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The equation for the wavefunction in nonrelativistic quantum mechanics *

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Abstract The momentum operator may be generalized for the correspondence of the limit of quantum commutators of observables with the classical commutators. The substitution of the momentum operator into the nonrelativistic equation for the conservation of energy yields a generalization of the stationary wave equation. Replacing also the energy operator, a generalization of the time-dependent Schrödinger equations, there exists a special class of wavefunctions that satisfies the nonrelativistic wave equation in quantum theory.

Key words commutator, generalized momentum operator, wave equation.

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1 Introduction

The wavefunction determines the probability distribution for the location of a particle which belongs to space of square integrable functions and it is normalized such that the integral of the probability density equals one. The development of quantum mechanics begins with the operator formulation of the fundamental coordinate and momentum operators, which then provide a definition of the Hamiltonian. When these operators are Hermitian, the eigenvalues are real and represent the measurements of real observables. The operators have nontrivial commutator relations [1] in contrast with the multiplication of complex numbers and there exists a differential operator which can represent the momentum. The equation for the conservation of energy may then be written as a second-order differential equation for the wavefunction [2].
The equivalence between the limit $\hbar \to 0$ of the quantum commutators and the classical commutator defined by the Poisson bracket requires a generalization of the differential operator representing the momentum. The replacement of the coefficient $-i\hbar$ of the derivative $\frac{d}{dx}$ by $\kappa_0 - i\hbar$ does not provide a Hermitian operator, and another formula is required [3]. Given the new expression for the momentum operator, a new wave equation is derived from the energy principle. It is a matrix equation and can be written as a system of four differential equations for the components of a spinor.
Given a set of conditions on the components, the differential equations have different forms. There is one condition that yields the Schrödinger equation. An interpretation of these relations is given for

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radial potentials. The nonrelativistic wave equation for these potentials is recovered when one of the coordinate directions is identified with the radial coordinate.
When the wavefunction represents a spinor field such as the electron, the matrix equation is a system of coupled differential equations for the components. Various conditions such as the reality of the spinor, Majorana and anti-Majorana conditons yield a reduced set of equations in §4. It is found that the effective mass is decreased by a factor of 3 which occurs for real Majorana fermions. The time-dependent nonrelativistic wave equations are formulated in §4. A difference between the solutions for Majorana and anti-Majorana spinors is described.

2 A nonrelativistic equation for the wavefunction

The generalization of the quantum commutator of the coordinate and momentum operators [3] for correspondence with the classical commutator [4–8] follows from $\tilde{p}_i = \kappa_0\gamma^i - i\hbar\partial_i$ with the adjoint

$$\begin{aligned}\tilde{p}_i^\dagger &= \kappa_0(\gamma^i)^\dagger\overleftarrow{\partial_i} + i\hbar\overleftarrow{\partial_i} \\ &= \kappa_0\gamma^0\gamma^i\gamma^0\overleftarrow{\partial_i} + i\hbar\overleftarrow{\partial_i} \\ &= -\kappa_0\gamma^i\overleftarrow{\partial_i} + i\hbar\overleftarrow{\partial_i}\,,\end{aligned} \tag{2.1}$$

which is equivalent in the space of square integrable functions, with the boundary conditions of the wavefunction and its derivative vanishing at infinity, to

$$\kappa_0\gamma^i\partial_i - i\hbar\partial_i. \tag{2.2}$$

The momentum operator is hermitian and the product with the adjoint is

$$\begin{aligned}\tilde{p}_i^\dagger\tilde{p}_i &= (\kappa_0\gamma^i\partial_i - i\hbar\partial_i)^\dagger(\kappa_0\gamma^i\partial_i - i\hbar\partial_i) \\ &= (\kappa_0\gamma^i\partial_i - i\hbar\partial_i)^2 \\ &= -(\kappa_0^2\mathbb{I} + \hbar^2)\partial_i^2 - 2\kappa_0\hbar\gamma^i\partial_i^2.\end{aligned} \tag{2.3}$$

The nonrelativistic energy conservation law

$$\frac{p^2}{2m} + V(x) = E \tag{2.4}$$

would be translated to

$$\sum_i\left[-\frac{(\kappa_0^2+\hbar^2)}{2m}\partial_i^2\psi - i\frac{\kappa_0\hbar}{m}\gamma^i\partial_i^2\psi\right] + V(x)\psi = E\psi. \tag{2.5}$$

