Bull. Pure Appl. Sci. Sect. E Math. Stat.
40E(1), 18–25 (2021)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2021.00003.5

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

Some fixed point theorems on *b*-metric spaces *

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Abstract In this paper, we introduce new contractive mappings in the setup of completeness and uniqueness of fixed point theorem on *b*-metric space. We improve the recent fixed point results established by Agrawal et al. (Agrawal, S., Qureshi, K. and Nema, J., A fixed point theorems for *b*-metric space, IJPAM, 9(1), 2016, 45–50). We also show that different contractive type mappings exist in *b*-metric space.

Key words *b*-metric space, fixed point, contraction mappings.

2020 Mathematics Subject Classification 47H09, 47H10, 54H25.

1 Introduction

Fixed points theory has become an important field in mathematics due to its variety of applications in science, economics and game theory. Metric fixed point theory has been an area of vigorous scientific activity since the basic result of Banach [4] in 1922. It is well known that the Banach contraction principle is a fundamental result in the fixed point theory which has been used and extended in many different directions.

In 1989 the concept of *b*-metric space was introduced by I.A. Bakhtin [3] as a generalization of metric space and he proved an analogue of the Banach contraction principle in *b*-metric space. In 1993, Czerwik [8] extended the results of *b*-metric spaces. Since then, several papers have dealt with the generalization of the renowned Banach fixed point theorem in the *b*-metric space such as [2,5–7,9].

In this paper we extend some well know fixed point theorems which are also valid in the *b*-metric space. An analogy result of Agrawal et al. [1] is also proved.

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received December 27, 2019 / Revised March 12, 2021 / Accepted April 02, 2021. Online First Published on June 30, 2021 at https://www.bpasjournals.com/.
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2 Preliminaries

Definition 2.1. Let X be a non-empty set and $s \geq 1$ be a given real number. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s\big[d(x, y) + d(y, z)\big]$ for all $x, y, z \in X$,

then the pair (X, d) is called a b-metric space.

Definition 2.2. Let (X, d) be a b-metric space. Then

(a) a sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

(b) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(c) the b-metric space (X, d) is called b-complete if every b-Cauchy sequence in X is b-convergent.

Example 2.3. [6]. The set $L_p(\mathbb{R})$ (with $0 < p < 1$), where $l_p(\mathbb{R}) = \Big\{(x_n) \subset \mathbb{R} \text{ s.t. } \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\}$ together with function $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$$

where $x = x_n,\ y = y_n \in l_p(\mathbb{R})$ is a b-metric space. By an elementary calculation we obtain that

$$d(x, z) \leq 2^{1/p}\Big[d(x, y) + d(y, z)\Big].$$

Example 2.4. [6]. The space $L_p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p\, dt < \infty$ is a b-metric space if we take

$$d(x, y) = \Big(\int_0^1 |x(t) - y(t)|^p\, dt\Big)^{1/p}, \quad \text{for each } \ x, y \in L_p[0, 1]$$

The following two examples also show the importance of the b-metric space.

Example 2.5. Let $X = \mathbb{N}$, define $d : X \times X \to X$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 4\alpha, & \text{if } x, y \in \{1, 2\} \text{ and } x \neq y, \\ \alpha, & \text{if } x \text{ or } y \notin \{1, 2\} \text{ and } x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a b-metric space with coefficient $s = \frac{4}{2} > 1$ but (X, d) is not a metric space as $d(1, 2) = 4\alpha > 2\alpha = d(1, 3) + d(3, 2)$.

Example 2.6. Let $X = \mathbb{N}$, define $d : X \times X \to X$ such that $d(x, y) = d(y, x)\ \forall\, x, y \in X$ and

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 5\alpha, & \text{if } x = 1, y = 2, \\ \alpha, & \text{if } x \in \{1, 2\} \text{ and } y \in \{3\}, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2} > 1$ but (X, d) is not a metric space as $d(1, 2) = 5\alpha > 2\alpha = d(1, 3) + d(3, 2)$.

