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A New Triple Integral Transform for Solving Volterra Integro-Differential Equations

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ABSTRACT
In this work points, we study the definition of the (TLSET) triple Laplace-Sumudu-Elzaki transform called new triple integral transform some important properties and theorems like convolution theorem property, triple integral property, linearity property, differential property, and the application examples for solution of linear Volterra Integro-differential equations (LVIDEs) in 3-dimensions.

Keyword: (TLSET) Triple Integral Transform, (IC) Initial Conditions, Inverse (TLSET) Triple Integral Transform.

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1. INTRODUCTION
In recent times, integral equations have been widely used to model many fundamental phenomena in astronomy, physics, engineering, chemistry, and other scientific fields. Note that most integral differential equations give analytical solutions in a closed form. [(14)]. It is therefore, important to suggest new combine methods for finding solutions to integral differential equations is important. [(1-3)]
The public three-dimensional (LVIDEs) written in form [(12-14)],

$$\frac{\partial^3 \widehat{u}(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}+\widehat{u}(\chi,\tau,\hbar)=\Phi(\chi,\tau,\hbar)+\int_{\chi_0}^{\chi}\int_{\tau_0}^{\tau}\int_{\hbar_0}^{\hbar}H(\chi,\tau,\hbar,\lambda,s,\ell,\widehat{u}(\lambda,s,\ell))d\lambda ds d\ell$$

(1)

Where, $\widehat{u}(\chi,\tau,\hbar)$ is the undefined function, and H and Φ are the functions analytical in the required domain.

Definition.1 [(5-8)] The (LT) symbolize by the $\mathbf{L}[.]$ and defined by,

$$\mathbf{L}\left[g(\chi):\delta)\right]=\int_{\chi=0}^{\chi=\infty}e^{-\delta\chi}g(\chi)d\chi=G(\delta)\,,\quad \chi>0$$

Definition .2 [(4)] The (ST) symbolize by the $\mathbf{S}[.]$ defined by,

$$\mathbf{S}\left[g(\tau):\varepsilon)\right]=\frac{1}{\varepsilon}\int_{\tau=0}^{\tau=\infty}e^{-\frac{\tau}{\varepsilon}}g(\tau)d\tau=G(\varepsilon)\,,\quad \tau>0$$

Definition .3 [(9-13)] The (ET) symbolize by the $\mathbf{E}[.]$ defined by,

$$\mathbf{E}\left[g(\hbar):\rho)\right]=\rho\int_{\hbar=0}^{\hbar=\infty}e^{-\frac{\hbar}{\rho}}g(\hbar)d\hbar=G(\rho)\,,\quad \hbar>0$$

Definition .4 [(2-6)] The (TLSET) new triple *integral* transform denoted by:

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}\left[g(\chi,\tau,\hbar):(\delta,\varepsilon,\rho)\right]=\frac{\varepsilon}{\rho}\int_{\hbar=0}^{\hbar=\infty}\int_{\tau=0}^{\tau=\infty}\int_{\chi=0}^{\chi=\infty}e^{-\left(\delta\chi+\frac{\tau}{\varepsilon}+\frac{\hbar}{\rho}\right)}g(\chi,\tau,\hbar)d\chi d\tau d\hbar$$

(2)

Where, δ,ε,ρ are variables (TLSET) and $\chi,\tau,\hbar>0$, where,

$$g(\chi,\tau,\hbar)=\frac{i}{8\pi^3}\int_{\lambda-i\infty}^{\lambda+i\infty}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{\varepsilon}\rho e^{\delta\chi+\frac{\tau}{\varepsilon}+\frac{\hbar}{\rho}}G(\delta,\frac{1}{\varepsilon},\frac{1}{\rho})d\rho d\varepsilon d\delta \tag{3}$$

the new inverse (TLSET) symbolize by the $\mathbf{L}_{\chi}^{-1}\mathbf{S}_{\tau}^{-1}\mathbf{E}_{\hbar}^{-1}[.]$.

The following structure of this work : In section 2, we examine unique solution and the existence of Eq. (1); in section 3, we study the fundamental definitions of the linearity property, convolution theorem of $X(\chi,\tau,\hbar),Y(\chi,\tau,\hbar)$, and Integral property, in section 4,we study an applied some examples for solving (LVIDEs).

Finally, we present the conclusion of this work .