The Schrödinger equation results in the limit $\kappa_0 \to 0$.
It may be recalled that the generalized uncertainty relation is

$$\mathrm{Tr}(\Delta x_i\Delta p_j) \geq \frac{\sqrt{\kappa_0^2+\hbar^2}}{2}\delta_{ij} \tag{2.6}$$

after tracing matrix components in the momentum operator. This inequality can be verified for $\kappa_0 = \hbar$. It can be verified also to be equivalent to the uncertainty relation for matrix elements in the product $\Delta x_i\Delta p_j \geq \frac{\sqrt{\kappa_0^2+\hbar^2}}{2}$ with κ_0 replaced by $\hbar$ in the traced relation.
If κ_0 is set equal to $\hbar$, the matrix differential equation is

$$-\frac{\hbar^2}{m}\nabla^2\psi - i\frac{\hbar^2}{2m}\sum_i\gamma^i\partial_i^2\psi + V(x)\psi = E\psi. \tag{2.7}$$

Given the representation of the gamma matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix} \tag{2.8}$$

where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
The Schrödinger equation can be derived only in one dimension when $\kappa_0 = \hbar$ with the coordinate being x_2 and the central elements being selected. Then

$$-\frac{\hbar^2}{m}\frac{d^2\psi}{dy^2} - \frac{i\hbar}{2m}(i)\frac{d^2\psi}{dy^2} + V(y)\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dy^2} + V(y)\psi = E\psi. \tag{2.9}$$

This equation is compatible with the solutions to the nonrelativistic wave equation with a radial potential and the coordinate y being identified with r.
The spectrum of the hydrogen atom, for example, can be found with the radial direction aligned along the y axis. However, the system of differential equations with the generalized momentum operator can generate new effects. The solutions to the Schrödinger equation for the hydrogen atom consist of products of Legendre polynomials and spherical harmonics. The directions $\hat{x}^1$ and $\hat{x}^3$ introduce an imaginary term in the differential equation which can produce damped solutions. Amongst the real equations, there are two. In addition to the conventional Schrödinger equation, there is an equation of the form

$$-\frac{3\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\tilde{\psi} + V(x)\tilde{\psi} = E\tilde{\psi}. \tag{2.10}$$

The mass m has been replaced by $\frac{m}{3}$. The Legendre polynomials increase with powers of m and would be larger for a particle with mass m rather than $\frac{m}{3}$.
Consider another entry 1 in a gamma matrix γ^i, $i = 1, 2, 3$. Then there is a component of the equation that has the form

$$-\frac{\hbar^2}{2m}(2+i)\partial_i^2\hat{\psi} + V(x)\hat{\psi} = E\hat{\psi}. \tag{2.11}$$

3 A system of differential equations in the quantum theory

These equations may be systematized by considering a column vector representation $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$,

and the matrix differential equation is

$$-\frac{\kappa_0^2 + \hbar^2}{2m}\nabla^2 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} - \frac{i\kappa_0\hbar}{m} \begin{pmatrix} 0 & 0 & \partial_3^2 & \partial_1^2 - i\partial_2^2 \\ 0 & 0 & \partial_1^2 - i\partial_2^2 & \partial_3^2 \\ \partial_3^2 & \partial_1^2 + i\partial_2^2 & 0 & 0 \\ \partial_1^2 - i\partial_2^2 & \partial_3^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{3.1}$$
$$+ V(x)\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

This matrix equation is equivalent to the following system of differential equations

$$\begin{aligned}
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_1 - i\frac{\kappa_0\hbar}{m}\partial_3^2\psi_3 - i\frac{\kappa_0\hbar}{m}\partial_1^2\psi_4 - \frac{\kappa_0\hbar}{m}\partial_2^2\psi_4 + V(x)\psi_1 = E\psi_1,\\
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_2 - i\frac{\kappa_0\hbar}{m}\partial_1^2\psi_3 + \frac{\kappa_0\hbar}{m}\partial_2^2\psi_3 + i\frac{\kappa_0\hbar}{m}\partial_3^2\psi_4 + V(x)\psi_2 = E\psi_2,\\
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_3 + i\frac{\kappa_0\hbar}{m}\partial_3^2\psi_1 + i\frac{\kappa_0\hbar}{m}\partial_1^2\psi_2 + \frac{\kappa_0\hbar}{m}\partial_2^2\psi_2 + V(x)\psi_1 = E\psi_3,\\
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_4 + i\frac{\kappa_0\hbar}{m}\partial_1^2\psi_1 - \frac{\kappa_0\hbar}{m}\partial_2^2\psi_1 - i\frac{\kappa_0\hbar}{m}\partial_3^2\psi_2 + V(x)\psi_4 = E\psi_4.
\end{aligned} \tag{3.2}$$

