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On the twin primes *

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Abstract This paper brings up a few possible approaches to solving the twin primes conjecture.

Key words Building-blocks, symmetry, infinity, composite numbers, unique factorization, prime factors.

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1 Introduction

The theory of numbers is an intriguing and fascinating field of mathematics [1–4]. Many believe that the twin primes are infinite. In fact, the twin primes pairs could easily be found among the integers. There is evidently no region of the natural number system so remote that it lies beyond the largest twin primes pair. It is even possible to forecast the approximate number of twin primes pairs found in any region of the natural number system.

The occurrence of twin primes pairs is evidently unpredictable or random. This means that the chance of two numbers x and $x+2$ being prime (twin primes) is somewhat similar to the chance of getting heads on two successive tosses of a coin. If two successive tosses of a coin are independent, the chance of success of obtaining heads for the two successive tosses of the coin is the product of the chances of success of obtaining a head for each toss of the coin. As each coin has probability $\frac{1}{2}$ of coming up heads with a toss, two coins would have probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of coming up a pair of heads with a toss.

The prime number theorem, which had been proven, states that if n is a large number, and we select a number x at random between 0 and n, the chance that x is prime would be approximately $\frac{1}{\log n}$, the larger n is, the better would be the approximation given by $\frac{1}{\log n}$ to the proportion of primes in the numbers up to n. Like two coins coming up heads, the chance that both x and $x+2$ are prime (twin primes) would be approximately $\frac{1}{(\log n)^2}$. That is, there would be approximately $\frac{n}{(\log n)^2}$ twin primes pairs between 0 and n. As n goes to infinity, this fraction approaches infinity. This represents a quantitative version of the twin primes conjecture.

As $x+2$ being prime depends on the fact that x is already prime, we should modify the estimate $\frac{n}{(\log n)^2}$ to $\frac{1.32032\ldots\times n}{(\log n)^2}$.

The following, Table 1, is a comparison between the twin primes predicted by the above formula and the twin primes found, where the agreement is evidently very good.

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Table 1: The number of predicted and found twin primes in intervals of large numbers.

Interval	Twin primes	
	Predicted	Found
$100,000,000 - 100,150,000$	584	601
$1,000,000,000 - 1,000,150,000$	461	466
$10,000,000,000 - 10,000,150,000$	374	389
$100,000,000,000 -$ $100,000,150,000$	309	276
$1,000,000,000,000 -$ $1,000,000,150,000$	259	276
$10,000,000,000,000 -$ $10,000,000,150,000$	221	208
$100,000,000,000,000 -$ $100,000,000,150,000$	191	186
$1,000,000,000,000,000 -$ $1,000,000,000,150,000$	166	161

All this represents the numerical evidence that the twin primes are infinite as we can find more twin primes pairs whenever we look for them. But the proof is lacking.

2 The main results

We explain now why the list of the twin primes pairs should be infinite.

Lemma 2.1. *According to the precepts of fractal geometry and group theory, symmetry is a very important intrinsic part of nature. There is symmetry all around us and within us. There is evident symmetry in human bodies, the structures of viruses and bacteria, polymers and ceramic materials, the permutations of numbers, the universe and many others, even the movements of prices in financial markets, the growths of populations, the sound of music, the flow of blood through our circulatory system, the behavior of people en masse, etc. In other words, regularity, pattern, order, uniformity or symmetry is evident everywhere.*

The reasoning here makes use of a very important idea in fractal geometry [5] and group theory, namely, symmetry [6].

A prime number is an integer which is divisible only by 1 and itself, e.g., $2, 3, 7, 19, \ldots$ etc. A twin primes pair consists of two primes which differ from one another by 2, e.g., 5 and 7, 11 and 13, 17 and 19, and, 29 and 31, etc. A composite number or non-prime is a product of primes or prime factors, e.g., the composite numbers 15 is the product of two primes, 3 and 5 (i.e., $15 = 3 \times 5$), and 231 is the product of three primes, 3, 7 and 11 (i.e., $231 = 3 \times 7 \times 11$), etc. The integers or whole numbers are either primes or composites and are infinite.

