

## The classical limit of quantum commutation relations \*

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**Abstract** The commutation relations of quantum mechanics have a classical limit equal to the Poisson brackets if the coefficient is generalized to be a complex rather than purely imaginary. The effect on the uncertainty relations is described. The complex number may be identified with a modular variable and the quantization is derived from topology.

**Key words** quantum commutator, classical limit, generalized momentum, Hermiticity.

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### 1 Introduction

It is well known that the classical commutation relations, defined for the momentum and space variables through the Poisson bracket, are valid for any two differentiable functions on phase space. The quantum commutator is constructed to be the operator equivalent of the Poisson brackets, and represents the transformation from fields that are functions on the space-time coordinates to operator-valued distributions. A difference that is immediately evident in the commutation relations is the presence of real and imaginary classical and quantum brackets respectively. The limit of a vanishing coefficient of the quantum commutation relations is an Abelian algebra. Consistency with the Poisson brackets of classical field theory can be achieved if the quantum commutator is a complex number with a real classical limit.

This complex number might be regarded as a modular variable. Deformations of the complex structure on the torus are parametrized by the modulus with a range that is the upper half-plane. The formulation of the operator commutators in terms of this modular variable reflects a connection between quantization and topology. The discreteness of the topology of surfaces therefore will result in a quantization of the coefficients in the commutators and the spectrum of the operators.

### 2 A modular variable in the quantum commutation relations

The classical Poisson bracket of two functions on phase space,  $F(q, p)$  and  $G(q, p)$ , is

$$\{F, G\}_{P.B.} = \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \quad (2.1)$$

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and

$$\{q_i, p_j\}_{P.B.} = \delta_{ij}. \quad (2.2)$$

The quantum commutation relation between the position and momentum variables is given by the substitution  $p_i \rightarrow -i\hbar \frac{d}{dq_i}$ . Then

$$[q_i, p_j]f(q) \rightarrow q_i \left( -i\hbar \frac{d}{dq_j} \right) f(q) + \hbar \frac{d}{dq_j} (q_i f(q)) = i\hbar \delta_{ij} f(q) \quad (2.3)$$

and

$$[q_i, p_j] = i\hbar \delta_{ij} \quad (2.4)$$

as an operator relation. The limit  $\hbar \rightarrow 0$  of this commutator is an Abelian algebra and does not produce the classical Poisson bracket.

Therefore, a consistent classical limit is derived if the coefficient is replaced by a complex number,  $\kappa = \kappa_0 + i\hbar$ . Although  $\kappa_0$  would have to be set equal to one for the position and momentum coordinates, more general values could occur for other functions on phase space.

Suppose that  $\tilde{p}_j = -i\gamma^j \partial_j - i\hbar \partial_j$ . Then  $[x_i, \tilde{p}_j] = (i\gamma^j + i\hbar)\delta_{ij}$ . In the limit  $\hbar \rightarrow 0$ ,  $\tilde{p}_j \rightarrow -i\gamma^j \partial_j$ , where  $\gamma^j, j = 1, 2, 3$  are Dirac gamma matrices. The adjoint of this operator is

$$(\tilde{p}_j)_{\hbar \rightarrow 0}^\dagger = (-i\gamma^j \partial_j)^\dagger = i(\partial_j)^\dagger (\gamma^j)^\dagger = -i(\partial_j)^\dagger \gamma^j \quad (2.5)$$

since  $\gamma^j \dagger = \gamma^0 \gamma^j \gamma^0 = -\gamma^j$ . The adjoint of the commutation relation  $[x_i, (\tilde{p}_j)_{\hbar \rightarrow 0}] = i\gamma^j \delta_{ij}$  is

$$\begin{aligned} (x_i (\tilde{p}_j)_{\hbar \rightarrow 0} - (\tilde{p}_j)_{\hbar \rightarrow 0} x_i)^\dagger &= (\tilde{p}_j)_{\hbar \rightarrow 0}^\dagger x_i^\dagger - x_i^\dagger (\tilde{p}_j)_{\hbar \rightarrow 0}^\dagger \\ &= (\tilde{p}_j)_{\hbar \rightarrow 0}^\dagger x_i - x_i (\tilde{p}_j)_{\hbar \rightarrow 0}^\dagger \\ &= (i\gamma^j \delta_{ij})^\dagger = -i(\gamma^j)^\dagger \delta_{ij} = i\gamma^j \delta_{ij}. \end{aligned} \quad (2.6)$$

The consistency of the adjoint relation and that of the commutator  $[x_i, p_j] = -i\hbar \delta_{ij}$  both depend on the interpretation of the adjoint of the derivative operator.