When $\kappa_0 = \hbar$,

$$\begin{aligned}
&-\frac{\hbar^2}{m}\nabla^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_3 - i\frac{\hbar^2}{2m}\partial_1^2\psi_4 - \frac{\hbar^2}{2m}\partial_2^2\psi_4 + V(x)\psi_1 = E\psi_1,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_2 - i\frac{\hbar^2}{2m}\partial_1^2\psi_3 + \frac{\hbar^2}{2m}\partial_2^2\psi_3 + i\frac{\hbar^2}{2m}\partial_3^2\psi_4 + V(x)\psi_2 = E\psi_2,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_3 + i\frac{\hbar^2}{2m}\partial_3^2\psi_1 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_3 = E\psi_3,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_4 + i\frac{\hbar^2}{2m}\partial_1^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2 + V(x)\psi_4 = E\psi_4.
\end{aligned} \tag{3.3}$$

To obtain the equations depending only on the coordinate x_2, let $\partial_1\psi = \partial_3\psi = 0$. Then the system of equations in flat space is

$$\begin{aligned}
&-\frac{\hbar^2}{m}\partial_2^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_4 + V(x)\psi_1 = E\psi_1,\\
&-\frac{\hbar^2}{m}\partial_2^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_3 + V(x)\psi_2 = E\psi_2,\\
&-\frac{\hbar^2}{m}\partial_2^2\psi_3 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_3 = E\psi_3,\\
&-\frac{\hbar^2}{m}\partial_2^2\psi_4 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_4 = E\psi_4.
\end{aligned} \tag{3.4}$$

If $\psi_1 = \psi_4$ and $\psi_2 = \psi_3$, the equations reduce to

$$\begin{aligned}
&-\frac{3\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_1 = E\psi_1\\
&-\frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_2 = E\psi_2,
\end{aligned} \tag{3.5}$$

which are the real equations in the x^2 coordinate found in §2.
Consider a spinor that satisfies the condition $\bar{\psi} = \psi^T C$, where C is the charge conjugation matrix $\begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$. Since $\bar{\psi} = \psi^\dagger\gamma^0 = (\psi_1^* \ \psi_2^* \ -\psi_3^* \ \psi_4^*)$ and $\psi^T C = (\psi_4 \ -\psi_3 \ \psi_2 \ -\psi_1)$, we have

$$\begin{aligned}
\psi_1^* &= \psi_4,\\
\psi_2^* &= -\psi_3,\\
-\psi_3^* &= \psi_2,\\
-\psi_4^* &= -\psi_1.
\end{aligned} \tag{3.6}$$

The last two equalities follow from the first two and there are two independent components in the spinor $\begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_2^* \\ \psi_1^* \end{pmatrix}$. It is equivalent to the spinor $\begin{pmatrix} \alpha_A \\ \tilde{\alpha}^{A'} \end{pmatrix}$ where $\alpha_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, since

$$\tilde{\alpha}^{A'} = \varepsilon^{A'B'}\tilde{\alpha}_{B'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1^* \\ \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} -\psi_2^* \\ \\ \psi_1^* \end{pmatrix}. \tag{3.7}$$

The set of differential equations for a spinor with $\psi_3 = -\psi_2^*$ and $\psi_4 = \psi_1^*$ is

$$\begin{aligned} -\frac{\hbar^2}{m}\nabla^2\psi_1 + i\frac{\hbar^2}{2m}\partial_3^2\psi_2^* - i\frac{\hbar^2}{2m}\partial_1^2\psi_1^* - \frac{\hbar^2}{2m}\partial_2^2\psi_1^* + V(x)\psi_1 = E\psi_1, \\ -\frac{\hbar^2}{m}\nabla^2\psi_2 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2^* - \frac{\hbar^2}{2m}\partial_2^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1^* + V(x)\psi_2 = E\psi_2, \\ \frac{\hbar^2}{m}\nabla^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 - V(x)\psi_2^* = -E\psi_2^*, \\ -\frac{\hbar^2}{m}\nabla^2\psi_1^* + i\frac{\hbar^2}{2m}\partial_1^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2 + V(x)\psi_1 = E\psi_1^*. \end{aligned} \tag{3.8}$$

