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$\boldsymbol{P}$-third order generalized Finsler space in the Berwald sense *

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Abstract The generalized recurrent and birecurrent spaces for P^i_{jkh} are discussed by Finslerian geometrics. Now, in this paper, we generalize these spaces by using the covariant derivative of third order in the sense of Berwald. The necessary and sufficient condition for P^i_{jkh} satisfying the generalized trirecurrence property is obtained. In addition, we show that the covariant derivative in the sense of Berwald coincides with the h-covariant derivative in the sense of Cartan. Various identities concerning such space are established.

Key words Generalized $\mathcal{B}P$-trirecurrent space, Berwald covariant derivative $\mathcal{B}_k$, h-covariant derivative, Cartan's second curvature tensor P^i_{jkh}.

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1 Introduction

The concept of trirecurrent and generalized trirecurrent spaces is discussed by [13, 15] and [27]. The generalized recurrent space for H^i_{jkh}, $R^i_{jkh}$and P^i_{jkh} in the sense of Berwald was studied in [1, 7, 9, 10, 18] and [25]. Also, the generalized property for normal projective curvature tensor N^i_{jkh} in the sense of Berwald was introduced by [11]. Furthermore, the generalized birecurrent space for different curvature tensors in the sense of Berwald is studied in [3, 8, 14] and [16]. Srivastava [24] defined the special R-generalized recurrent Finsler spaces of order two. The first author has earlier studied the $\mathcal{B}P$-recurrent and birecurrent Finsler spaces [2, 3, 20].
In the same regard, Qasem and Ahmed [12] studied the generalized $\mathcal{B}H$-trirecurrent space. The generalized trirecurrent space for K^i_{jkh} and R^i_{jkh} in the sense of Cartan is discussed by Nasr [21] and Husien [23] respectively. In this paper we focus on a Finsler space that P^i_{jkh} satisfies the generalized trirecurrent characteristic. Additionally we obtain the relationship between two types of curvature tensors and covariant derivatives.

2 Preliminaries

This section will provide some definitions which we need in this study. Let F_n be an n-dimensional Finsler space equipped with the metric function $F(x, y)$ satisfying the request conditions [4, 26]. The

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vector y_i is defined by

$$y_i = \ g_{ij}\,(x, y)\, y^j. \tag{2.1}$$

Two sets of quantities g_{ij} and its associative g^{ij} are connected by

$$g_{ij} g^{ik} = \delta_j^k = \begin{cases} 1, & \text{if } \ j = k \ , \\ 0, & \text{if } \ j \neq k \ . \end{cases} \tag{2.2}$$

In view of (2.1) and (2.2), we have

$$\text{a) } \delta_k^i \ y^k = y^i, \ \text{ b) } \delta_j^i g_{ir} = \ g_{jr} \text{ and c) } \delta_k^i y_i = y_k. \tag{2.3}$$

The $(h)\,hv$-torsion tensor C^i_{jk} is the associate tensor of the tensor C_{ijk} which are defined by [6,22]:

$$\text{a) } C_{\ ijk}\ y^i = C_{\ kij}\ y^i = C_{\ jki}\ y^i = 0, \ \text{ b) } C^i_{jk} y^j = C^i_{kj}\ y^j = 0 \text{ and c) } \delta^i_j C_{kil} = C_{kjl}, \tag{2.4}$$

where,

$$C_{\ ijk} = \frac{1}{2} \dot{\partial}_i\ g_{jk} = \frac{1}{4} \dot{\partial}_i\ \dot{\partial}_j\ \dot{\partial}_k\ F^2.$$

The Cartan h-*covariant derivative* (*Cartan's second kind covariant differentiation*) with respect to x^k is given by

$$X^i_{|k} = \partial_k X^i - \left(\dot{\partial}_r\ X^i\right) G^{\ r}_{\ k} + X^r \Gamma^{*\ i}_{rk}, \tag{2.5}$$

where $\Gamma^{*\ i}_{rk}$ is a function called the *Cartan's connection parameter*. The h-covariant derivative of the vector y^i vanish identically, i.e.,

$$y^i_{|k} = 0. \tag{2.6}$$

The connection parameter G^i_{jk} of Berwald connected with the Cartan's connection parameter Γ^{*i}_{jk} is given by

$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|h}\ y^h\ . \tag{2.7}$$

The Berwald's covariant derivative $\mathcal{B}_k$ of an arbitrary tensor field T^i_j with respect to x^k is given by [17]

$$\mathcal{B}_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}. \tag{2.8}$$

The processes of Berwald's covariant differentiation and the partial differentiation commute according to

$$\left(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k\right) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj} \tag{2.9}$$

for an arbitrary tensor field T^i_j .
The Berwald's covariant derivative of the vector y^i and metric tensor g_{ij} satisfy

$$\text{a) } \mathcal{B}_k y^i = 0 \text{ and b) } \mathcal{B}_k g_{ij} = -2C_{ijk|h} y^h = -2y^h \mathcal{B}_h C_{ijk}. \tag{2.10}$$

