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Some New Results on Commutative Algebra

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ABSTRACT

In this paper, we shall discuss and give applications of a polarization formula, which yields estimates of$x_1 \dots \dots \dots \dots \dots x_n$ when estimates of x^n are given, and estimates of the polarized forms of the terms of the Taylor expansion of a bounded holomorphic function over a bell in Banach space. The polarization formula can be used to obtain new properties of commutative algebras, with a convex topology or a convex bounded structure. The polarization formula will be proved in the first section,a few application will be given in section 2,3,4 in which we obtain estimates of $x_1 \dots \dots \dots \dots \dots x^n$ from estimates of x^n. We shall also define the infinite dimensional holomorphic functional calculus.

KEYWORDS: Polarization, Convex Algebra, Equiregular.

1. INTRODUCTION

The polarization formula: Let E and F be linear spaces over fields of characteristic zero
$\hat{u}$: E X ....X, E→F be a symmetric multilinear mapping, and
let u(x) = $\hat{u}$(x,.............x).Polarization is the operation which allows us to recover $\hat{u}$ when u is given.

Theorem (1.1)
Let e_1.......................,e_n be elements of E.We have
n!$\hat{u}$ (e_1.......................,e_n)=$\sum_1$ ≤(1,2.........n) ($(-1)^{n-c(I)}$u($\sum j \in$ I e_j
where C(I) is the number of element of I.

We shall first prove the formula when E=F=A is a commutative algebra. Once this is done, the reader will have a choice .He may read the proof and observe that it also yields the result in general or he may read further and find a soft method which shows why theorem (1.1) is a corollary of its special case.
a is a commutative algebra. We consider the multilinear form $\hat{u}$ (a_1,.............a_n)=a_1.............a_n

Let W_k=$\sum_{c(I)=k}$ ($\sum_{i \in I} a_i$)
We want to show that
$(-1)^n$n!a_1...........................a_n=$\sum_o^n (-1)^k w_k$
Consider anyterm $a_{i_1}^{p_1}$....................$a_{i_r}^{p_r}$ in the development of w_k.We take
$i_1 < i_2 <$..................................$< i_r$, also
P_1+..P_r=n. This term occurs with the coefficients

$$\frac{n! \ !}{P_1^1 \dots \dots \dots \dots \dots \dots \dots \dots \dots P_r^1} \binom{n-r}{k-r}$$

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Since $\binom{n-r}{k-r}$ of the subsets I of (1,2,.....................................,n) which have k elements contain $i_1 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots i_r$, and each of these subsets yields a term similar to the one we consider, with the coefficient $\frac{n!\ !}{P_1^! \dots\dots\dots\dots\dots\dots P_r^!}$.
The coefficient of $a_{i_1}^{p_1}$...................$a_{i_r}^{p_r}$ in
$\sum(-1)^k w_k = \frac{n!\ !}{P_1^! \dots\dots\dots\dots\dots\dots P_r^!} \sum_k (-1)^k \binom{n-r}{k-r}$.

This contain the factor $(1-1)^{n-r}$ .It vanishes for n$\neq r$.For n=r,it is equal to $(-1)^n n!$.
We may apply the above proof when **a** is the algebra of polynomials in n indeteminates a_1............................a_n
We obtain a relation

$(-1)^n$n ! a_1............................a_n = $\sum_k (-1)^k \sum_{C(I)=k} (\sum_{i\in I} a_i)^n$

In the space $\boldsymbol{P_n}$ (a_1............................a_n).Let now $\hat{u}$: E x.................................x E→ F be a Symmetric multi-linear mapping, ande_1.......................,e_n be elements of E. We obtain a linear mapping

$\boldsymbol{P_n} \to \boldsymbol{F}$ which mapsa_{i_1}...a_{i_n} onto $\hat{u}(a_{i_1}, \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots ., a_{i_1})$.

This mapping maps the above relation in $\boldsymbol{P_n}$ onto a relation of F.It is easy to see that this is the required one.This formula allows us to obtain convex estimates of $\hat{u}$ when convex estimates of u are assumed.