The third equation in the set of equations (3.8) is the negative of the complex conjugate of the second and the fourth equation of this set is that of the first. There are two differential equations for two variables ψ_1, ψ_2 and the system is not overdetermined.
For real spinors with $\psi_1 = \psi_4$, $\psi_2 = -\psi_3$ and $\partial_1\psi = \partial_3\psi = 0$, the set of equations with real coefficients is

$$\begin{aligned} -\frac{\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_1 = E\psi_1, \\ -\frac{3\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_2 = E\psi_2. \end{aligned} \tag{3.9}$$

Suppose instead that $\psi_1 = -\psi_4$ and $\psi_2 = \psi_3$, which could be derived for real spinors and the identification of $\bar{\psi}$ with $-\psi^T$, then the equations for the two components become

$$\begin{aligned} -\frac{\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_1 = E\psi_1, \\ -\frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_2 = E\psi_2. \end{aligned} \tag{3.10}$$

4 The time-dependent wave equation in nonrelativistic quantum mechanics

When the Hamiltonian $H = -i\hbar\frac{\partial}{\partial t}$ is used in the time-dependent equation we get

$$-\frac{\hbar^2}{2m} + V(x)\psi = -i\hbar\frac{\partial}{\partial t}. \tag{4.1}$$

The commutator $\left[t, -i\hbar\frac{\partial}{\partial t}\right] f = t\left(-i\hbar\frac{\partial f}{\partial t}\right) + i\hbar\frac{\partial}{\partial t}(tf) = i\hbar f$, then $[t, H] = i\hbar$ and $\Delta E \Delta t \geq \frac{\hbar}{2}$.
Recovery of a real classical commutator in the limit $\hbar \to 0$ would suggest consideration of the operator $\kappa_0\gamma^0\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}$, which is an analogue of the momentum operator $\kappa_0\gamma^i\partial_i - i\hbar\frac{\partial}{\partial x_i}$. The adjoint of this operator is

$$\begin{aligned} (\kappa_0\gamma^0\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t})^\dagger &= \kappa_0(\gamma_0)^\dagger\frac{\overleftarrow{\partial}}{\partial t} + i\hbar\frac{\overleftarrow{\partial}}{\partial t}, \\ &= \kappa_0\gamma^0\frac{\overleftarrow{\partial}}{\partial t} + i\hbar\frac{\overleftarrow{\partial}}{\partial t}. \end{aligned} \tag{4.2}$$

In the space of square-integrable functions in the time interval $(-\infty, \infty)$, this operator is equivalent to $-\kappa_0\gamma^0\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}$. If κ_0 is real, this operator is non-Hermitian.
By contrast, the operator $i\kappa_0\gamma^0\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}$ is Hermitian. The commutator is

$$\left[t, i\kappa_0\gamma^0\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}\right] f = -(i\kappa_0\gamma^0 - i\hbar)f. \tag{4.3}$$

Since γ^0 is real, the commutator is purely imaginary and does not yield a real classical commutator in the limit $\hbar \to 0$.
Since the Poisson brackets in nonrelativistic quantum mechanics are defined as commutators of functions derivatives with respect to the position coordinates q^i and the momentum variables p^i, rather than $t,\ E$, it suffices to use $-i\hbar\frac{\partial}{\partial t}$ here for the Hamiltonian.
The system of differential equations for the nonrelativistic spinor field then would be