2. ON THE EXISTENCE CONDITION FOR THE (LVIDEs) [(13)], [(14)]

In this section, we examine unique solution and the existence of equation (1), on the perfect metric space of complex valued continuous functions as follows:

$$K=\left[C(M,d)\right],d\left(Y,X\right)=\sup\left\{\left|Y(\chi,\tau,\hbar)-X(\chi,\tau,\hbar):(\chi,\tau,\hbar)\in M\right|\right\}$$

Where $M=[0,1]\times[0,1]\times[0,1]$

Theorem 2.1 Let the two functions Φ and Ψ be continued on $[0,1]\times[0,1]\times C$ and $[0,1]\times[0,1]\times[0,1]$ respectively and there exists nonnegative constant $\mathsf{L}\leq 1$ becomes,

$$\left|\Psi(\chi,\tau,\hbar,\lambda,s,\ell,\hat{u}(\lambda,s,\ell))-\Psi(\chi,\tau,\hbar,\lambda,s,\ell,\hat{v}(\lambda,s,\ell))\right|$$
$$\leq\mathsf{L}\left|\hat{u}(\lambda,s,\ell)-\hat{v}(\lambda,s,\ell)\right|$$

Then

$$\hat{u}(\chi,\tau,\hbar)=\Phi(\chi,\tau,\hbar)+\int_{\lambda=0}^{\lambda=\chi}\int_{s=0}^{s=\tau}\int_{\ell=0}^{\ell=\hbar}H(\chi,\tau,\hbar,\lambda,s,\ell,\hat{u}(\lambda,s,\ell))d\lambda ds d\ell\,,$$

hold one and only continued solution $\hat{u}$ on M .

Proof, view [(13)] to proof

Lemma 2.2 if the thesis of theorem 2.1 is holding, then Equation (1) we conclude,

$$\frac{\partial^3\hat{u}(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}+\hat{u}(\chi,\tau,\hbar)=\Phi(\chi,\tau,\hbar)+\int_{\chi_0}^{\chi}\int_{\tau_0}^{\tau}\int_{\hbar_0}^{\hbar}H(\chi,\tau,\hbar,\lambda,s,\ell,\hat{u}(\lambda,s,\ell))d\lambda ds d\ell$$

(4)

With the initial conditions

$\widehat{u}(\chi,0,0)=c_1(\chi), \widehat{u}(0,0,\hbar)=c_3(\hbar), \widehat{u}(\chi,0,\hbar)=c_5(\chi,\hbar),$

$\widehat{u}(0,\tau,0)=c_2(\tau), \widehat{u}(\chi,\tau,0)=c_4(\chi,\tau), \widehat{u}(0,\tau,\hbar)=c_6(\tau,v), \widehat{u}(0,0,0)=c_0$

Thus through theorem 2.1, the equation hold a unique continued solution. View [(15)].

3. PROPERTIES OF NEW (TLSET) TRIPLE INTEGRAL TRANSFORM [(7)],[(9)]

Theorem 3.1 linearity of the New Triple Integral Transform

Let, $X(\chi,\tau,\hbar)$ and $Y(\chi,\tau,\hbar)$ be functions who's the new triple integral transform exists then

$$\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[cX(\chi,\tau,\hbar)+bY(\chi,\tau,\hbar)\right]=c\,\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[X(\chi,\tau,\hbar)\right]+b\,\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[Y(\chi,\tau,\hbar)\right]$$

Where c and b are constants

Theorem 3.2 the convolution product theorem

If, $\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[W(\chi,\tau,\hbar)\right]=w(\delta,\varepsilon,\rho)$, $\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[Z(\chi,\tau,\hbar)\right]=z(\delta,\varepsilon,\rho)$

And, $\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[(W *** Z)(\chi,\tau,\hbar)\right]=\int_0^\chi\int_0^\tau\int_0^\hbar W(\chi-c,\tau-b,\hbar-n)Z(c,b,n)\,d\chi\, d\tau\, d\hbar$

then,

$$\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[(W *** Z)(\chi,\tau,\hbar)\right]=\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[W(\chi,\tau,\hbar)\right].\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[Z(\chi,\tau,\hbar)\right]=w(\delta,\varepsilon,\rho).z(\delta,\varepsilon,\rho)$$