The h-curvature tensor (Cartan's third curvature tensor) is defined by [17] as

$$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + \left(\partial_l \Gamma^{*i}_{jk}\right) G^l_h + C^i_{jm} \left(\partial_k G^m_h - G^m_{kl} G^l_h\right) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} - k/h\ ^*\ .$$

The curvature tensor R^i_{jkh}, R-Ricci tensor R_{jk} and curvature vector R_k satisfy

$$\text{a) } R^i_{jkh} y^j = H^i_{kh}\ = K^i_{jkh} y^j, \ \text{ b) } R^i_{jkh} = K^i_{jkh} + C^i_{js} H^s_{kh}, \ \text{ c) } R_{jk} = K_{jk} + C^r_{js} H^s_{kr} \text{ and d) } R_{jk} y^j = R_k. \tag{2.11}$$

The tensor P^i_{jkh}called *hv-curvature tensor* (*Cartan's second curvature tensor*) is positively homogeneous of degree -1 in y^i and is defined by [17,19]

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk}\ + C^i_{jr} P^{\ r}_{kh}\ - C^i_{jh|k}.$$

The associate tensor P_{ijkh}, torsion tensor P^i_{kh} and P-Ricci tensor P_{jk} of hv-curvature tensor P^i_{jkh} satisfies the relations

$$\text{a) } P^i_{jkh}\ y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^{\ i}_{\ kh|r} y^r, \ \text{ b) } P^i_{jki} = P_{jk}, \ \text{ c) } P^i_{ki} = P_k, \ \text{ d) } P_k\ y^k = P, \ \text{ and e) } P^i_{jk} y^j = 0. \tag{2.12}$$

Using (2.12) in (2.7), we get

$$P^i_{kh} = G^i_{kh} - \Gamma^{*i}_{kh}. \tag{2.13}$$

The hv-curvature tensor P^i_{jkh} satisfies the following:

$$P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r} y^r, \tag{2.14}$$

and

$$P^i_{jkh} - P^i_{kjh} = C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k \ . \tag{2.15}$$

Alaa et al. [2,3,20] introduced the generalized $\mathcal{B}P$-recurrent space and generalized $\mathcal{B}P$-birecurrent space which are characterized by the conditions

$$\mathcal{B}_n P^i_{jkh} = \lambda_n P^i_{jkh} + \mu_n \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right), \qquad P^i_{jkh} \neq 0 \tag{2.16}$$

and

$$\mathcal{B}_m \mathcal{B}_n P^i_{jkh} = u_{mn} P^i_{jkh} + v_{mn} \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2y^t \mu_n \mathcal{B}_t \left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right), \tag{2.17}$$

where $u_{mn} = \mathcal{B}_m \lambda_n + \lambda_m \lambda_n$ and $v_{mn} = \lambda_n \mu_m + \mathcal{B}_m \mu_n$ are non-zero covariant tensors field of second order.

3 Main results

In this section, we discuss a Finsler space that P^i_{jkh} satisfies the generalized trirecurrent characteristic. Important theorems concerned with this space are proved. Taking $\mathcal{B}$-covariant derivative for the condition (2.17) with respect to x^l and using the condition (2.16) and (2.10), we get

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n P^i_{jkh} = {} & a_{lmn} P^i_{jkh} + b_{lmn} \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2y^t c_{mn} \mathcal{B}_t \left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right) \\ & -2\ y^t d_{ln} \mathcal{B}_t \left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t \left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right), \end{aligned} \tag{3.1}$$

where $a_{lmn} = \mathcal{B}_l u_{mn} + u_{mn} \lambda_l$ and $b_{lmn} = \mathcal{B}_l v_{mn} + u_{mn} \mu_l$ are non-zero covariant tensor fields of the third order, $c_{mn} = v_{mn}$ and $d_{ln} = \mathcal{B}_l \mu_n$ are non-zero covariant tensor field of the second order. Also $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n$ is the differential operator in the sense of Berwald with respect to x^l, x^m and x^n, successively.

Definition 3.1. A Finsler space F_n in which Cartan's second curvature tensor P^i_{jkh} satisfies the condition (3.1) will be called *a generalized $\mathcal{B}P$-trirecurrent space* and the tensor will be called *a generalized $\mathcal{B}$-trirecurrent*. This space and the tensor are denoted by $G(\mathcal{B}P)-TRF_n$ and $G\mathcal{B}-TRF_n$, respectively.

In the next theorems below we establish that some tensors are non-vanishing.