The inner product for two quantum mechanical state vectors represented by complex wavefunctions is

$$\langle \psi | \eta \rangle = \int d^3 x \bar{\psi} \eta. \quad (2.7)$$

If the state vectors are spinors, then the integral is  $\int d^3 x \bar{\psi}^T \eta$ . Representing the Hilbert space as the space of square integrable wavefunctions in  $\mathbb{C}^n$ , where  $n$  is the number of components of the spinors,

$$\langle \psi A^\dagger | \eta \rangle = \int d^3 x \left( \overline{\psi A^\dagger} \right)^T \eta = \int d^3 x \overline{(A^\dagger \psi)}^T \eta = \int d^3 x \bar{\psi}^T A \eta = \langle \psi | A \eta \rangle. \quad (2.8)$$

Then  $\langle \psi A | \eta \rangle = \langle \psi | A \eta \rangle$  for a Hermitian operator.

The commutation relation  $[x_i, p_j] = i\hbar \delta_{ij}$  would be consistent under the adjoint operation only if  $p_j$  is Hermitian, which requires a differentiation between  $(\partial_j)^\dagger$  and  $\partial_j$ . It is evident for two square integral functions  $\psi_1, \psi_2$  which vanish at infinity, that

$$\begin{aligned} \int d^3 x \psi_1 \overleftarrow{\partial_j} \psi_2 &= - \int d^3 x \psi_1 \partial_j \psi_2 + \int dx_i dx_k \psi_1 \psi_2 \Big|_{\substack{x_j = -\infty \\ j \neq i, k}}^\infty. \\ &= - \int d^3 x \psi_1 \partial_j \psi_2. \end{aligned} \quad (2.9)$$

Therefore, the operator  $(\partial_j)^\dagger$  can be interpreted to be the negative of the operator  $\partial_j$  in the space of square integrable functions with the above inner product. It follows that the operator  $i\hbar \partial_j$  is equivalent to  $-i\hbar \overleftarrow{\partial_j}$ , and

$$\begin{aligned}
\langle \psi | \tilde{p}_j | \eta \rangle &= \int d^3x \psi(x) \overline{\left( -i\hbar \tilde{\partial}_j \right) \eta(x)} = \int d^3x i\hbar \partial_j \bar{\psi}(x) \eta(x) \\
&= \int d^3x \bar{\psi}(x) (-i\hbar \partial_j) \eta(x) = \langle \psi | p_j | \eta \rangle
\end{aligned} \tag{2.10}$$

after integration by parts, which verifies the Hermiticity of  $p_j$  as an operator in this space. Similarly,

$$\begin{aligned}
\langle \psi | (\tilde{p}_j)_{\hbar \rightarrow 0} | \eta \rangle &= \int d^3x \psi(x) \overline{\left( -i\gamma^j \tilde{\partial}_j \right) \eta(x)} = \int d^3x x \partial_j \bar{\psi}(x) i(\gamma^j)^\dagger \eta(x) \\
&= \int d^3x x \partial_j \bar{\psi}(x) (-i\gamma^j) \eta(x) = \int d^3x \bar{\psi}(x) i\gamma^j \partial_j \eta(x) \\
&= - \int d^3x \bar{\psi}(-i\gamma^j \partial_j) \eta(x) = - \langle \psi | (\tilde{p}_j)_{\hbar \rightarrow 0} | \eta \rangle
\end{aligned} \tag{2.11}$$

after integration by parts. Given the anti-Hermiticity of  $(\tilde{p}_j)_{\hbar \rightarrow 0}$ , the validity of (2.6) follows, with  $(\tilde{p}_j)_{\hbar \rightarrow 0} x_i - x_i (\tilde{p}_j)_{\hbar \rightarrow 0} = [x_i, (\tilde{p}_j)_{\hbar \rightarrow 0}]$ .