Property 3.3

If $g(\chi,\tau,\hbar)=\dfrac{\partial^3 g(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}$, Then

$$\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[\frac{\partial^3 g(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}:(\delta,\varepsilon,\rho)\right]=\frac{\delta}{\varepsilon\rho}\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar(\chi,\tau,\hbar)+\frac{\delta\rho}{\varepsilon}\mathbf{L}_\chi(\chi,0,0)+\frac{1}{\varepsilon\rho}\mathbf{E}_\hbar(0,0,\hbar)$$
$$+\frac{\rho}{\varepsilon}\mathbf{S}_\tau(0,\tau,0)-\frac{\delta}{\varepsilon\rho}\mathbf{L}_\chi\mathbf{E}_\hbar(\chi,0,\hbar)-\frac{1}{\varepsilon\rho}\mathbf{S}_\tau\mathbf{E}_\hbar(0,\tau,\hbar)-\frac{\delta\rho}{\varepsilon}\mathbf{L}_\chi\mathbf{S}_\tau(\chi,\tau,0)-\frac{\rho}{\varepsilon}g(0,0,0)$$

Property 3.4

If $\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[\Phi(\chi,\tau,\hbar)\right]=\overline{\Phi}(\delta,\sigma,\rho)$ and $\varphi(\chi,\tau,\hbar)=\int_{v=0}^{v=\chi}\int_{w=0}^{w=\tau}\int_{u=0}^{u=\hbar}\Phi(v,w,u)\,dv\,dw\,du$

Then, $\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left\{\int_0^\chi\int_0^\tau\int_0^\hbar\Phi(v,w,u)\,dv\,dw\,du\right\}=\varepsilon\rho\dfrac{\overline{\Phi}(\delta,\varepsilon,\rho)}{\delta}$

4. NEW (TLSET) OF SOME SIMPLE FUNCTIONS [(8)],[(10)]

[i]. let, $\widehat{u}(\chi,\tau,\hbar)=1$ then,

$$\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[1:(\delta,\varepsilon,\rho)\right]=\frac{\rho}{\varepsilon}\int_{\hbar=0}^{\infty}\int_{\tau=0}^{\infty}\int_{\chi=0}^{\infty}e^{-\delta\chi-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}d\chi\, d\tau\, d\hbar$$
$$=\frac{\rho}{\varepsilon}\int_{\hbar=0}^{\infty}\int_{\tau=0}^{\infty}e^{-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}\left(\int_{\chi=0}^{\infty}e^{-\delta\chi}d\chi\right)d\tau\, d\hbar=\frac{\rho}{\varepsilon}\int_{\hbar=0}^{\infty}\int_{\tau=0}^{\infty}e^{-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}\left\{-\frac{1}{\delta}\left(e^{-\delta\chi}\right)\Big|_{\chi=0}^{\infty}\right\}d\tau\, d\hbar=\frac{\rho^2}{\varepsilon}$$

[ii]. Let $\widehat{u}(\chi,\tau,\hbar)=\chi\tau\hbar$, $\chi,\tau,\hbar>0$ then

$$\mathbf{L}_\chi\mathbf{S}_\tau\mathbf{E}_\hbar\left[[\chi\tau\hbar]:(\delta,\varepsilon,\rho)\right]=\frac{\rho}{\varepsilon}\int_{\chi=0}^{\infty}\int_{\tau=0}^{\infty}\int_{\hbar=0}^{\infty}\chi\tau\hbar e^{-\delta\chi-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}d\chi\, d\tau\, d\hbar$$
$$=\frac{\rho}{\varepsilon}\int_{\hbar=0}^{\infty}\int_{\tau=0}^{\infty}\tau\hbar e^{-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}\left\{\int_{\chi=0}^{\infty}\chi e^{-\delta x}d\chi\right\}d\tau\, d\hbar=\frac{\varepsilon\rho^3}{\delta^2}$$

[iii].let $\widehat{u}(\chi,\tau,\hbar)=e^{a\chi+b\tau+c\hbar}$, $\chi,\tau,\hbar>0$, then

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}\left[e^{a\chi+b\tau+c\hbar}:(\delta,\varepsilon,\rho)\right]=\frac{\rho}{\varepsilon}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{a\chi+b\tau+c\hbar}\,e^{-\delta\chi-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}d\chi d\tau d\hbar$$

$$=\frac{\rho}{\varepsilon}\int_0^{\infty}\int_0^{\infty}e^{-\left(\frac{1-c\rho}{\rho}\right)\hbar}e^{-\left(\frac{1-b\varepsilon}{\varepsilon}\right)\tau}\left\{\int_0^{\infty}e^{-(\delta-a)\chi}d\chi\right\}d\hbar d\tau=\frac{\rho^2}{(\delta-a)(1-b\varepsilon)(1-c\rho)}$$