Theorem 3.2. *In $G(\mathcal{B}P) - TRF_n$, Berwald's covariant derivative of the third order for the torsion tensor P^i_{kh} and $(-S^{\ i}_{jkh|r} y^r)$ are given by*

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n P^i_{kh} = {} & a_{lmn} P^i_{kh} + b_{lmn} \left(y^i g_{kh} - \delta^i_k y_h\right) - 2y^t c_{mn} \mathcal{B}_t \left(y^i C_{khl}\right) \\ & -2\ y^t d_{ln} \mathcal{B}_t \left(y^i C_{khm}\right) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t \left(y^i C_{khm}\right), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n (-S^{\ i}_{jkh|r} y^r) = {} & a_{lmn} (-S^{\ i}_{jkh|r} y^r) + b_{lmn} \left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - 2y^t c_{mn} \mathcal{B}_t \left(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}\right) \\ & -2\ y^t d_{ln} \mathcal{B}_t \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right), \end{aligned} \tag{3.3}$$

respectively.

Proof. Let us consider a $G(\mathcal{B}P) - TRF_n$ which is characterized by the condition (3.1). Transvecting the condition (3.1) by y^j, using (2.10), (2.12), (2.3), (2.1) and (2.4), we get (3.2).
Taking the $\mathcal{B}$-covariant derivative for (2.14) thrice with respect to x^n, x^m and x^l, successively, using the condition (3.1), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(-S\,{}^{i}_{jkh|r}y^r) = a_{lmn}(P^i_{jkh} - P^i_{jhk}) + b_{lmn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) -2y^t c_{mn}\mathcal{B}_t\left(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}\right)$$

$$-2\ y^t d_{ln}\mathcal{B}_t\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right).$$

Using (2.14) in above equation, we get (3.3). Hence, we have proved this theorem. ☐

Theorem 3.3. *The tensor* $(C^i_{jk|h} + C^i_{hs}P^s_{jk} - h/j)$ *is a generalized* $\mathcal{B}-$ *trirecurrent in* $G(\mathcal{B}P) - TRF_n$.

Proof. Assume that $G(\mathcal{B}P) - TRF_n$. Taking the $\mathcal{B}$-covariant derivative of (2.15) thrice with respect to x^n, x^m and x^l, successively, using the condition (3.1), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(C^i_{kh|j} + C^i_{sj}P^s_{kh} - j/k) = a_{lmn}(P^i_{jkh} - P^i_{hjk}) + \propto_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right)$$

$$-2y^t\gamma_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right) - 2\ y^t\Omega_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right) - 2y^t\sigma_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right),$$

where $2b_{lmn} = \propto_{lmn}$, $2c_{mn} = \gamma_{mn}$, $2d_{ln} = \Omega_{ln}$ and $2\mu_n = \sigma_n$.
Using (2.15) in above equation, we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(C^i_{kh|j} + C^i_{sj}P^s_{kh} - j/k) = a_{lmn}(C^i_{kh|j} + C^i_{sj}P^s_{kh} - j/k) + \propto_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) \tag{3.4}$$

$$-2y^t\gamma_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right) - 2\ y^t\Omega_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right) - 2y^t\sigma_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right)$$

thus proving the theorem. ☐

Now, we have a corollary related to the previous theorems. Contracting the indices i and h in the condition (3.1) and (3.2), respectively and using (2.12), (2.3), (2.4) and (2.1), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P_{jk} = a_{lmn}P_{jk} \tag{3.5}$$

and

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P_k = a_{lmn}P_k. \tag{3.6}$$

Transvecting (3.6) by y^k, using (2.10) and (2.12), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P = a_{lmn}P. \tag{3.7}$$

The equations (3.5), (3.6) and (3.7) show that P_{jk}, P_k and P behave as trirecurrent. Thus, we conclude the following corollary:

Corollary 3.4. *In* $G(\mathcal{B}P) - TRF_n$, *the behavior of Ricci tensor* P_{jk}, *curvature vector* P_k *and curvature scalar* P *are trirecurrent.*

In the next three theorems we focus on the necessary and sufficient conditions for R^i_{jkh}, H^i_{kh} and K^i_{jkh} that satisfy the generalized trirecurrence property.

Theorem 3.5. *In* $G(\mathcal{B}P) - TRF_n$, *for* $n = 4$, *Cartan's third curvature tensor* R^i_{jkh} *is a generalized trirecurrent if and only if the tensor* $\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)$ *is trirecurrent.*

Proof. Since in the Riemannian space V_4, the projective curvature tensor P^i_{jkh} is defined as follows [5]

$$P^i_{jkh} = R^i_{jkh} - \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right), \tag{3.8}$$

taking the $\mathcal{B}-$covariant derivative of (3.8) thrice with respect to x^n, x^m and x^l, successively, using the condition (3.1), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n R^i_{jkh} = a_{lmn}P^i_{jkh} + b_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) -2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right)$$

$$-2\ y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)+\frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right).$$