The primes, which Euclid had proven to be infinite, are the atoms or building-blocks of the infinite integers or whole numbers, which comprise of the infinite list of the odd numbers that are all either primes or products of primes (i.e., composites), and, the infinite list of the even numbers that are all products of primes (i.e., composites, with the exception of 2 which is a prime, e.g., $6 = 2 \times 3, 8 = 2 \times 2 \times 2$ and $10 = 2 \times 5$, etc.). The infinite list of the integers or whole numbers may be classified as an infinite group, with various symmetries, subgroups and infinite elements, hidden within it. These various symmetries, subgroups and infinite elements, within this infinite group may be classified as follows:-

1. **Subgroup A**: The infinite consecutive primes such as $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$ etc. to infinity, separated by 2 integers (twin primes), 4 integers, 6 integers, 8 integers, 10 integers, etc. to infinity, which, incidentally, except for 2, are all odd numbers; this splitting up of the subgroup into infinite elements is as shown below:

   1(i) Element A1: The infinite list of all the primes pairs separated by 2 integers (twin primes) ..., (e.g., 17 and 19);

   1(ii) Element A2: The infinite list of all the primes pairs separated by 4 integers/1 odd composite - single composite ..., (e.g., 79 and 83 separated by 81);

   1(iii) Element A3: The infinite list of all the primes pairs separated by 6 integers/2 consecutive odd composites - twin composites ..., (e.g., 47 and 53 separated by 49 and 51);

   1(iv) Element A4: The infinite list of all the primes pairs separated by 8 integers/3 consecutive odd composites - "triple" composites ..., (e.g., 359 and 367 separated by 361, 363 and 365);

   1(v) Element A5: The infinite list of all the primes pairs separated by 10 integers/4 consecutive odd composites - "quadruple" composites ..., (e.g., 709 and 719 separated by 711, 713, 715 and 717).

$$\vdots$$

2. **Subgroup B**: The infinite consecutive odd composites such as $9, 15, 21, 25, 27, 33, 35, 39, 45, 49, \ldots$ etc. to infinity, of "infinite sizes" sandwiched between 2 primes; this splitting up of the subgroup into infinite elements is shown below:

   2(i) Element B1: The infinite list of all "1 odd composite sandwiched between 2 primes - single composite" ..., (e.g., 9 sandwiched between the primes 7 and 11);

2(ii) Element B2: The infinite list of all "2 consecutive odd composites sandwiched between 2 primes - twin composites" ..., (e.g., 253 and 255 sandwiched between the primes 251 and 257);

2(iii) Element B3: The infinite list of all "3 consecutive odd composites sandwiched between 2 primes - "triple" composites" ..., (e.g. 685, 687 and 689 sandwiched between the primes 683 and 691);

2(iv) Element B4: The infinite list of all "4 consecutive odd composites sandwiched between 2 primes - "quadruple" composites" ..., (e.g., 2,769, 2,771, 2,773 and 2,775 sandwiched between the primes 2,767 and 2,777);

2(v) Element B5: The infinite list of all "5 consecutive odd composites sandwiched between 2 primes - "quintuple" composites" ..., (e.g., 19,291, 19,293, 19,295, 19,297 and 19,299 sandwiched between the primes 19,289 and 19,301).

$\vdots$

3. **Subgroup C**: The infinite consecutive odd composites separated by 4 integers and 6 integers respectively; this splitting up of the subgroup into the 2 infinite elements is shown below:

   3(i) Element C1: The infinite list of all "2 consecutive odd composites separated by 4 integers/1 prime" ..., (e.g., 209 and 213 separated by the prime 211);

   3(ii) Element C2: The infinite list of all "2 consecutive odd composites separated by 6 integers/2 primes" ..., (e.g., 279 and 285 separated by the twin primes 281 and 283).