Note: Every metric space is a b-metric space with coefficient $s = 1$ but the converse of this implication is not true.

3 Main Result

Theorem 3.1. *Let (X, d) be a complete b-metric space. Let T be a self map on X and satisfying for any $x, y \in X$ such that*

$$d(Tx, Ty) \leqslant a \max\{d(x,y), \tfrac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\} + b\{d(x,Tx) + d(y,Ty)\} \tag{3.1}$$

where $a, b > 0$ such that $a + 2bs \leq 1$ and $s \geq 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined by recursion

$$x_n = Tx_{n-1} = T^n x_0; \; n = 1, 2, 3, \dots. \tag{3.2}$$

From (3.1) and (3.2), we obtain that

$$\begin{aligned} d\big(x_n, x_{n+1}\big) = d\big(Tx_{n-1}, Tx_n\big) \leq a \max\Big\{d\big(x_{n-1}, x_n\big), \frac{d\big(x_{n-1}, Tx_{n-1}\big)d\big(x_n, Tx_n\big)}{1 + d\big(Tx_{n-1}, Tx_n\big)} \\ + b\Big\{d\big(x_{n-1}, Tx_n\big) + d\big(x_n, Tx_{n-1}\big)\Big\} \end{aligned}$$

$$\begin{aligned} d\big(Tx_{n-1}, Tx_n\big) \leq a \max\left\{d\big(x_{n-1}, x_n\big), \frac{d\big(x_{n-1}, x_n\big)d\big(x_n, x_{n+1}\big)}{1 + d\big(x_n, x_{n+1}\big)}\right\} \\ + b\Big\{d\big(x_{n-1}, x_{n+1}\big) + d\big(x_n, x_n\big)\Big\} \end{aligned}$$

Since

$$\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} < d(x_{n-1}, x_n),$$

then we get

$$\begin{aligned} d\big(Tx_{n-1}, Tx_n\big) &\leq ad\big(x_{n-1}, x_n\big) + bd\big(x_{n-1}, x_{n+1}\big) \\ d\big(x_n, x_{n+1}\big) &\leq ad\big(x_{n-1}, x_n\big) + sb\Big[d\big(x_{n-1}, x_n\big) + d\big(x_n, x_{n+1}\big)\Big] \quad \big(\text{by } (b_3) \text{ of Definition 2.1}\big) \\ &= ad\big(x_{n-1}, x_n\big) + sbd\big(x_{n-1}, x_n\big) + sbd\big(x_n, x_{n+1}\big) \\ (1 - bs)d\big(x_n, x_{n+1}\big) &\leq (a + bs)d\big(x_{n-1}, x_n\big) \\ d\big(x_n, x_{n+1}\big) &\leq \frac{(a + bs)}{1 - bs} d\big(x_{n-1}, x_n\big) \\ \Rightarrow d\big(x_n, x_{n+1}\big) &\leq kd\big(x_{n-1}, x_n\big), \qquad \text{where} \qquad k = \frac{a + bs}{1 - bs} \leq 1 \\ &\leq k^2 d\big(x_{n-2}, x_{n-1}\big). \end{aligned}$$