Corollary: Let E,F be real or complex linear spaces. Let X⊆ E,Y⊆F be absolutely convex .Let $u_\alpha: E \to F(\alpha \varepsilon A)$ be homogeneous polynomial mappings, u_α of degree n_α , and assume that $u_\alpha\ x \subseteq M_\alpha\ Y$ for all α.

Then

$\hat{u}_\alpha(X, \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots ., X) \subseteq (2\ e)^n \alpha\ M_\alpha Y$ for all α
C(I)≤ $n_\alpha, \sum_{i\in I} e_1 \in n_\alpha\ X$

If each $e_1 \in X$,and u($\sum_{i\in I} e_i) \in M_\alpha n_\alpha^{n_\alpha} Y$.

Now,$n_\alpha!\ \hat{u}$ (e_1.......................,e_n is a linear combination of 2^{n_α} such terms

$\widehat{u_\alpha}$ (e_1.......................,e_{n_α}) $\in M_\alpha \frac{(2n_\alpha)^{n_\alpha}}{n_\alpha!}$Y
Stirling 's formula shows that $n_\alpha! \geq \left(\frac{n_\alpha}{e}\right)^{n_\alpha}$ i.e

$\widehat{u_\alpha}$(e_1.......................,e_{n_α}) $\in (2\ e)^{n_\alpha} M_\alpha$ Y.

2. FRECHET ALGEBRAS ON WHICH ENTIRE FUNCTIONS OPERATE

Let $\mathfrak{a}$be a topological algebra, a $\in \mathfrak{a}$.Let $f = \sum f_n a^n$ be an entire function.It is responsible to say that f operates on $\mathfrak{a}$ if$\sum f_n a^n$ converges.It is even more responsible to say that entire functions operate on $\mathfrak{a}$ if $\sum f_n a^n$ converges each time(f_n) is the sequence of Taylors coefficients of an entire function and each time a $\in \mathfrak{a}$

It is clear that entire functions operate on $\mathfrak{a}$if $\mathfrak{a}$ is complete ,Locally m-convex if $\mathfrak{a}$is an inverse limit of Banach algebras.

Theorem (2.1)

Let $\mathfrak{a}$ be acommutative, Frechet algebra on which analytical functions operate. Then $\mathfrak{a}$ is locally m-convex.
The proof goes in two steps. We first prove the following lemma by a double category argument:

Lemma (1.1)
Let $\mathfrak{a}$ be acomplete metrizable locally convex algebra.Assume that $\lambda_n a^n \to 0$ for every sequence of positive scalers λ_n such that To $\lambda_n^{\frac{1}{n}} \to 0$ each neighbourhood V of th origin in $\mathfrak{a}$we can associate a neighbourhood W such that

V $\supseteq \{x^n : x \in W, n \in N, n \neq 0\}$

We start out with a closed convex balanced neighbourhood U of the origin and a sequence λ_n such that $\lambda_n^{\frac{1}{n}}$ is decreasing and tends to zero. Let V be the set of a $\in \mathfrak{a}$ such that $\lambda_n a^n \in U$for all n. Then V is closed and balanced, V is absorbing. Consider any $a \in \mathfrak{a}$, Since $\lambda_n a^n \to 0, \lambda_n a^n \in U$ for all but a finite number of values of n. For those n,$\lambda_n a^n \in \in^{-1}\ U\ for\ \in > 0$, small .We take $\in < 1, \lambda_n (\in a)^n \in U$ for all n.So V has a non-empty interior.

Let U_1be a new neighbourhood of the origin such that U$\supseteq U_1 . U_1$. Let V_1 be the set of a$\in$ $\mathfrak{a}\lambda_n a^n \in U_1$for all n$\geq$ 1.Then V_1has a non –empty interior.
Let $X = \frac{1}{2(a+b)}$ with a$\epsilon V_1, b\epsilon V_2, then$
$\lambda_n x^n = 2^{-n} \sum \binom{n}{p} \frac{\lambda_n}{\lambda_r . \lambda_{n-r}} \lambda_r a^r \lambda_{n-r} b^{n-r}$. This belongs to U because

$\lambda_r a^r \lambda_{n-r} b^{n-r} \in U, \lambda_n \leq \lambda_r \lambda_{n-r}$.
(because $\lambda_n^{\frac{1}{2}}$ is decreasing)and $2^{-n} \sum \binom{n}{r} \frac{\lambda_r}{\lambda_r . \lambda_{n-r}} \leq 1$
So V$\supseteq$ $\frac{1}{2(V_1 + v_2)}$ is a neighbourhood of the origin.