[iv].Let $\hat{u}(\chi,\tau,\hbar)=\sin\chi\sin\tau\sin\hbar$ then

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}\left[\sin\chi\sin\tau\sin\hbar:(\delta,\varepsilon,\rho)\right]=\frac{\rho}{\varepsilon}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\delta\chi-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}\sin\chi\sin\tau\sin\hbar d\chi d\tau d\hbar=\frac{\varepsilon\rho^3}{\left[1+\delta^2\right]\left[1+\varepsilon^2\right]\left[1+\rho^2\right]}$$

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}\left[\cos x\,\cos\tau\cos\hbar:(\delta,\varepsilon,\rho)\right]=\frac{\rho}{\varepsilon}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\delta\chi-\frac{\tau}{\varepsilon}-\frac{\hbar}{\rho}}\cos x\,\cos\tau\cos\hbar dx d\tau d\hbar=\frac{\delta\rho^2}{\left[1+\delta^2\right]\left[1+\varepsilon^2\right]\left[1+\rho^2\right]}$$

5. APPLICATION OF SOLUTION OF (LVIDES) BY NEW TRIPLE INTEGRAL TRANSFORM

In this section, here we examine the employment of the (LVIDE) three important problems.

Problem 5.1 Let us examine the (LVIDE) ,

$$\frac{\partial^3\hat{u}(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}+\hat{u}(\chi,\tau,\hbar)=\chi cos\hbar-\frac{\chi^2\tau\sin\hbar}{2}+\int_0^{\chi}\int_0^{\tau}\int_0^{\hbar}\hat{u}(\lambda,s,\ell)d\lambda ds d\ell,$$

(5)

With

$$\hat{u}(0,0,\hbar)=0,\hat{u}(\chi,0,\hbar)=\chi\cos\hbar,\hat{u}(\chi,\tau,0)=\chi,\hat{u}(0,\tau,\hbar)=0,$$
$$\hat{u}(0,0,0)=0,\hat{u}(x,0,0)=\chi,\hat{u}(0,\tau,0)=0,$$

(6)

Employing the $\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}[.]$new (TLSET) to equation (5), and following dual integral transform of initial score equation (6), we have,

$$\frac{\delta}{\varepsilon\rho}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)+\frac{\delta\rho}{\varepsilon}\mathbf{L}_{\chi}(\chi,0,0)+\frac{1}{\varepsilon\rho}\mathbf{E}_{\hbar}(0,0,\hbar)+\frac{\rho}{\varepsilon}\mathbf{S}_{\tau}(0,\tau,0)-\frac{\delta}{\varepsilon\rho}\mathbf{L}_{\chi}\mathbf{E}_{\hbar}(\chi,0,\hbar)-\frac{\delta\rho}{\varepsilon}\mathbf{L}_{\chi}\mathbf{S}_{\tau}(\chi,\tau,0)-$$
$$\frac{1}{\varepsilon\rho}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(0,\tau,\hbar)+\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)-\frac{\rho}{\varepsilon}g(0,0,0)=\frac{1}{\delta^2}\left[\frac{\rho^2}{1+\rho^2}\right]+\frac{\varepsilon\rho}{\delta}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)-\frac{\varepsilon}{\delta^3}\left[\frac{\rho^3}{1+\rho^2}\right]$$

(7)

$$\mathbf{L}_{\chi}\mathbf{E}_{\hbar}(\chi,0,\hbar)=\frac{1}{\delta^2}\left[\frac{\rho^2}{\rho^2+1}\right],\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(0,\tau,\hbar)=0\ ,\ \mathbf{E}_{\hbar}(0,0,\hbar)=0,\mathbf{S}_{\tau}(0,\tau,0)=0,\mathbf{L}_{\chi}\mathbf{S}_{\tau}(\chi,\tau,0)=\frac{1}{\delta^2},$$