Continuing the above process, we get

$$d\big(x_n, x_{n+1}\big) \leq k^n d\big(x_0, x_1\big) \tag{3.3}$$

On taking the limit as $n \to \infty$, we obtain that

$$d\big(x_n, x_{n+1}\big) \to 0 \;\text{ as }\; n \to \infty.$$

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N},\ m > n$, then

$$\begin{aligned}
d(x_n, x_m) &\le s\Big\{d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\Big\} \\
&\le s\Big\{d(x_n, x_{n+1})\Big\} + s^2\Big\{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\Big\} \\
&\le s\Big\{d(x_n, x_{n+1})\Big\} + s^2\Big\{d(x_{n+1}, x_{n+2})\Big\} + s^2\Big\{d(x_{n+2}, x_m)\Big\} \\
&\le s\Big\{d(x_n, x_{n+1})\Big\} + s^2\Big\{d(x_{n+1}, x_{n+2})\Big\} + s^3\Big\{d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_m)\Big\} \\
&\le sk^n\Big\{1 + sk + s^2k^2 + s^3k^3 + \ldots\Big\}d(x_0, x_1) \\
&\le sk^n d(x_0, x_1)\Big\{1 + sk + (sk)^2 + (sk)^3 + \ldots\Big\} \\
&\le \frac{sk^n d(x_0, x_1)}{(1 - sk)}.
\end{aligned}$$

On taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_n, x_m) = 0 \ \text{ as } \ n, m \to \infty, \ \ \text{since} \ \ \frac{sk^n}{1 - sk} < 1,$$

hence $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in X.
Since X is complete then $\exists\, u$ in X s.t.

$$\lim_{n\to\infty} x_n = u (\in X).$$

Now, we show that u is the fixed point of T. As

$$\begin{aligned}
d(u, Tu) &\le s\Big\{d(u, x_{n+1}) + d(x_{n+1}, Tu)\Big\} \qquad (\text{by } (b_3) \text{ of Definition 2.1}) \\
&\le s\Big\{d(u, x_{n+1})\Big\} + sa\max\left\{d(x_n, u), \frac{d(x_n, Tx_n)d(u, Tu)}{1 + d(Tx_n, Tu)}\right\} \\
&\quad + sb\Big\{d(x_n, Tu) + d(u, Tx_n)\Big\} \\
&= sd(u, x_{n+1}) + sa\max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, Tu)}{1 + d(x_{n+1}, Tu)}\right\} \\
&\quad + sb\Big\{d(x_n, Tu) + d(u, x_{n+1})\Big\} \\
&\le sd(u, x_{n+1}) + sa\max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, Tu)}{1 + d(x_{n+1}, Tu)}\right\} \\
&\quad + s^2b\Big\{d(x_n, u) + d(u, Tu)\Big\} + sbd(u, x_{n+1})
\end{aligned}$$

$$\begin{aligned}
(1 - s^2b)d(u, Tu) &\le s(1 + b)d(u, x_{n+1}) + sa\max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, Tu)}{1 + d(x_{n+1}, Tu)}\right\} \\
&\quad + s^2bd(x_n, u)
\end{aligned}$$

$$\Rightarrow (1 - s^2b)d(u, Tu) \le s(1 + b)d(u, x_{n+1}) + saM_2 + s^2bd(x_n, u) \tag{3.4}$$

where

$$\begin{aligned}
M_2 &= \max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, Tu)}{1 + d(x_{n+1}, Tu)}\right\} \\
&\le \max\Big\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu)\Big\}.
\end{aligned}$$

Case-(i): If $M_2 = d(x_n, u)$ then from (3.4), we get

$$\therefore (1-s^2b)d(u,Tu) \le s(1+b)d(u,x_{n+1}) + sa\, d(x_n,u) + s^2bd(x_n,u)$$
$$(1-s^2b)d(u,Tu) \le s(1+b)d(u,x_{n+1}) + (sa+s^2b)d(x_n,u)$$
$$\Rightarrow d(u,Tu) \le \frac{s(1+b)}{(1-s^2b)}d(u,x_{n+1}) + \frac{(sa+s^2b)}{(1-s^2b)}d(x_n,u).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(u,Tu) = 0, \quad \left(\text{since } \frac{s(1+b)}{(1-s^2b)}, \frac{sa+s^2b}{(1-s^2b)} < 1\right).$$

Hence u is the fixed point of T.