Using (3.8) in the above equation, we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n R^i_{jkh}=a_{lmn}R^i_{jkh}+b_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)-2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)$$

$$-2\ y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)=a_{lmn}\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right).$$

The above equation means that the tensor $\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)$ behaves as trirecurrent. Hence, we have proved this theorem. □

Theorem 3.6. *In $G(\mathcal{B}P)-TRF_n$, for $n=4$, Berwald's covariant derivative of third order for the torsion tensor H^i_{kh} of Berwald curvature tensor are given by*

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n H^i_{kh}=a_{lmn}H^i_{kh}+b_{lmn}\left(y^i g_{kh}-\delta^i_k y_h\right)-2y^t c_{mn}\mathcal{B}_t\left(y^i C_{khl}\right) \tag{3.9}$$

$$-2\ y^t d_{ln}\mathcal{B}_t\left(y^i C_{khm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(y^i C_{khm}\right)$$

if and only if the tensor $\left(\delta^i_h R_{jk}-\ \delta^i_k R_{jh}\right)$ is trirecurrent.

Proof. Transvecting (3.8) by y^j, using (2.10), (2.12) and (2.11), we get

$$P^i_{kh}=H^i_{kh}-\frac{1}{3}\left(\delta^i_h R_k-\delta^i_k R_h\right). \tag{3.10}$$

Taking the $\mathcal{B}-$covariant derivative for (3.10) thrice with respect to x^n, x^m and x^l, successively, using (3.2), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n H^i_{kh}=a_{lmn}P^i_{kh}+b_{lmn}\left(y^i g_{kh}-\delta^i_k y_h\right)-2y^t c_{mn}\mathcal{B}_t\left(y^i C_{khl}\right)-2\ y^t d_{ln}\mathcal{B}_t\left(y^i C_{khm}\right)$$

$$-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(y^i C_{khm}\right)+\frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_k-\delta^i_k R_h\right).$$

Using (3.10) in the above equation, we get (3.9) if and only if

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_k-\delta^i_k R_h\right)=a_{lmn}\left(\delta^i_h R_k-\ \delta^i_k R_h\right).$$

The above equation shows that the tensor $\left(\delta^i_h R_k-\ \delta^i_k R_h\right)$ behaves as trirecurrent. Hence, we have proved the theorem. □

Theorem 3.7. *In $G(\mathcal{B}P)-TRF_n$, for $n=4$, Cartan's fourth curvature tensor K^i_{jkh} is a generalized $\mathcal{B}-$ trirecurrent if and only if*

$$\frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left\{\left((K_{jk}+C^r_{jm}\ H^m_{kr})\delta^i_h-(K_{jh}+C^r_{jm}\ H^m_{hr}\right)\delta^i_k\right\}-\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(C^i_{jm}\ H^{\ m}_{kh}) \tag{3.11}$$

$$=\frac{1}{3}a_{lmn}\left\{\left((K_{jk}+C^r_{jm}\ H^m_{kr})\delta^i_h-(K_{jh}+C^r_{jm}\ H^m_{hr}\right)\delta^i_k\right\}-a_{lmn}(C^i_{jm}\ H^{\ m}_{kh}).$$

Proof. Using (2.11) in (3.8), we get

$$P^i_{jkh}=K^i_{jkh}+C^i_{jm}\ H^{\ m}_{kh}-\frac{1}{3}\{\left(K_{jk}+C^r_{jm}\ H^m_{kr}\right)\delta^i_h-\left(K_{jh}+C^r_{jm}\ H^m_{hr}\right)\delta^i_k\}. \tag{3.12}$$

Taking the $\mathcal{B}-$covariant derivative for (3.12) thrice with respect to x^n, x^m and x^l, successively, using (3.8), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n K^i_{jkh}=a_{lmn}P^i_{jkh}+b_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)-2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)$$

$$-2\ y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)$$
$$-\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(C^i_{jm}\ H^m_{kh})+\frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left\{\left(K_{jk}+C^r_{jm}\ H^m_{kr})\delta^i_h-(K_{jh}+C^r_{jm}\ H^m_{hr}\right)\delta^i_k\right\}.$$

Using (3.12) in the above equation, we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n K^i_{jkh}=a_{lmn}K^i_{jkh}+b_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)-2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)$$
$$-2\ y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)$$

if and only if (3.11) holds. Hence we have proved this theorem. □

In next theorem we find the necessary and sufficient condition for the tensors $\left(\dot\partial_h P_{jk}\right)$ and $\left(\dot\partial_h P_k\right)$ to behave as trirecurrent.