$$\begin{aligned}
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_1 - i\frac{\kappa_0}{2m}\partial_3^2\psi_3 - i\frac{\kappa_0\hbar}{2m}\partial_1^2\psi_4 - \frac{\kappa_0\hbar}{2m}\partial_2^2\psi_4 + V(x)\psi_1 = -i\hbar\frac{\partial\psi_1}{\partial t},\\
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_2 - i\frac{\kappa_0\hbar}{2m}\partial_1^2\psi_3 + \frac{\kappa_0\hbar}{2m}\partial_2^2\psi_3 + i\frac{\kappa_0\hbar}{2m}\partial_3^2\psi_4 + V(x)\psi_2 = -i\hbar\frac{\partial\psi_2}{\partial t},\\
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_3 + i\frac{\kappa_0\hbar}{2m}\partial_3^2\psi_1 + i\frac{\kappa_0\hbar}{2m}\partial_1^2\psi_2 + \frac{\kappa_0\hbar}{2m}\partial_2^2\psi_2 + V(x)\psi_3 = -i\hbar\frac{\partial\psi_3}{\partial t},\\
&-\frac{\kappa_0^2+\hbar^2}{2m}\nabla^2\psi_4 + i\frac{\kappa_0\hbar}{2m}\partial_1^2\psi_1 - \frac{\kappa_0\hbar}{2m}\partial_2^2\psi_1 - i\frac{\kappa_0\hbar}{2m}\partial_3^2\psi_2 + V(x)\psi_4 = -i\hbar\frac{\partial\psi_4}{\partial t}.
\end{aligned} \tag{4.4}$$

When $\kappa_0 = \hbar$,

$$\begin{aligned}
&-\frac{\hbar^2}{m}\nabla^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_3 - i\frac{\hbar^2}{2m}\partial_1^2\psi_4 - \frac{\hbar^2}{2m}\partial_2^2\psi_4 + V(x)\psi_1 = -i\hbar\frac{\partial\psi_1}{\partial t},\\
&-\frac{\hbar^2}{m}\nabla^2\psi_2 - i\frac{\hbar^2}{2m}\partial_1^2\psi_3 + \frac{\hbar^2}{2m}\partial_2^2\psi_3 + i\frac{\hbar^2}{2m}\partial_3^2\psi_4 + V(x)\psi_2 = -i\hbar\frac{\partial\psi_2}{\partial t},\\
&-\frac{\hbar^2}{m}\nabla^2\psi_3 + i\frac{\hbar^2}{2m}\partial_3^2\psi_1 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_3 = -i\hbar\frac{\partial\psi_3}{\partial t},\\
&-\frac{\hbar^2}{m}\nabla^2\psi_4 + i\frac{\hbar^2}{2m}\partial_1^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2 + V(x)\psi_4 = -i\hbar\frac{\partial\psi_4}{\partial t}.
\end{aligned} \tag{4.5}$$

For a complex spinor satisfying the condition $\bar{\psi} = \psi^T C$,

$$\begin{aligned}
&-\frac{\hbar^2}{m}\nabla^2\psi_1 + i\frac{\hbar^2}{2m}\partial_3^2\psi_2^* - i\frac{\hbar^2}{2m}\partial_1^2\psi_1^* - \frac{\hbar^2}{2m}\partial_2^2\psi_1^* + V(x)\psi_1 = -i\hbar\frac{\partial}{\partial t}\psi_1,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_2 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2^* - \frac{\hbar^2}{2m}\partial_2^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1^* + V(x)\psi_2 = -i\hbar\frac{\partial}{\partial t}\psi_2,\\
&\frac{\hbar^2}{m}\nabla^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 - V(x)\psi_2^* = i\hbar\frac{\partial}{\partial t}\psi_2^*,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_1^* + i\frac{\hbar^2}{2m}\partial_1^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2 + V(x)\psi_1^* = -i\hbar\frac{\partial}{\partial t}\psi_1^*.
\end{aligned} \tag{4.6}$$

when $\kappa_0 = \hbar$. The conjugate of the fourth equation and the negative of the conjugate of the third equation in (4.6) are

$$\begin{aligned}
&-\frac{\hbar^2}{m}\nabla^2\psi_1 + i\frac{\hbar^2}{2m}\partial_3^2\psi_2^* - i\frac{\hbar^2}{2m}\partial_1^2\psi_1^* - \frac{\hbar^2}{2m}\partial_2^2\psi_1^* + V(x)\psi_1 = i\hbar\frac{\partial}{\partial t}\psi_1,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_2 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2^* - \frac{\hbar^2}{2m}\partial_2^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1^* + V(x)\psi_2 = i\hbar\frac{\partial}{\partial t}\psi_2.
\end{aligned} \tag{4.7}$$