Case-(ii): If $M_2 = d(x_n, x_{n+1})d(u,Tu)$, then from (3.3) we obtain that

$$\therefore\ M_2 \le k^n d(x_0,x_1)d(u,Tu), \qquad \text{since } k<1.$$

On taking the limit as $n \to \infty$, we have $M_2 \to 0$. Then as

$$(1-s^2b)d(u,Tu) \le s(1+b)d(u,x_{n+1}) + s^2b\, d(x_n,u)$$
$$\Rightarrow d(u,Tu) \le \frac{s(1+b)}{(1-s^2b)}d(u,x_{n+1}) + \frac{s^2b}{(1-s^2b)}d(x_n,u).$$

On taking the limit as $n \to \infty$, we get that u is a fixed point of T in X.

Uniqueness of the fixed point: Assume that u and v are two distinct fixed points of T. Then from (3.2), we have that

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &= a\max\left\{d(u,v), \frac{d(u,Tu)d(v,Tv)}{1+d(Tu,Tv)}\right\} + b\Big\{d(u,Tv) + d(v,Tu)\Big\} \\ &= (a+2b)d(u,v) < d(u,u), \end{aligned}$$

which is a contradiction. Thus $u = v$. □

Theorem 3.2. *Let (X,d) be a complete b-metric space. Let T be a self map on X satisfying for any $x,y \in X$ such that*

$$\begin{aligned} d(Tx,Ty) \leqslant{}& a\max\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\} + b\{d(x,Ty)+d(y,Tx)\} \\ &+ \min\{d(x,Tx), d(x,Ty), d(y,Tx), d(y,Ty)\} \end{aligned} \tag{3.5}$$

where $a,b>0$ such that $a+2bs \le 1$ and $s \ge 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined by the recursion relation

$$x_n = Tx_{n-1} = T^n x_0;\ n = 1,2,3,4,\ldots. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\begin{aligned} d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le{}& a\max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{1+d(x_{n-1},x_n)},\right. \\ &\left.\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{1+d(Tx_{n-1},Tx_n)}\right\} \\ &+ b\Big\{d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})\Big\} \\ &+ \min\Big\{d(x_{n-1},Tx_{n-1}), d(x_{n-1},Tx_n), \\ &d(x_n,Tx_{n-1}), d(x_n,Tx_n)\Big\} \end{aligned}$$

$$d(Tx_{n-1},Tx_n) \le a\max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\right\}$$
$$+ b\Big\{d(x_{n-1},x_{n+1})+d(x_n,x_n)\Big\} + \min\Big\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}),$$
$$d(x_n,x_n), d(x_n,x_{n+1})\Big\}$$

$$\begin{aligned} d(Tx_{n-1},Tx_n) &\le a\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\} + b\Big\{d(x_{n-1},x_{n+1}) \\ &\quad + \min\Big\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}), 0, d(x_n,x_{n+1})\Big\} \\ &\le a\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\} + b\, d(x_{n-1},x_{n+1}) \\ &\le a\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\} + sb\Big\{d(x_{n+1},x_n), d(x_n,x_{n+1})\Big\} \end{aligned}$$

Then

$$(1-sb)d(x_n,x_{n+1}) \le a\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\} + sb\Big\{d(x_{n+1},x_n)\Big\}. \tag{3.7}$$

Now the following two cases arise:

Case-(i): If $\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n-1})\Big\} = d(x_n,x_{n+1})$, then from (3.7) we have that

$$\begin{aligned} (1-sb)d(x_n,x_{n+1}) &\leqslant ad(x_n,x_{n+1}) + sbd(x_{n-1},x_n) \\ (1-a-bs)d(x_n,x_{n+1}) &\leqslant bsd(x_{n-1},x_n) \\ &= \frac{bs}{(1-a-bs)}d(x_{n-1},x_n) = kd(x_{n-1},x_n) \end{aligned}$$

where, $k = \frac{bs}{(1-a-bs)} \le 1$, thus,

$$d(x_n,x_{n+1}) \le k\, d(x_{n-1},x_n) \le k^2 d(x_{n-2},x_{n-1}) \le \cdots \le k^n d(x_0,x_1).$$