4. **Subgroup D**: The infinite single primes and twin primes separating 2 consecutive odd composites; this splitting up of the subgroup into the 2 infinite elements is shown below:

   4(i) Element D1: The infinite list of all the single primes separating 2 consecutive odd composites ..., (e.g., 23 separating the 2 consecutive odd composites 21 and 25);

   4(ii) Element D2: The infinite list of all the twin primes separating 2 consecutive odd composites ..., (e.g., 11 and 13 separating the 2 consecutive odd composites 9 and 15).

5. **Subgroup E**: The infinite consecutive even composites such as $4, 6, 8, 10, 12, 14, 16, 18, 20, 22, \ldots$ etc. to infinity, all separated by only 2 integers; this subgroup may be classified as a single infinite element. There is always 1 even number between a twin primes pair, which is separated by 2 integers, a prime and a composite which are separated by 2 integers, and, 2 composites which are separated by 2 integers. That is, the even numbers are always found in the Subgroup A, Subgroup B, Subgroup C and Subgroup D above always evenly spaced out in consecutive order by 2 integers.

There is an evident symmetry in the above-mentioned infinite group, which would be broken if any of the elements within them were to be finite. There are close interlinks between all the various infinite elements in all the five subgroups above, e.g., the infinity of the list of all the primes pairs each separated by 6 integers (viz., the Element A3 of the Subgroup A) implies the infinity of the list of all the "2 consecutive odd composites sandwiched between 2 primes - twin composites", (the Element B2 of the Subgroup B) and vice versa, the infinity of the list of all the primes pairs each separated by 2 integers (twin primes) (the Element A1 of the Subgroup A) implies the infinity of the list of all the "2 consecutive odd composites separated by 6 integers/2 primes" (the Element C2 of the Subgroup C) and vice versa, the infinity of the list of all the infinite elements (e.g., the Element A1, Element A2, Element A3, etc. to infinity) in the Subgroup A above, which represents the infinity of the list of the primes which Euclid had in fact proven, implies the infinity of the list of all the "2 consecutive odd composites separated by 4 integers/1 prime", (the Element C1 of the Subgroup C) and vice versa, the infinity of the list of all the "2 consecutive odd composites separated by 6 integers/2 primes" (the Element C2 of the Subgroup C) implies the infinity of the list of all the twin primes separating 2 consecutive odd composites (the Element D2 of the Subgroup D) and vice versa, the infinity of all the lists of all the infinite elements (viz., the Elements A1, A2, A3, ..., B1, B2, B3 ..., C1 and C2, D1 and D2) in the Subgroup A, Subgroup B, Subgroup C and Subgroup D above respectively implies the infinity of the list of the consecutive even composites, i.e., $4, 6, 8, 10, 12, 14, 16, 18, 20, 22, \ldots$ to infinity (Subgroup E), which we know to be true in any case, and vice versa.

Subgroup A and Subgroup B above are practically “mirror” images of one another - they represent the viewing of the primes and the composites from 2 variant angles - the infinitude, or, finiteness of either implies the infinitude, or, finiteness of the other; the same applies to both Subgroup C and Subgroup D above. It is similar to the following way of viewing a glass which is partially filled: this glass could be described as “half full” or “half empty” if it is half filled, “three-quarter full” or “one-quarter empty” if it is three-quarter filled, or, “one-quarter full” or “three-quarter empty” if it is one-quarter filled, etc.
It is evident that the infinitude, or, finiteness of any one of the above-mentioned elements would imply the infinitude, or, finiteness of the other element that is interlinked with it and vice versa. All these infinite elements are evidently entangled together and complementary, being all the infinite building-blocks of the infinite integers or whole numbers. The infinity of the list of the integers or whole numbers, the primes included, in fact implies that all these various elements within it should be infinite, and, vice versa, since all these various elements are closely interlinked and could not do without each other. Therefore, the breaking of the evident intrinsic symmetry of this whole infinite group, i.e., the infinite list of the integers or whole numbers, due to the finiteness of any of the elements within it, could not be possible.
We now pose a very important question here: Besides questioning whether the infinite list of all the primes pairs separated by 2 integers (twin primes) is really infinite, should we not also be questioning whether the following are really infinite?:

(a) The infinite lists of all the primes pairs separated respectively by 4 integers, 6 integers, 8 integers, 10 integers and sequentially larger integers to infinity (as in the Subgroup A above).