Addition of these equations to the previous equations for ψ_1 and ψ_2 yields

$$\begin{aligned}
&-\frac{\hbar^2}{m}\nabla^2\psi_1 + i\frac{\hbar^2}{2m}\partial_3^2\psi_2^* - i\frac{\hbar^2}{2m}\partial_1^2\psi_1^* - \frac{\hbar^2}{2m}\partial_2^2\psi_1^* + V(x)\psi_1 = 0,\\
&-\frac{\hbar^2}{m}\nabla^2\psi_2 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2^* - \frac{\hbar^2}{2m}\partial_2^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1^* + V(x)\psi_2 = 0.
\end{aligned} \tag{4.8}$$

which have a dependence only on the spatial coordinates.

Suppose instead that the spinor satisfies the condition $\bar{\psi} = -\psi^T C$. Then $\psi_3 = \psi_2^*$, $\psi_4 = -\psi_1^*$ and the four differential equations are

$$\begin{aligned}
-\frac{\hbar^2}{m}\nabla^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_1^2\psi_1^* + \frac{\hbar^2}{2m}\partial_2^2\psi_1^* + V(x)\psi_1 &= -i\hbar\frac{\partial}{\partial t}\psi_1, \\
-\frac{\hbar^2}{m}\nabla^2\psi_2 - i\frac{\hbar^2}{2m}\partial_1^2\psi_2^* + \frac{\hbar^2}{2m}\partial_2^2\psi_2^* - i\hbar^2 2m\partial_3^2\psi_1^* + V(x)\psi_2 &= -i\hbar\frac{\partial}{\partial t}\psi_2, \\
\frac{\hbar^2}{m}\nabla^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_2^* &= -i\hbar\frac{\partial}{\partial t}\psi_2^*, \\
\frac{\hbar^2}{m}\nabla^2\psi_1^* + i\frac{\hbar^2}{2m}\partial_1^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 - i\hbar^2 2m\partial_3^2\psi_2 - V(x)\psi_1^* &= i\hbar\frac{\partial}{\partial t}\psi_1^*.
\end{aligned} \tag{4.9}$$

Adding the negative of the conjugate of the third equation to the second equation and the conjugate of the fourth equation to the first equation in (4.9) gives

$$\begin{aligned}
-\frac{\hbar^2}{m}\nabla^2\psi_1 + \frac{\hbar^2}{2m}\partial_2^2\psi_1^* &= -i\hbar\frac{\partial}{\partial t}\psi_1, \\
-\frac{\hbar^2}{m}\nabla^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2^* &= -i\hbar\frac{\partial}{\partial t}\psi_2.
\end{aligned} \tag{4.10}$$

The general solution is a function that is harmonic in the variables x_1, x_2 and solves the heat equation in the variables x_2, t. For real spinors, these equations with $\partial_1\psi = \partial_3\psi = 0$ reduce to the Schrödinger equation in the coordinate x_2 with no potential

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_2^2} = -i\hbar\frac{\partial}{\partial t}\psi. \tag{4.11}$$

which is solved by theta functions of x_2 and t.

5 Conclusion

The differential equation for the wavefunction in quantum mechanics is generalized to a matrix equation with a new Hermitian operator representing the momentum variable. It is demonstrated that the system of equations can be reduced to the Schrödinger equation when the coefficient κ_0 of the gamma matrix γ^i equals zero. If $\kappa_0 = \hbar$, the nonrelativistic wave equation of quantum mechanics is derived after $\bar{\psi}$ is equated with $-\psi^T C$, with C being the standard charge conjugation matrix and the spinor is required to be real. Under other conditions, different wave equations are found for some of the components yielding quantum effects of lesser magnitude.

The time-independence of the differential equations for Majorana spinors yields stationary solutions, which have been experimentally detected [9] The dynamics of a real anti-Majorana fermion under no potential is governed by a heat equation with an imaginary specific heat. Replacing t by $\tau = it$, the heat equation then has a real specific heat. It may be recalled that thermal states are defined in Euclidean quantum field theory with a periodicity given by the inverse of the temperature. The equation for the real free anti-Majorana fermion therefore describes the thermal quantum states for period boundary conditions on the wavefunction.

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