On taking the limit as $n \to \infty$, we get

$$d(x_n,x_m) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.8}$$

Case-(ii): If $\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\} = d(x_n,x_{n-1})$, then from (3.7) it follows that

$$\begin{aligned} (1-bs)d(x_n,x_{n+1}) &\leqslant (a+bs)d(x_{n-1},x_n) \\ d(x_n,x_{n+1}) &\leqslant \frac{a+bs}{(1-bs)}d(x_{n-1},x_n) \end{aligned}$$

where, $k = \frac{a+bs}{(1-bs)} < 1$,

$$\Rightarrow d(x_n,x_{n+1}) \le k\, d(x_{n-1},x_n) \le k^2 d(x_{n-2},x_{n-1}) \le \cdots \le k^n d(x_0,x_1).$$

On taking the limit as $n \to \infty$, we get that

$$d(x_n,x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty.$$

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N},\ m > n$, then

$$\begin{aligned} d(x_n,x_m) &\le s\Big\{d(x_n,x_{n+1}) + d(x_{n+1},x_m)\Big\} \\ d(x_n,x_m) &\le s\, d(x_n,x_{n+1}) + s^2\Big\{d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)\Big\} \\ &\le s\, d(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^2 d(x_{n+2},x_m) \\ &\le s\, d(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^3 d(x_{n+2},x_{n+3}) + \ldots \\ &\le sk^n d(x_0,x_1) + s^2 k^{n+1} d(x_0,x_1) + s^3 k^{n+2} d(x_0,x_1) + \ldots \\ &\le sk^n d(x_0,x_1)\Big[1 + sk + (sk)^2 + (sk)^3 + \ldots\Big] \\ &\le \frac{sk^n}{1-sk} d(x_0,x_1). \end{aligned}$$

Then $\lim_{n\to\infty} d(x_n, x_m) = 0$ as $n, m \to \infty$, and $k < 1$,

$$\lim_{n\to\infty} \frac{sk^n}{1-sk} d(x_0, x_1) = 0 \text{ as } n, m \to \infty.$$

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X and since X is complete then $\exists\, u \in X$ such that $\lim_{n\to\infty} x_n = u(\in X)$. Now, we show that u is a fixed point of T. For this we have,

$$\begin{aligned}
d(u, Tu) &\le s\Big\{d(u, x_{n+1}) + d(x_{n+1}, Tu)\Big\}\\
&\le sd(u, x_{n+1}) + sd(Tx_n, Tu)\\
&\le sd(u, x_{n+1}) + sa\max\left\{d(x_n, u), \frac{d(x_n, Tx_n)d(u, Tu)}{1+d(x_n, u)}, \frac{d(x_n, Tx_n)d(u, Tu)}{1+d(Tx_n, Tu)}\right\}\\
&\quad + sb\Big\{d(x_n, Tu) + d(u, Tx_n)\Big\} + s\min\Big\{d(x_n, Tx_n), d(x_n, Tu), d(u, Tx_n), d(u, Tu)\Big\}\\
&= sd(u, x_{n+1}) + sa\max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, Tu)}{1+d(x_n, u)}, \frac{d(x_n, x_{n+1})d(u, Tu)}{1+d(x_{n+1}, Tu)}\right\}\\
&\quad + sb\Big\{d(x_n, Tu) + d(u, x_{n+1})\Big\} + s\min\Big\{d(x_n, x_{n+1}), d(x_n, Tu), d(u, x_{n+1}), d(u, Tu)\Big\}\\
&\le sd(u, x_{n+1}) + sa\max\Big\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu)\Big\} + sb\Big\{d(x_n, Tu) + d(u, x_{n+1})\Big\}\\
&\le sd(u, x_{n+1}) + sa\max\Big\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu)\Big\} + s^2b\Big\{d(x_n, u) + d(u, Tu)\Big\}\\
&\quad + sb\, d(u, x_{n+1})
\end{aligned}$$