Theorem 3.8. *In $G(\mathcal{B}P)-TRF_n$, the behavior of the tensors $\left(\dot\partial_h P_{jk}\right)$ and $\left(\dot\partial_h P_k\right)$ are trirecurrent if and only if*

$$(\mathcal{B}_l\mathcal{B}_m P_{rk})G^r_{hnj}+(\mathcal{B}_l P_{rk})(\mathcal{B}_m G^r_{hnj})+(\mathcal{B}_m P_{rk})(\mathcal{B}_l G^r_{hnj})+P_{rk}(\mathcal{B}_l\mathcal{B}_m G^r_{hnj})+(\mathcal{B}_l\mathcal{B}_m P_{jr})G^r_{hnk}\quad(3.13)$$
$$+(\mathcal{B}_l P_{jr})(\mathcal{B}_m G^r_{hnk})+(\mathcal{B}_m P_{jr})(\mathcal{B}_l G^r_{hnk})+P_{jr}(\mathcal{B}_l\mathcal{B}_m G^r_{hnk})+(\mathcal{B}_l\mathcal{B}_r P_{jk})G^r_{hmn}+(\mathcal{B}_r P_{jk})(\mathcal{B}_l G^r_{hmn})$$
$$+(\mathcal{B}_l\mathcal{B}_n P_{rk})G^r_{hmj}+(\mathcal{B}_n P_{rk})(\mathcal{B}_l G^r_{hmj})+(\mathcal{B}_l\mathcal{B}_n P_{jr})G^r_{hmk}+(\mathcal{B}_n P_{jr})(\mathcal{B}_l G^r_{hmk})$$
$$+(\mathcal{B}_r\mathcal{B}_n P_{jk})G^r_{hlm}+(\mathcal{B}_m\mathcal{B}_r P_{jk})G^r_{hln}+(\mathcal{B}_m\mathcal{B}_n P_{rk})G^r_{hlj}+(\mathcal{B}_m\mathcal{B}_n P_{jr})G^r_{hlk}-(\dot\partial_h a_{lmn})\ P_{jk}=0,$$

and

$$(\mathcal{B}_l\mathcal{B}_m P_r)G^r_{hnk}+(\mathcal{B}_l P_r)(\mathcal{B}_m G^r_{hnk})+(\mathcal{B}_m P_r)(\mathcal{B}_l G^r_{hnk})+P_r(\mathcal{B}_l\mathcal{B}_m G^r_{hnk})+(\mathcal{B}_r\mathcal{B}_l P_k)G^r_{hmn}\quad(3.14)$$
$$+(\mathcal{B}_r P_k)(\mathcal{B}_l G^r_{hmn})+(\mathcal{B}_l\mathcal{B}_n P_r)G^r_{hmk}+(\mathcal{B}_n P_r)(\mathcal{B}_l G^r_{hmk})+(\mathcal{B}_r\mathcal{B}_n P_k)G^r_{hlm}$$
$$+(\mathcal{B}_m\mathcal{B}_r P_k)G^r_{hln}+(\mathcal{B}_m\mathcal{B}_n P_r)G^r_{hlk}-(\dot\partial_h a_{lmn})P_k=0,$$

respectively.

Proof. Differentiating (3.5) partially with respect to y^h, we get

$$\dot\partial_h(\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P_{jk})=(\dot\partial_h a_{lmn})\ P_{jk}+a_{lmn}\dot\partial_h P_{jk}$$

Using the commutation formula exhibited by (2.9) for the tensor $(\mathcal{B}_m\mathcal{B}_n P_{jk})$ in the above equation, we get

$$\mathcal{B}_l\dot\partial_h\left(\mathcal{B}_m\mathcal{B}_n P_{jk}\right)-(\mathcal{B}_r\mathcal{B}_n P_{jk})G^r_{hlm}-(\mathcal{B}_m\mathcal{B}_r P_{jk})G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_{rk})G^r_{hlj}$$
$$-(\mathcal{B}_m\mathcal{B}_n P_{jr})G^r_{hlk}=(\dot\partial_h a_{lmn})\ P_{jk}+a_{lmn}\dot\partial_h P_{jk}.$$

Again applying the commutation formula exhibited by (2.9) for the tensor $(\mathcal{B}_n P_{jk})$ in the above equation, we get

$$\mathcal{B}_l[\mathcal{B}_m\dot\partial_h\left(\mathcal{B}_n P_{jk}\right)-(\mathcal{B}_r P_{jk})G^r_{hmn}-(\mathcal{B}_n P_{rk})G^r_{hmj}-(\mathcal{B}_n P_{jr})G^r_{hmk}]-(\mathcal{B}_r\mathcal{B}_n P_{jk})G^r_{hlm}$$
$$-(\mathcal{B}_m\mathcal{B}_r P_{jk})G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_{rk})G^r_{hlj}-(\mathcal{B}_m\mathcal{B}_n P_{jr})G^r_{hlk}=(\dot\partial_h a_{lmn})\ P_{jk}+a_{lmn}\dot\partial_h P_{jk}.$$

