\,\mathcal{B}_m v^j$$
$$+\dot{\partial}_j a_{sqlnm}\,\mathcal{B}_r v^j y^r = -\left(\frac{\sum_{k=1}^n H_k y^k}{H^i_{kh}}\right) a_{sqlnm}.$$

Using (1.6) in above equation, we get

$$v^j\mathcal{B}_j a_{sqlnm} = -\left(\frac{\sum_{k=1}^n H_k y^k}{H^i_{kh}}\right) a_{sqlnm}.$$

Which can be written as

$$H^i_{kh}\, v^j\mathcal{B}_j a_{sqlnm} = -\left(\sum_{k=1}^n H_k y^k\right) a_{sqlnm},$$

where *the $h(v)$ −torsion tensor* $H^i_{kh} \neq 0$. Thus,

(2.24) $\quad v^j = \sum_{k=1}^n H_k y^k,$

if

(2.25) $\quad H^i_{kh}\mathcal{B}_j a_{sqlnm} = -a_{sqlnm}.$

Exactly as required.

3. CONCLUSION

This study has established new identities and values for certain type of curvature tensors in $G\mathcal{B}K - 5RF_n$ that admits K −curvature inheritance and H −curvature inheritance. This paper has examined the directional and extensional changes that occur to curvature tensors in fifth recurrent Finsler space when a force acts upon them. We obtained the tensors are vanishing when flowing in a certain direction under specific condition. Our study aligns with the mysteries of the universe, as any extended particles within black holes vanish, a phenomenon of disappearance that scientists continue to investigate to this day.

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