Suppose that the imaginary coefficient of the gamma matrix in the commutator is replaced by  $\kappa_0 \in \mathbb{R}$ . Then  $\tilde{p}_j = -\kappa_0 \gamma^j \partial_j - i\hbar \partial_j$  is Hermitian. The adjoint of the relation  $[x_i, \tilde{p}_j] = (\kappa_0 \gamma^j + i\hbar) \delta_{ij}$  yields

$$[x_i, \tilde{p}_j]^\dagger = \left\{ (\kappa_0 \gamma^j + i\hbar) \delta_{ij} \right\}^\dagger = (i\gamma^j - i\hbar) \delta_{ij}. \tag{2.12}$$

### 3 The uncertainty principle

The uncertainty principle [1] may be derived from the inequality

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle| \tag{3.1}$$

when  $\langle \{A, B\} \rangle \in \mathbb{R}$  and  $\langle [A, B] \rangle \in i\mathbb{R}$  [2], since  $AB = \frac{1}{2}\{A, B\} + \frac{1}{2}[A, B]$ , and then,  $|\langle AB \rangle|^2 = \frac{1}{4} \langle \{A, B\} \rangle^2 + \frac{1}{4} |\langle [A, B] \rangle|^2$ , since the Cauchy-Schwarz inequality  $|\langle AB \rangle|^2 \leq \langle A^2 \rangle \langle B^2 \rangle$  yields  $\frac{1}{4} |\langle [A', B'] \rangle|^2 \leq \langle A'^2 \rangle \langle B'^2 \rangle$ , which gives (2.1) after setting  $A' = A - \langle A \rangle \mathbb{I}$  and  $B' = B - \langle B \rangle \mathbb{I}$ .

Another inequality [3]

$$(\Delta A)^2 (\Delta B)^2 \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|^2 \tag{3.2}$$

may be used for the generalized commutation relations. Beginning with the commutator  $[x_i, \tilde{p}_j] = (\kappa_0 \gamma^j + i\hbar) \delta_{ij}$ , and including a trace for matrix elements in  $[x_i, \tilde{p}_i]$ ,

$$\begin{aligned}
\text{Tr} \left| \left\langle \frac{1}{2i} (\kappa_0 \gamma_i + i\hbar) \right\rangle \right|^2 &= \frac{1}{4} \text{Tr} \langle (-i\kappa_0 \gamma_i + \hbar)^\dagger (-i\kappa_0 \gamma_i + \hbar) \rangle \\
&= \frac{1}{4} \text{Tr} \langle (i\kappa_0 (\gamma^j)^\dagger + \hbar) (-i\kappa_0 \gamma^j + \hbar) \rangle \\
&= \frac{1}{4} \text{Tr} \langle (-i\kappa_0 (\gamma^j) + \hbar)^2 \rangle \\
&= \frac{1}{4} \text{Tr} \langle (-\kappa_0^2 (\gamma^j)^2 - 2i\kappa_0 \hbar \gamma^j + \hbar^2) \rangle \\
&= \frac{1}{4} \langle (\kappa_0^2 \mathbb{I} - 2i\kappa_0 \hbar \gamma^j + \hbar^2) \rangle \\
&= \frac{1}{4} (4\kappa_0^2 + \hbar^2).
\end{aligned} \tag{3.3}$$

It follows that

$$\text{Tr} (\Delta x_i)^2 (\Delta \tilde{p}_i)^2 \geq \frac{1}{4} (4\kappa_0^2 + \hbar^2). \tag{3.4}$$

Since  $\tilde{p}_i$  is represented by a differential operator that includes matrix coefficients,  $\Delta p_i$  also will have matrix coefficients multiplying the uncertainty in the eigenvalues of the derivative and the trace of the square will be a numerical value. Then the generalized uncertainty principle is

$$\left[ \text{Tr} (\Delta x_i)^2 (\Delta \tilde{p}_j)^2 \right]^{\frac{1}{2}} \equiv (\Delta x_i)(\Delta \tilde{p}_j)_{\text{Tr}} \geq \frac{1}{2}(4\kappa_0^2 + \hbar^2)^{\frac{1}{2}} \delta_{ij}. \quad (3.5)$$

Since  $\kappa_0$  has the units of action, its minimum value can be taken to be  $\hbar$ . Then the lower bound for the product of uncertainties summed over coordinates of the gamma matrix would be

$$(\Delta x_i)(\Delta \tilde{p}_j)_{\text{Tr}} \geq \frac{\sqrt{5}}{2} \hbar \delta_{ij}. \quad (3.6)$$

An increase in the minimum value of the product of the uncertainties of conjugate variables would be consistent with the various entropic and experimental effects discussed in [4–6].

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