Also, applying the commutation formula exhibited by (2.9) for the tensor P_{jk} in the above equation, we get

$$\mathcal{B}_l[\mathcal{B}_m\{\mathcal{B}_n\dot\partial_h P_{jk}-P_{rk}G^r_{hnj}-P_{jr}G^r_{hnk}\}-(\mathcal{B}_r P_{jk})G^r_{hmn}-(\mathcal{B}_n P_{rk})G^r_{hmj}-(\mathcal{B}_n P_{jr})G^r_{hmk}]$$
$$-(\mathcal{B}_r\mathcal{B}_n P_{jk})G^r_{hlm}-(\mathcal{B}_m\mathcal{B}_r P_{jk})G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_{rk})G^r_{hlj}-(\mathcal{B}_m\mathcal{B}_n P_{jr})G^r_{hlk}=(\dot\partial_h a_{lmn})\ P_{jk}+a_{lmn}\dot\partial_h P_{jk},$$

which can be written as

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(\dot\partial_h P_{jk})-(\mathcal{B}_l\mathcal{B}_m P_{rk})G^r_{hnj}-(\mathcal{B}_l P_{rk})(\mathcal{B}_m G^r_{hnj})-(\mathcal{B}_m P_{rk})(\mathcal{B}_l G^r_{hnj})-P_{rk}(\mathcal{B}_l\mathcal{B}_m G^r_{hnj})$$

$$-(\mathcal{B}_l\mathcal{B}_m P_{jr})G^r_{hnk}-(\mathcal{B}_l P_{jr})(\mathcal{B}_m G^r_{hnk})-(\mathcal{B}_m P_{jr})(\mathcal{B}_l G^r_{hnk})-P_{jr}(\mathcal{B}_l\mathcal{B}_m G^r_{hnk})-(\mathcal{B}_l\mathcal{B}_r P_{jk})G^r_{hmn}$$
$$-(\mathcal{B}_r P_{jk})(\mathcal{B}_l G^r_{hmn})-(\mathcal{B}_l\mathcal{B}_n P_{rk})G^r_{hmj}-(\mathcal{B}_n P_{rk})(\mathcal{B}_l G^r_{hmj})-(\mathcal{B}_l\mathcal{B}_n P_{jr})G^r_{hmk}-(\mathcal{B}_n P_{jr})(\mathcal{B}_l G^r_{hmk})$$
$$-(\mathcal{B}_r\mathcal{B}_n P_{jk})\,G^r_{hlm}-(\mathcal{B}_m\mathcal{B}_r P_{jk})\,G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_{rk})\,G^r_{hlj}-(\mathcal{B}_m\mathcal{B}_n P_{jr})\,G^r_{hlk}=(\dot\partial_h a_{lmn})\,P_{jk}+a_{lmn}\dot\partial_h P_{jk}.$$

This shows that

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\dot\partial_h P_{jk}\right)=a_{lmn}\left(\dot\partial_h P_{jk}\right) \tag{3.15}$$

if and only if (3.13) holds.
Differentiating (3.6) partially with respect to y^h, we get

$$\dot\partial_h(\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P_k)=(\dot\partial_h a_{lmn})P_k+a_{lmn}\dot\partial_h P_k.$$

Using the commutation formula exhibited by (2.9) for the tensor $(\mathcal{B}_m\mathcal{B}_n P_k)$ in the above equation, we get

$$\mathcal{B}_l\dot\partial_h\left(\mathcal{B}_m\mathcal{B}_n P_k\right)-(\mathcal{B}_r\mathcal{B}_n P_k)\,G^r_{hlm}-(\mathcal{B}_m\mathcal{B}_r P_k)\,G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_r)\,G^r_{hlk}=(\dot\partial_h a_{lmn})P_k+a_{lmn}\dot\partial_h P_k.$$

Again applying the commutation formula exhibited by (2.9) for the tensor $(\mathcal{B}_n P_k)$ in the above equation, we get

$$\mathcal{B}_l[\mathcal{B}_m\dot\partial_h\left(\mathcal{B}_n P_k\right)-(\mathcal{B}_r P_k)\,G^r_{hmn}-(\mathcal{B}_n P_r)\,G^r_{hmk}]-(\mathcal{B}_r\mathcal{B}_n P_k)\,G^r_{hlm}-(\mathcal{B}_m\mathcal{B}_r P_k)\,G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_r)\,G^r_{hlk}$$
$$=(\dot\partial_h a_{lmn})\,P_k+a_{lmn}\dot\partial_h P_k.$$