$$g(0,0,0)=0,\mathbf{L}_{\chi}(\chi,0,0)=\frac{1}{\delta^2},\tag{8}$$

Replace equation (8) in equation (7), we have,

$$\left[1+\frac{\delta}{\varepsilon\rho}-\frac{\varepsilon\rho}{\delta}\right]\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)+\frac{\delta\rho}{\varepsilon}\left[\frac{1}{\delta^2}\right]-\frac{\delta\rho}{\varepsilon}\left[\frac{1}{\delta^2}\right]-\frac{\delta}{\varepsilon\rho}\left[\frac{1}{\delta^2}\right]\left[\frac{\rho^2}{\rho^2+1}\right]=\frac{1}{\delta^2}\left[\frac{\rho^2}{1+\rho^2}\right]-\frac{\varepsilon}{\delta^3}\left[\frac{\rho^3}{1+\rho^2}\right]$$

$$\left[1+\frac{\delta}{\varepsilon\rho}-\frac{\varepsilon\rho}{\delta}\right]\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{1}{\delta^2}\left[\frac{\rho^2}{1+\rho^2}\right]-\frac{\varepsilon}{\delta^3}\left[\frac{\rho^3}{1+\rho^2}\right]+\frac{1}{\varepsilon}\left[\frac{1}{\delta}\right]\left[\frac{\rho}{\rho^2+1}\right]$$

$$\left[1+\frac{\delta}{\varepsilon\rho}-\frac{\varepsilon\rho}{\delta}\right]\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{1}{\delta^2}\left[\frac{\rho^2}{1+\rho^2}\right]-\frac{\varepsilon}{\delta^3}\left[\frac{\rho^3}{1+\rho^2}\right]+\frac{1}{\varepsilon}\left[\frac{1}{\delta}\right]\left[\frac{\rho}{\rho^2+1}\right]$$

$$\left[1+\frac{\delta}{\varepsilon\rho}-\frac{\varepsilon\rho}{\delta}\right]\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{1}{\delta^2}\left[\frac{\rho^2}{\rho^2+1}\right]\left(1+\frac{\delta}{\varepsilon\rho}-\frac{\varepsilon\rho}{\delta}\right)$$

Applying the $\mathbf{L}_{\chi}^{-1}\mathbf{S}_{\tau}^{-1}\mathbf{E}_{\hbar}^{-1}[.]$ we find the solution

$$\hat{u}(\chi,\tau,\hbar)=\chi\cos\hbar$$

Problem 5.2 Let us examine the (LVIDE) ,

$$\frac{\partial^3\hat{u}(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}+\hat{u}(\chi,\tau,\hbar)=\chi\tau\hbar+1-\frac{\chi^2\tau^2\hbar^2}{8}+\int_0^{\chi}\int_0^{\tau}\int_0^{\hbar}\hat{u}(\lambda,s,\ell)d\lambda ds d\ell,$$

(9)

With $\hat{u}(\chi,0,\hbar)=\hat{u}(0,\tau,\hbar)=\hat{u}(\chi,\tau,0)=0$ (10)

Employing the $\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}[.]$new (TLSET) to equation (9), and following dual integral transform of initial score equation (10), we have,

$$\frac{\delta}{\varepsilon\rho}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)+\frac{\rho}{\varepsilon}\mathbf{S}_{\tau}(0,\tau,0)+\frac{1}{\varepsilon\rho}\mathbf{E}_{\hbar}(0,0,\hbar)+\frac{\delta\rho}{\varepsilon}\mathbf{L}_{\chi}(\chi,0,0)-\frac{\delta}{\varepsilon\rho}\mathbf{L}_{\chi}\mathbf{E}_{\hbar}(\chi,0,\hbar)-\frac{\delta\rho}{\varepsilon}\mathbf{L}_{\chi}\mathbf{S}_{\tau}(\chi,\tau,0)-$$

$$\frac{1}{\varepsilon\rho}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(0,\hbar,\tau)-\frac{\rho}{\varepsilon}g(0,0,0)+\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\varepsilon\rho^3}{\delta^2}+\frac{\rho^2}{\delta}-\frac{\varepsilon^2\rho^4}{\delta^3}+\frac{\varepsilon\rho}{\delta}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)$$

(11)

$$\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(0,\tau,\hbar)=\mathbf{L}_{\chi}\mathbf{S}_{\tau}(\chi,\tau,0)=\mathbf{L}_{\chi}\mathbf{E}_{\hbar}(\chi,0,\hbar)=0$$

(12)

