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A New Series of Triangular Design

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ABSTRACT

This paper proposes a simple method for constructing a new series of Triangular designs with block-size 4 for all designs, starting from a Triangular association scheme T_2 $(n \geq 6)$. When $n = 5$, using the block construction method proposed, the Triangular design becomes that of Takeuchi (1962). The advantages of such designs lies where the experimental field is of unique 1-directional limit (out of 2-directions viz.; length and breadth) i.e., block-size can be of unique size though its number of treatments as well as that of blocks may vary as experimenter wishes whenever it needs.

Keywords: PBIB design, Triangular association scheme, BIB design.

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1. INTRODUCTION

Partially Balanced Incomplete Block (PBIB) design was first developed in 1939 by Bose and Nair (1939) [1] as an extension of the Balanced Incomplete Block (BIB) design [2, 3, 4, 5]. Triangular design is an important kind of PBIB design based on Triangular association scheme. Comprehensive tables of Triangular designs and a review of some construction methods of Triangular design have been found in Clatworthy (1973) [6].

A Triangular association scheme is the arrangement of $v = n(n-1)/2$ treatments on the upper-diagonal (and symmetrically on the lower diagonal) of an $n \times n$ square array with the diagonal entries left blank. Two treatments are first associates if they lie in the same row or column; otherwise, they are second associates. The parameters of the association scheme are

$$v = \frac{n(n-1)}{2}, n_1 = 2(n-2), n_2 = \frac{(n-2)(n-3)}{2},$$

$$P_1 = \begin{bmatrix} n-2 & n-3 \\ n-3 & (n-3)(n-4)/2 \end{bmatrix}, P_2 = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{bmatrix}, \text{where } n \geq 5.$$

Constructions of Triangular designs were done by Shrikhande (1952) [7], Youden (1951) [8], Bose and Shimamoto (1952) [9], Chang, Liu and Liu (1965) [10], Masuyama (1965) [11], Liu and Chang (1964) [12] and others. The main purpose of this paper is to point out a general and simple method to construct a new series of Triangular designs with block-size 4 for all designs starting from a Triangular association scheme, whose blocks are constructed by considering all the common 1st associates for any two 2nd associates to one another.

2. CONSTRUCTION OF TRIANGULAR DESIGN

Let T_2 be the $n \times n$ square symmetrical array with its diagonal entries left blank and the $n(n-1)/2$ treatments arranged in the upper (lower) diagonal of the square symmetrical array.

Theorem 2.1: For $n \geq 6$, a new series of Triangular designs with the parameters can be constructed, $v = \binom{n}{2}, b = n(n-1)(n-2)(n-3)/8, r = (n-2)(n-3), k = 4, \lambda_1 = n-3, \lambda_2 = 2$.
Proof: Consider a Triangular association scheme T_2 and two 2nd associates θ and ψ to one another. A block with reference to θ and ψ is defined as
$B_{\theta,\psi}$ = {all the common 1st associates to both θ and ψ} …(1)

There are n_2 second associates to a treatment θ of a Triangular association scheme. The treatment θ can paired with one of its 2nd associates in n_2 i.e., $(n-2)(n-3)/2$ different ways. As association relation are symmetric, total number of different pairs of 2nd associates is $\left[\binom{n}{2} \times (n-2)(n-3)/2\right]/2$ i.e., $n(n-1)(n-2)(n-3)/8$. Thus, $b = n(n-1)(n-2)(n-3)/8$

The treatment θ has n_1 i.e., $2(n-2)$ 1st associates $\theta_1, \theta_2, \dots, \theta_{(n-2)}$ and $\psi_1, \psi_2, \dots, \psi_{(n-2)}$ in two different rows (also in two different columns). As $p_{12}^1 = n-3$, corresponding to two 1st associates θ, θ_1, let $\psi_2, \psi_3 \dots, \psi_{(n-2)}$ be those $(n-3)$ elements which are 1st associates to θ and 2nd associates to θ_1. Each of the blocks constructed based on each pair of 2nd associates $(\theta_1, \psi_2), (\theta_1, \psi_3), \dots, (\theta_1, \psi_{n-2})$ contains θ. Similarly, those blocks constructed corresponding to pairs of 2nd associates $(\theta_2, \psi_1), (\theta_2, \psi_3), \dots, (\theta_2, \psi_{n-2})$; $(\theta_3, \psi_1), (\theta_3, \psi_2), (\theta_3, \psi_4), \dots, (\theta_3, \psi_{n-2})$; … ,; $(\theta_{n-2}, \psi_1), (\theta_{n-2}, \psi_2), \dots, (\theta_{n-2}, \psi_{n-3})$ contains θ. It can be seen that there are $(n-2)(n-3)$ such pairs of 2nd associates whose both components are 1st associates to θ. So, each of the blocks constructed based on such each pair of 2nd associates contain θ in $(n-2)$ $(n-3)$ blocks. Thus, $r = (n-2)(n-3)$.
By (1), it is evident that $k = p_{11}^2 = 4$.

Consider any two 1st associates θ_1 and θ_2 to one another for counting their concurrence. Since there are p_{11}^1 i.e., $(n-2)$ common 1st associates elements i.e., $\psi, \theta_3, \theta_4, \dots, \theta_{n-1}$, say, to θ_1 and θ_2, out of which 1 element ψ, say WLOG is 2nd associates to all the remaining $\theta_3, \theta_4, \dots, \theta_{n-1}$ where $\theta_3, \theta_4, \dots, \theta_{n-1}$ are in the same row or column. So, it is possible to have $(n-3)$ pairs of 2nd associates $(\theta_3, \psi), (\theta_4, \psi), \dots, (\theta_{n-1}, \psi)$ which are 1st associates to both θ_1 and θ_2. Thus, each of the blocks constructed based on such $(n-3)$ pairs of 2nd associates contain θ_1 and θ_2. Therefore, $\lambda_1 = (n-3)$.