$$\begin{aligned}
(1-s^2b)d(u, Tu) &\le s(1+b)d(u, x_{n+1}) + sa\max\Big\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu)\Big\}\\
&\quad + s^2b\, d(x_n, u)
\end{aligned}$$

$$(1-s^2b)d(u, Tu) \le s(1+b)d(u, x_{n+1}) + saM_2 + s^2b\, d(x_n, u) \tag{3.9}$$

where, $M_2 = \max\Big\{d(x_n, u), d(x_n, x_{n+1})d(u, Tu)\Big\}$, which gives rise to the following two cases for discussion depending on the value of M_2 in this expression:

Case I: If $M_2 = d(x_n, u)$ then from (3.9), we get

$$\begin{aligned}
(1-s^2b)d(u, Tu) &\le s(1+b)d(u, x_{n+1}) + sa\, d(x_n, u) + s^2b\, d(x_n, u)\\
&= s(1+b)d(u, x_{n+1}) + (sa + s^2b)d(x_n, u)\\
d(u, Tu) &\le \frac{s(1+b)}{(1-s^2b)}d(u, x_{n+1}) + \frac{(sa+s^2b)}{(1-s^2b)}d(x_n, u).
\end{aligned}$$

On taking the limit as $n \to \infty$, we get $d(u, Tu) = 0$. Hence u is a fixed point of T in X.

Case II: If $M_2 = d(x_n, x_{n+1})d(u, Tu)$ then from (3.8), we get $d(x_n, x_{n+1}) \le k^n d(x_0, x_1)$

$$\therefore\ M_2 = k^n d(x_0, x_1)d(u, Tu) \to 0 \text{ as } n \to \infty \text{ since } k < 1.$$

Then from (3.9), we obtain that

$$\begin{aligned}
&(1-s^2b)d(u, Tu) \le s(1+b)d(u, x_{n+1}) + s^2b\, d(x_n, u)\\
\Rightarrow\, &d(u, Tu) \le \frac{s(1+b)}{(1-s^2b)}d(u, x_{n+1}) + \frac{s^2b}{(1-s^2b)}d(x_n, u).
\end{aligned}$$

Now taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(u, Tu) = 0.$$

Hence u is a fixed point of T.

Uniqueness of the fixed point: We have to show that u is a unique fixed point of T. Suppose on the contrary that there are two distinct fixed points u and v of T. Then $Tu = u$ and $Tv = v$ and $u \neq v$, thus follows that

$$\begin{aligned}
d(u,v) &= d(Tu,Tv) \\
&\leqslant a\max\{d(u,v), \frac{d(u,Tu)d(v,Tv)}{1+d(u,v)}, \frac{d(u,Tu)d(v,Tv)}{1+d(Tu,Tv)}\} + b\{d(u,Tv)+d(v,Tu)\} \\
&\quad + \min\{d(u,Tu), d(u,Tv), d(v,Tu), d(v,Tv)\} \\
&\leqslant a\max\{d(u,v), d(u,u)d(v,v), d(u,u)d(v,v)\} + b\{d(u,v)+d(v,u)\} \\
&\quad + \min\{d(u,u), d(u,v), d(v,u), d(u,v)\} \\
&\leqslant ad(u,v) + 2bd(u,v) \\
&\Rightarrow d(u,v) \leqslant (a+2b)d(u,v),
\end{aligned}$$

which is a contradiction. Therefore, $u = v$. Hence u is a unique fixed point of T. □

Acknowledgments The authors are thankful to Prof. K.N. Rajeshwari and Prof. A S. Randive for their useful suggestions in the preparation of this paper. The authors also express their gratitude to the referees and to the Editor-in-Chief for suggesting modifications and corrections in the original draft of this paper which have improved the quality of this manuscript.

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