Also, applying the commutation formula exhibited by (2.9) for the tensor P_k in the above equation, we get

$$\mathcal{B}_l[\mathcal{B}_m\{\mathcal{B}_n\dot\partial_h P_k-P_r G^r_{hnk}\}-(\mathcal{B}_r P_k)\,G^r_{hmn}-(\mathcal{B}_n P_r)\,G^r_{hmk}]-(\mathcal{B}_r\mathcal{B}_n P_k)\,G^r_{hlm}$$
$$-(\mathcal{B}_m\mathcal{B}_r P_k)\,G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_r)\,G^r_{hlk}=(\dot\partial_h a_{lmn})\,P_k+a_{lmn}\dot\partial_h P_k.$$

which can be written as

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(\dot\partial_h P_k)-(\mathcal{B}_l\mathcal{B}_m P_r)G^r_{hnk}-(\mathcal{B}_l P_r)(\mathcal{B}_m G^r_{hnk})-(\mathcal{B}_m P_r)(\mathcal{B}_l G^r_{hnk})-P_r(\mathcal{B}_l\mathcal{B}_m G^r_{hnk})$$
$$-(\mathcal{B}_r\mathcal{B}_l P_k)\,G^r_{hmn}-(\mathcal{B}_r P_k)\,(\mathcal{B}_l G^r_{hmn})-(\mathcal{B}_l\mathcal{B}_n P_r)\,G^r_{hmk}-(\mathcal{B}_n P_r)\,(\mathcal{B}_l G^r_{hmk})-(\mathcal{B}_r\mathcal{B}_n P_k)\,G^r_{hlm}$$
$$-(\mathcal{B}_m\mathcal{B}_r P_k)\,G^r_{hln}-(\mathcal{B}_m\mathcal{B}_n P_r)\,G^r_{hlk}=(\dot\partial_h a_{lmn})\,P_k+a_{lmn}\dot\partial_h P_k.$$

This shows that

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\dot\partial_h P_k\right)=a_{lmn}\left(\dot\partial_h P_k\right) \tag{3.16}$$

if and only if (3.14) holds.
(3.15) and (3.16) show that the tensors $\left(\dot\partial_h P_{jk}\right)$ and $\left(\dot\partial_h P_k\right)$ behave as trirecurrent if and only if (3.13) and (3.14) hold. □

In the next theorems we obtain the relation between the covariant derivative in the sense of Berwald and Cartan for different tensors.

Theorem 3.9. *In $G\,(\mathcal{B}P)-TRF_n$, when the covariant derivative in the sense of Berwald with respect to x^l, x^m coincide with the $h-$covariant derivative in the sense of Cartan with respect to x^n, then P^i_{jkh} is given by*

$$\mathcal{B}_l\mathcal{B}_m P^i_{jkh|n}=a_{lmn}P^i_{jkh}+b_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)-2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right) \tag{3.17}$$
$$-2\,y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)$$

is a generalized birecurrent if and only if

$$\mathcal{B}_l\mathcal{B}_m(P^r_{jkh}P^i_{rn}-P^i_{rkh}P^r_{jn}-P^i_{jrh}P^r_{kn}-P^i_{jkr}P^r_{hn})=0. \tag{3.18}$$

Proof. Using the definition of the covariant derivative in the sense of Berwald for P^i_{jkh} exhibited by (2.8) with respect to x^l, we get

$$\mathcal{B}_n P^i_{jkh} = \partial_n P^i_{jkh} + P^r_{jkh} G^i_{rn} - P^i_{rkh} G^r_{jn} - P^i_{jrh} G^r_{kn} - P^i_{jkr} G^r_{hn} - \left(\dot{\partial}_r P^i_{jkh}\right) G^r_n .$$

Using (2.13) in the above equation, we get

$$\mathcal{B}_n P^i_{jkh} = \partial_n P^i_{jkh} + P^r_{jkh} \Gamma^{*i}_{rn} - P^i_{rkh} \Gamma^{*r}_{jn} - P^i_{jrh} \Gamma^{*r}_{kn} - P^i_{jrh} \Gamma^{*r}_{hn} - \left(\dot{\partial}_r P^i_{jkh}\right) \mathrm{G}^r_n$$
$$+ P^r_{jkh} P^i_{rn} - P^i_{rkh} P^r_{jn} - P^i_{jrh} P^r_{kn} - P^i_{jkr} P^r_{hn}.$$

Using the definition of h-covariant derivative in the sense of Cartan exhibited by (2.5) with respect to x^n in the above equation, we get

$$\mathcal{B}_n P^i_{jkh} = P^i_{jkh|n} + P^r_{jkh} P^i_{rn} - P^i_{rkh} P^r_{jn} - P^i_{jrh} P^r_{kn} - P^i_{jkr} P^r_{hn}. \tag{3.19}$$