This is half the proof of the lemma. We observe that we can associate to every sequence of complex numbers c_n such that $|c_n|^{\frac{1}{n}} \to 0$ a sequence of positive reals λ_n such that $|c_n| \leq \lambda_n$ and $\left(\frac{1}{n}\right)^{\frac{1}{n}}$decreases to zero. So, for every such sequences c_n and every neighbourhood U of the origin, we can find a neighbourhood V of the origin such that $c_n x_n \in$ U each time x$\in V$.

To Prove the second half, we consider a neighbourhood U of the origin ,take U closed and convex ,and then a basis $V_1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .., V_k, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$...of neighbourhood of the origin in $\mathfrak{a}$.

We let X_k be the set of sequences of scalars c_n,with $|c_n|^{\frac{1}{n}} \to 0$ and for all $x \in V_k : c_n x^n \in U$.Then $x_k$is absolutely convex ,closed in the space of entire functions, and $U\ \ X_k$ is the space of all entire functions. Baire's theorem shows that X_k must have an interior for large K.

This gives constants $M_1, M_2,$ and a neighbourhood V_k of the origin, such that $x^n \in M_1 . M_2^n\ U$, whenever $x \in V_k$. And $x^n \in U$if $x \in \frac{V_k}{M_{1.} M_2}$.
So the lemma is proved.

To complete the proof, we shall use a rewording of the corollary theorem.

Lemma (2.1): Let U and V be absolutely convex subsets of a commutative algebra. Assume that $x^n \in U$ as soon as $x \in V, n = 1,2,3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$....An absolutely convex V_1can be found with $V_1^2 \subseteq V_1$and $\supseteq V_1 \supseteq (2e)^{-1}$V.
$V_1$is the absolutely convex hull of the set of $x_1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ., X_n$,with $\forall\ n : x_1 \in \frac{V}{2} e$, and n$\geq$ 1.The corollary gives the result.

Combining Theorem (1.1) and (2.1), we see that each neighbourhood U of the origin in $\mathfrak{a}$
Contains a neighbourhood V, which is absolutely convex and idempotent. The algebra is locally multiplicatively convex.

3. CONTINUOUS INVERSE LOCALLY CONVEX ALGEBRAS

Theorem (3.1)
A commutative, locally convex, continuous inverse algebra is locally multiplicatively convex. It is possible to associate to each neighbourhood U of the origin in $\mathfrak{a}$a neighbourhood V in such a way that
U⊇ $\{x^n: n \in \boldsymbol{N}, n \neq 0$,x∈ $U.\}$

Once this has been shown, we observe that U and V may be taken absolutely convex, Lemma (2.1) shows that an absolutely convex, idempotent V_1can be found in such a way that $U \supseteq V_1 \supseteq (2e)^{-1}v,$ and theorem (3.1) follows.

So we must show that $x^n \to 0$ and $\in \boldsymbol{N}$,$n \neq 0$. We choose a neighbourhood of the origin U_1, such that $\rho(x) < 1/2$. When $x \in U_1$, then by the holomorphic functional calculus

$$x^n = \frac{1}{2\pi}\int_0^{2\pi} e^{int}(1 - e^{it}x)^{-1}dt.$$

If $x \to o, (1 - e^{it}x)^{-1} \to 1$ uniformly, t∈ IR and $x^n \to 0$ uniformly,$n \in \boldsymbol{N}, n \neq 0$.

4. THE EQUIREGULAR BOUNDEDNESS

Let now $\mathfrak{a}$be a commutative B-algebra with unit. An element $a\mathfrak{a}$ is regular if $(a - a)^{-1}$ is defined an bounded for a large.