Replace equation (12) in equation (11), we have,

$$\frac{\delta}{\varepsilon\rho}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)+\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)-\frac{\varepsilon\rho}{\delta}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\varepsilon\rho^3}{\delta^2}+\frac{\rho^2}{\delta}-\frac{\varepsilon^2\rho^4}{\delta^3}$$

$$\left[\frac{\delta}{\varepsilon\rho}+1-\frac{\varepsilon\rho}{\delta}\right]\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\varepsilon\rho^3}{\delta^2}+\frac{\rho^2}{\delta}-\frac{\varepsilon^2\rho^4}{\delta^3}$$

$$\left[\frac{\delta}{\varepsilon\rho}+1-\frac{\varepsilon\rho}{\delta}\right]\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\varepsilon\rho^3}{\delta^2}\left[1+\frac{\delta}{\varepsilon\rho}-\frac{\varepsilon\rho}{\delta}\right]$$

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\varepsilon\rho^3}{\delta^2}$$

Applying the $\mathbf{L}_{\chi}^{-1}\mathbf{S}_{\tau}^{-1}\mathbf{E}_{\hbar}^{-1}[.]$ we find the solution

$$\hat{u}(\chi,\tau,\hbar)=\chi\tau\hbar$$

Problem 5.3 Let us examine the (LVIDE) ,

$$\frac{\partial^3\hat{u}(\chi,\tau,\hbar)}{\partial\chi\partial\tau\partial\hbar}+\hat{u}(\chi,\tau,\hbar)=\frac{\chi^2\tau\hbar+\chi\tau^2\hbar+\chi\tau\hbar^2}{2}+\chi+\tau+\hbar-\int_0^{\chi}\int_0^{\tau}\int_0^{\hbar}\hat{u}(\lambda,s,\ell)d\lambda ds d\ell$$

(13)

With

$$\hat{u}(0,\tau,\hbar)=\hbar+\tau,\hat{u}(\chi,0,\hbar)=\chi+\hbar,\hat{u}(0,0,\hbar)=\hbar,\hat{u}(\chi,\tau,0)=\chi+\tau,$$

$$\hat{u}(0,\tau,0)=\tau,\hat{u}(0,0,0)=0,\hat{u}(\chi,0,0)=\chi,$$

(14)

Employing the $\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}[.]$new (TLSET) to equation (13), and following dual integral transform of initial score equation (14), we have,

$$\frac{\delta}{\varepsilon\rho}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)+\frac{\rho}{\varepsilon}\mathbf{S}_{\tau}(0,\tau,0)+\frac{\delta\rho}{\varepsilon}\mathbf{L}_{\chi}(\chi,0,0)+\frac{1}{\varepsilon\rho}\mathbf{E}_{\hbar}(0,0,\hbar)-\frac{\rho}{\delta\varepsilon}\mathbf{L}_{\chi}\mathbf{E}_{\hbar}(\chi,0,\hbar)-\frac{\delta\rho}{\varepsilon}\mathbf{L}_{\chi}\mathbf{S}_{\tau}(\chi,\tau,0)-$$

$$\frac{1}{\varepsilon\rho}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(0,\tau,\hbar)-\frac{\rho}{\varepsilon}g(0,0,0)+\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\varepsilon^2\rho^3}{\delta}+\frac{\varepsilon\rho^4}{\delta}+\frac{\varepsilon\rho^3}{\delta^3}+\frac{\rho^2}{\delta^2}$$

$$+\frac{\varepsilon\rho^2}{\delta}+\frac{\rho^3}{\delta}-\frac{\varepsilon\rho}{\delta}\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar) \tag{15}$$

$$\mathbf{L}_{\chi}\mathbf{E}_{\hbar}(\chi,0,)=\left[\frac{\rho^2}{\delta^2}+\frac{\rho^3}{\delta}\right],\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(0,\tau,\hbar)=\left[\varepsilon\rho^2+\rho^3\right],\mathbf{L}_{\chi}\mathbf{S}_{\tau}(\chi,\tau,0)=\left[\frac{1}{\delta^2}+\frac{\varepsilon^2}{\delta}\right],$$

$$\mathbf{S}_{\tau}(0,\tau,0)=\sigma\ ,\ \mathbf{E}_{\hbar}(0,0,\hbar)=\rho^3\ ,g(0,0,0)=0,\mathbf{L}_{\chi}(\chi,0,0)=\frac{1}{\delta^2}\ ,$$