Taking the $\mathcal{B}$-covariant derivative for (3.19) twice with respect to x^m and x^l, using the condition (3.1), we get

$$\mathcal{B}_l \mathcal{B}_m P^i_{jkh|n} = a_{lmn} P^i_{jkh} + b_{lmn} \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2y^t c_{mn} \mathcal{B}_t \left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right)$$
$$-2\, y^t d_{ln} \mathcal{B}_t \left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t \left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right)$$
$$+ \mathcal{B}_l \mathcal{B}_m (P^r_{jkh} P^i_{rn} - P^i_{rkh} P^r_{jn} - P^i_{jrh} P^r_{kn} - P^i_{jkr} P^r_{hn})$$

Thus, we obtain (3.17) if and only if (3.18) holds. Hence, we have proved this theorem. □

Theorem 3.10. *In $G\,(\mathcal{B}P) - TRF_n$, when the covariant derivative in the sense of Berwald with respect to x^l, x^m coincide with the $h-$ covariant derivative in the sense of Cartan with respect to x^n, then P^i_{kh} is given by*

$$\mathcal{B}_l \mathcal{B}_m P^i_{kh|n} = a_{lmn} P^i_{kh} + b_{lmn} \left(y^i g_{kh} - \delta^i_k y_h\right) - 2 y^t c_{mn} \mathcal{B}_t \left(y^i C_{khl}\right) \tag{3.20}$$
$$-2\, y^t d_{ln} \mathcal{B}_t \left(y^i C_{khm}\right) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t \left(y^i C_{khm}\right)$$

if and only if

$$\mathcal{B}_l \mathcal{B}_m (P^r_{kh} P^i_{rn} - P^i_{rh} P^r_{kn} - P^i_{kr} P^r_{hn}) = 0. \tag{3.21}$$

Proof. Transvecting (3.19) by y^j, using (2.6), (2.10) and (2.12), we get

$$\mathcal{B}_n P^i_{kh} = P^i_{kh|n} + P^r_{kh} P^i_{rn} - P^i_{rh} P^r_{kn} - P^i_{kr} P^r_{hn} . \tag{3.22}$$

Taking the $\mathcal{B}-$covariant derivative of (3.22) twice with respect to x^m and x^l, using (3.2), we get

$$\mathcal{B}_l \mathcal{B}_m P^i_{kh|n} = a_{lmn} P^i_{kh} + b_{lmn} \left(y^i g_{kh} - \delta^i_k y_h\right) - 2y^t c_{mn} \mathcal{B}_t \left(y^i C_{khl}\right) - 2y^t d_{ln} \mathcal{B}_t \left(y^i C_{khm}\right)$$
$$-2y^t \mu_n \mathcal{B}_l \mathcal{B}_t \left(y^i C_{khm}\right) + \mathcal{B}_l \mathcal{B}_m (P^r_{kh} P^i_{rn} - P^i_{rh} P^r_{kn} - P^i_{kr} P^r_{hn}).$$

Thus, we obtain (3.20) if and only if (3.21) holds. Hence, we have proved this theorem. □

Now, we have a corollary related to the previous theorems. Contracting the indices i and h in (3.19) and using (2.12), we get

$$\mathcal{B}_n P_{jk} = P_{jk|n} + P^r_{jki} P^i_{rn} - P_{rk} P^r_{jn} - P_{jr} P^r_{kn} - P^i_{jkr} P^r_{in} .$$

Taking the $\mathcal{B}$-covariant derivative of the above equation twice with respect to x^m and x^l, using (3.5), we get

$$\mathcal{B}_l \mathcal{B}_m P_{jk|n} = a_{lmn} P_{jk}$$

if and only if

$$\mathcal{B}_l \mathcal{B}_m (P_{rk} P^r_{jn} - P_{jr} P^r_{kn}) = 0. \tag{3.23}$$

Contracting the indices i and h in (3.22) and using (2.12), we get

$$\mathcal{B}_n P_k = P_{k|n} + P^r_{ki} P^i_{rn} - P_r P^r_{kn} - P^i_{kr} P^r_{in} \ .$$

Taking the $\mathcal{B}$-covariant derivative of the above equation twice with respect to x^m and x^l, using (3.6), we get

$$\mathcal{B}_l \mathcal{B}_m P_{k|n} = a_{lmn} P_k$$

if and only if

$$\mathcal{B}_l \mathcal{B}_m (P_r P^r_{kn}) = 0. \tag{3.24}$$

Thus, we conclude the following corollary:

Corollary 3.11. *The covariant derivative in the sense of Berwald with respect to x^l, x^m of the h-covariant derivative in the sense of Cartan with respect to x^n for P_{jk} and P_k behave as trirecurrent if and only if (3.23) and (3.24) hold, respectively.*

4 Conclusion

We introduced a Finsler space that P^i_{jkh} satisfies the generalized trirecurrence property in $G(\mathcal{B}R) - TRF_n$. Furthermore, we found that the Berwald's covariant derivatives of the third order for some tensors are non - vanishing. The relationship between P^i_{jkh} and R^i_{jkh} is studied in the above mentioned space.

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