Defination (4.1)
A subset B of regular elements of $\mathfrak{a}$is equiregular if it is possible to find M>0 in such a way that a-B has an nverse when $a \in B$,$|s'| \geq M$

Theorem (4.1):
B is an equiregular set if and only if it is possible to find some M such that
$\{\frac{b^n}{M^n}: b \in B, n \oplus \boldsymbol{N}\}$ is a bounded set.

Corollary: An equiregular set is bounded. Assume equiregular and assume B equiregular and assume $(a - s)^{-1}$ defined and bounded for $|s| \geq M$ and $a \in B$,Then for all $a \in B$, for all n,

$$a^n = \frac{M^n}{2\pi}\int_0^{2\pi} e^{int}(1 - Me^{-it}a)^{-1}dt, \ and \ \left\{\frac{a^n}{M^n}\right)$$

$$considered \ is \ bounded. If \ conversely, this \ set \ is \ bounded\ , and$$

$|a|$ ≥2M then$-\sum a^{-n-1}a^n = (a - s)^{-1}$ belongs to the completant hull of this bounded set.It follows that B is an equiregular set.

The set $\mathfrak{a}_r$of regular elements of a commutative B-algebra $\mathfrak{a}$is a sub-algebra of $\mathfrak{a}$.

Theorem (4.2):
The set of equiregular subsets of $\mathfrak{a}_r$is an algebra boundedness on $\mathfrak{a}_r$.

If B is equiregular, and if B_1 is a completant bounded subset of $\mathfrak{a}$, then $\cap_{\epsilon>0} (B + \epsilon B_1)$ is equiregular .

An analysis of the proof of the above theorem yields the uniform majorants that we need to prove the first part of the theorem (4.2). It is not more difficult to prove that the sum, or the product of equiregular sets is equiregular, then to show that the sum ,or the produ $Type\ equation\ here.$ ct of regular elements is regular.

Let now B be equiregular. Let $s \in \cap (B + \in B_1)$ with B_1completant,$B_1 \supseteq B$. We choose M real, and B_2 bounded in such a way that $(a - a)^{-1} \in B_1$ when $|s| \geq M, a \in B$.Choose $a_n \in B, a_n - s \in 2^{-n}B_1$.
Then $(a_n - a)^{-1} - (a_m - s)^{-1} = (a_m - a_n)(a_n - s)^{-1}(a_m - s)^{-1} \in (2^{-m} + 2^{-n})B_1B_2^2$.

This shows that $(a_n - s)^{-1}$ is Cauchy sequence of the Banach space $\mathfrak{a}_{B_3}$.$B_3$is completant. $B_3 \supseteq B_1B_2^2$. This sequence has a limit, which is an inverse of (a-s).This limit belongs to the closure of B_2 in the Banach space $\mathfrak{a}_{B_3}$
This completes the proof .It is yet to be known wheather the equiregular boundedness of a B-algebra is convex. A convex boundedness can be associated to every vector space boundedness, its elements are the sets with a bounded convex hull.

Theorem (4.3):
The convex boundedness associated to the equiregular bounddness is the allan boundedness of the algebra of regular elements.
This is an straightforward application of lemma (2.1) .Let B be equiregular and convex .Choose M large enough, thenB_1 convex, bounded and such that $\frac{x^n}{M^n} \in B_1 when\, x \in B_1, n \in \boldsymbol{N}$.Let B_1 be the absolutely convex hull of $\{\frac{x_1 \ldots\ldots x_n}{(2Me)^n} : n \in \boldsymbol{N}, \forall_i : \boldsymbol{x_1} \in \boldsymbol{B}\}$

Then B_2 is bounded, absolutely convex, and idempotent, and $B \subseteq MB_2$.

5. CONCLUSION

(1) Hence a commutative, locally convex, continuous inverse algebra is locally multiplicatively convex.
(2) It is clear that the bounded, absolutely convex idempotent sets are equiregular, so the Allan boundedness is the convex boundednedd associated to the equiregular one.

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