(16)

Replacing equation (16) in equation (15), we have,

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)\left[\frac{\delta}{\varepsilon\rho}+\frac{\varepsilon\rho}{\delta}+1\right]=\frac{\rho^2}{\delta^2}\left[\frac{\delta}{\varepsilon\rho}+\frac{\varepsilon\rho}{\delta}+1\right]+$$

$$\frac{\varepsilon\rho^2}{\delta}\left[\frac{\delta}{\varepsilon\rho}+\frac{\varepsilon\rho}{\delta}+1\right]+\frac{\rho^3}{\delta}\left[\frac{\delta}{\varepsilon\rho}+\frac{\varepsilon\rho}{\delta}+1\right]$$

$$\mathbf{L}_{\chi}\mathbf{S}_{\tau}\mathbf{E}_{\hbar}(\chi,\tau,\hbar)=\frac{\rho^2}{\delta^2}+\frac{\rho^3}{\delta}+\frac{\varepsilon\rho^2}{\delta}$$

Applying the $\mathbf{L}_{\chi}^{-1}\mathbf{S}_{\tau}^{-1}\mathbf{E}_{\hbar}^{-1}[.]$ we find the solution

$$\widehat{u}(\chi,\tau,\hbar)=\chi+\tau+\hbar$$

6. CONCLUSIONS

In this work points, we have discussed an combination of the transforms method called (TLSET) New triple integral transform method has been successfully applied to obtain solutions of (LVIDEs) regarding the (IC) initial conditions. For addressing problems in the traditional triple integral transform technique, which only deals with certain types of (LVIDEs) using the (TLSET) approach, it is no longer impossible to solve nonlinear (VIDEs).

REFERENCES

1. Wazwaz AM. Partial Differential Equations and Solitary Waves Theory. Beijing: Higher Education Press; Berlin, Heidelberg: Springer-Verlag; 2009.
2. Moghadam MM, Saeedi H. Application of differential transforms for solving the Volterra integro-partial differential equations. Iran J Sci Technol Trans A. 2010;34(A1):59.
3. Weatugala GK. Sumudu transforms: a new integral transform to solve differential equations and control engineering problems. Math Eng. 1993;61:319-29.
4. Elzaki TM, Mousa A. On the convergence of triple Elzaki transform. SN Appl Sci. 2019;1:275. doi:10.1007/s42452-019-0257-2.
5. Eltayeb H, Kilicman A. A note on double Laplace transform and telegraphic equation. Hindawi Publishing Corporation. 2013.
6. Ahmed S, Elzaki T. The solution of nonlinear Volterra integro-differential equations of second kind by combined Sumudu transforms and Adomian decomposition method. Int J Adv Innov Res. 2013;2(12):90-93.
7. Kiran G, Bhadane VH, Pradhan H, Desale SV. Elzaki transform solution of one-dimensional groundwater recharge through spreading. Int J Eng Res Appl. 2013;3(6).
8. Atangana A. A note on the triple Laplace transform and its applications to some kind of third-order differential equation. Abstr Appl Anal. 2013;2013:Article ID 769102.
9. Elzaki TM, Elzaki SM. Application of new transform "Elzaki transform" to partial differential equations. Glob J Pure Appl Math. 2011;7(1):65-70.

10. Mousa A. On the fractional triple Elzaki transform and its properties. Bull Pure Appl Sci Sect E Math Stat. 2019;38E(2):641-649.
11. Elzaki TM, Elzaki SM, Elnour EA. On the new integral transform "Elzaki transform": fundamental properties, investigation and application. Glob J Pure Appl Math. 2012;8(4):1-13.
12. Elzaki TM. The new integral transform "Elzaki transform". Glob J Pure Appl Math. 2011;7(1):57-64.
13. Mousa A, Elzaki TM. Solution of Volterra integro-differential equations by triple Laplace transform. Irish Interdiscip J Sci Res. 2019;3(4):67-72.
14. Mousa A, Elzaki TM. Solution of the Volterra integro-differential equations by the triple Elzaki transform. Bull Pure Appl Sci Sect E Math Stat. 2019;38E(2):563-569.
15. Manafian J, Bolghar P. Numerical solutions of nonlinear three-dimensional Volterra integro-differential equations with 3D block-pulse functions. Math Methods Appl Sci. 2018. doi:10.1002/mma.4936.