(b) The infinite lists of all the respective consecutive odd composites of “infinite sizes” sandwiched between 2 primes (as in the Subgroup B above).

(c) The 2 infinite lists with respectively “2 consecutive odd composites separated by 4 integers/1 prime” and “2 consecutive odd composites separated by 6 integers/2 primes” (as in the Subgroup C above).

(d) The 2 infinite lists of respective single primes and twin primes separating 2 consecutive odd composites (as in the Subgroup D above).

(e) The infinite list of the consecutive even composites all separated by only 2 integers (as in the Subgroup E above).

Could there possibly be any symmetry-breaking in the above-mentioned infinite group whence one or more of the elements within it would be finite? In particular, could there be a possibility for the symmetry of this infinite group to be broken due to the finiteness of Element A1 (i.e., the finiteness of the twin primes) within it? Since the above-mentioned group, i.e., the list of the integers or whole numbers, is infinite, it is indeed not possible for all of these elements to be finite. And, there is no evident reason to account for why any of these elements, especially Element A1, i.e., the list of primes separated by 2 integers, or, twin primes, should be finite. In fact, all these infinite elements are like the slabs of various sizes in a building. They are all necessary for the construction of the infinite building known as the “infinite list of the integers or whole numbers” and should thus all be infinite, wherein the symmetry of the infinite group, i.e., the infinite list of the integers or whole numbers, would be preserved.
Therefore, by Lemma 2.1, all the elements in the subgroups - Subgroup A, Subgroup B, Subgroup C, Subgroup D and Subgroup E above should be infinite.

Lemma 2.2. *The Fundamental Theorem of Arithmetic or the Unique Factorization Theorem states that there is only one possible combination of primes which will multiply together to produce any particular composite number, e.g., the only combination of primes which will produce the composite number 2,079 is:* $3 \times 3 \times 3 \times 7 \times 11$. *In the same manner, the following composite numbers are also uniquely factorized:*

(2.2.1) $63 = 3 \times 3 \times 7$ *(only).*

(2.2.2) $153 = 3 \times 3 \times 17$ *(only).*

(2.2.3) $1,021,020 = 2 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$ *(only).*

In other words, every positive whole number which is not prime (i.e., every positive whole number which is composite) can be broken up into prime factors, and, this can happen in only one way: i.e.,

$$c = \prod_{p \text{ prime}} p \text{ (in only one way).}$$

The 10 consecutive twin primes 3 and 5 to 107 and 109, e.g., give rise to the following 10 composite numbers which can be factorized in only one way, i.e., these can be factorized only by the respective twin primes:

2.2(1) $15 = 3 \times 5$ (only).

2.2(2) $35 = 5 \times 7$ (only) .

2.2(3) $143 = 11 \times 13$ (only).

2.2(4) $323 = 17 \times 19$ (only).

2.2(5) $899 = 29 \times 31$ (only).

2.2(6) $1,763 = 41 \times 43$ (only).

2.2(7) $3,599 = 59 \times 61$ (only).

2.2(8) $5,183 = 71 \times 73$ (only).

2.2(9) $10,403 = 101 \times 103$ (only).

2.2(10) $11,663 = 107 \times 109$ (only).

$$\vdots$$

As the composite numbers are infinite, this implies that there should be an infinitude of twin primes acting as prime factors for an infinitude of composite numbers in only one way as the twin primes are indispensable, i.e., necessary, as prime factors for the formation of the composite numbers which can only be formed through the product of twin primes in only one way - the twin primes can never be substituted as prime factors of these composite numbers by other primes.

3 Conclusion

Thus, the twin primes are possibly infinite.

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