Further, consider the two 2nd associates θ and ψ to one another for counting their concurrence. As p_{11}^2 i.e., 4 elements, let $\theta_1, \theta_2, \psi_1$ and ψ_2 be the common 1st associates to both θ and ψ. WLOG, from these 4 common 1st associates, only two pairs of 2nd associates can be made (θ_1, ψ_2), (θ_2, ψ_1), say. Clearly, only B_{θ_1,ψ_2} and B_{θ_2,ψ_1} contains θ and ψ because θ, ψ are 1st associates to the 2nd associate pairs (θ_1, ψ_2) and (θ_2, ψ_1). Thus, any two 2nd associates occur only in 2 blocks in the desired design i.e., $\lambda_2 = 2$.
Hence the Theorem.

Remark 1: It is noted that if $n = 5$, then $\lambda_1 = \lambda_2 = 2$. So, the proposed design becomes BIB design with the parameters $v = 10, b = 15, r = 6, k = 4, \lambda = 2$, which was proposed by Takeuchi (1962).
An illustration of the Theorem 2.1 is shown as an example below:

Example 2.1: For $n = 6$, the 45 blocks of the Triangular design with the parameters $v = 15, b = 45, r = 12, k = 4, \lambda_1 = 3, \lambda_2 = 2; n_1 = 8, p_{11}^1 = 3, p_{11}^2 = 4$, are
$B_{1,10} = (2,3,6,7)$; $B_{1,11} = (2,6,4,8)$; $B_{1,12} = (2,5,6,9)$; $B_{1,13} = (3,4,7,8)$; $B_{1,14} = (3,5,7,9)$;
$B_{1,15} = (4,8,5,9)$; $B_{2,7} = (1,6,3,10)$; $B_{2,8} = (1,4,6,11)$; $B_{2,9} = (1,5,6,12)$; $B_{2,13} = (3,4,10,11)$;
$B_{2,14} = (3,5,10,12)$; $B_{2,15} = (4,5,11,12)$; $B_{3,6} = (1,2,7,10)$; $B_{3,8} = (1,4,7,13)$; $B_{3,9} = (1,5,7,14)$;
$B_{3,11} = (2,4,10,13)$; $B_{3,12} = (2,5,10,14)$; $B_{3,15} = (4,5,13,14)$; $B_{4,6} = (1,2,8,11)$; $B_{4,7} = (1,3,8,13)$;
$B_{4,9} = (1,5,8,15)$; $B_{4,10} = (2,3,11,13)$; $B_{4,12} = (2,5,11,15)$; $B_{4,14} = (3,5,13,15)$; $B_{5,6} = (1,2,9,12)$;
$B_{5,7} = (1,3,9,14)$; $B_{5,8} = (14,9,15)$; $B_{5,10} = (2,3,12,14)$; $B_{5,11} = (2,4,12,15)$; $B_{5,13} = (3,4,14,15)$;
$B_{6,13} = (7,8,10,11)$; $B_{6,14} = (7,9,10,12)$; $B_{6,15} = (8,9,11,12)$; $B_{7,11} = (6,8,10,13)$; $B_{7,12} = (6,9,10,14)$;
$B_{7,15} = (8,9,13,14)$; $B_{8,10} = (6,7,11,13)$; $B_{8,12} = (6,9,11,15)$; $B_{8,14} = (7,9,13,15)$; $B_{9,10} = (6,7,12,14)$;
$B_{9,11} = (6,8,12,15)$; $B_{9,13} = (7,8,14,15)$; $B_{10,15} = (11,12,13,14)$; $B_{11,14} = (10,12,13,15)$; $B_{12,13} = (10,11,14,15)$.

REFERENCES

1. Bose RC, Nair KR. Partially balanced incomplete block designs. Sankhya. 1939;4:337-372.
2. Yates F. Incomplete randomised block. Ann Eugen. 1936;7:121-140. doi:10.1111/j.1469-1809.1936.tb02134.x.
3. Fisher RA, Yates F. *Statistical Tables for Biological, Agricultural and Medical Research*. Edinburgh: Oliver and Boyd; 1938.
4. Bose RC. On the construction of balanced incomplete block designs. Ann Eugen. 1939;9:353-399.
5. Yates F. A new method of arranging variety trials involving a large number of varieties. J Agric Sci. 1936;26:424-455.
6. Clatworthy WH. *Tables of Two-Associate-Class Partially Balanced Designs*. National Bureau of Standards Applied Mathematics Series No. 63. Washington (DC): National Bureau of Standards; 1973.
7. Shrikhande SS. On the dual of some balanced incomplete block designs. Biometrics. 1952;8:66-72.
8. Youden WJ. Linked blocks: A new class of incomplete block designs. Biometrics. 1951;7:124.
9. Bose RC, Shimamoto T. Classification and analysis of partially balanced designs with two associate classes. J Am Stat Assoc. 1952;47:151-184. doi:10.1080/01621459.1952.10500866.
10. Chang LC, Liu CW, Liu WR. Incomplete block design with triangular parameters for which k ≤10 and r ≤10. Sci Sin. 1965;14:329-338.
11. Masuyama M. Cyclic generation of triangular PBIB designs. Rep Stat Appl Res JUSE. 1965;12:73-81.
12. Liu CW, Chang LC. Some PBIB (2) designs induced by association schemes. Sci Sin. 1964;13:840-841.