Bull. Pure Appl. Sci. Sect. E Math. Stat.
40E(1), 87–98 (2021)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.5958/2320-3226.2021.00009.6
©Dr. A.K. Sharma, BPAS PUBLICATIONS,
115-RPS-DDA Flat, Mansarover Park,
Shahdara, Delhi-110032, India. 2021

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

Usage of the finite difference method for solving one-dimensional heat equations *

Kirtiwant P. Ghadle[1], Malayin A. Mohammed[2,†] and Aishwary K. Ghadle[3]

1,2,3. Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad, Maharashtra-431 004, India.
1. E-mail: drkp.ghadle@gmail.com
2. E-mail: ✉ malayinabdulqawi@gmail.com , 3. E-mail: aishwaryghadle@gmail.com

Abstract In this paper the one-dimensional heat equations with the heat generation arising in the associated fractal transient conduction is investigated. Analytical solutions are obtained by using the finite difference method (FDM). The method, in general, is easy to implement and it yields good results. In addition, the uniqueness and stability results are also discussed. Some illustrative examples are included to demonstrate the validity and applicability of the technique.

Key words Finite difference method, One-dimensional heat equation, Uniqueness and stability.

2020 Mathematics Subject Classification 34A07, 45J05, 49M27.

1 Introduction

The heat equation is of fundamental importance in diverse scientific fields as it describes the distribution of heat (or variation in temperature) in a given region over time. In mathematics it is a prototypical parabolic partial differential equation. In financial mathematics, the famous Black-Scholes option pricing models differential equation can be transformed into the heat equation allowing a relatively easy solution from a familiar body of mathematics [2, 4, 6, 18, 20]. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes and a direct practical application of the heat equation, in conjunction with the Fourier theory, in spherical coordinates, is the measurement of the thermal diffusivity in polymers [5, 7, 11, 14–16]. The heat equation can be efficiently solved numerically using the Bender-Schmidt method. Usually the solution of these equations is given in the Fourier series form. Monte [23] applied a natural analytical approach for solving the one dimensional transient heat conduction in a composite slab. Lu et al. [21] gave a novel analytical method applied to the transient heat conduction equation. In this paper, we find the solution of one-dimensional heat equations with certain initial and boundary conditions as a

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received December 19, 2019 / Revised February 12, 2021 / Accepted February 23, 2021. Online First Published on June 30, 2021 at https://www.bpasjournals.com/.
†Corresponding author Malayin A. Mohammed, E-mail: malayinabdulqawi@gmail.com

polynomial:

$$u_t = ku_{xx}, \ 0 < x < l, \ t > 0 \tag{1.1}$$
$$u(x,0) = f(x), \ 0 \leq x \leq l, \tag{1.2}$$
$$u(0,t) = u(l,t) = 0, \ t \geq 0 \tag{1.3}$$

where $u = u(x,t)$ is the dependent variable, and k is a constant coefficient. (1.1) is a model of transient heat conduction in a slab of material with thickness l. The domain of the solution is a semi-infinite strip of width l that continues indefinitely in time. The material property k is the thermal diffusivity. In a practical computation, the solution is obtained only for a finite time, say $t_{\max}$. The solution to (1.1) requires specification of boundary conditions at $x = 0$ and $x = l$ and initial conditions at $t = 0$. A set of simple boundary and initial conditions is given by (1.2) and (1.3).
The numerical solution of the heat equation is discussed in many textbooks, e.g., Ames [1], Cooper [9], Morton and Mayers [22] provide a more mathematical development of the finite difference method (FDM). See Cooper [9] for a modern introduction to the theory of partial differential equations along with a brief coverage of numerical methods. Fletcher [10], Golub and Ortega [12] and Hoffman [17] take a more applied approach that also introduces implementation issues. Fletcher provides Fortran code for several methods.
The finite difference method is one of the several techniques for obtaining numerical solutions to the boundary value problems [19], especially to solve partial differential equations, in which the partial derivatives are replaced by finite differences of two variables. Mortan and Mayer [24] and Cooper [25] provide a rich mathematical development of finite difference methods besides a modern introduction to the theory of partial differential equations along with a brief coverage of numerical methods. Fletcher [10] described the method to implement finite differences to solve boundary value problems. In this paper, we describe how to solve a one-dimensional heat equation using the finite difference method.

2 Numerical resolution of the problem

Consider the heat equation in one dimension,

$$u_t = ku_{xx}.$$

Recall that the partial derivative u_t, is defined by

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \longrightarrow 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}.$$

Therefore, we can use the approximation

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}. \tag{2.1}$$

This is called a forward difference approximation [3, 8, 13, 19]. In order to find an approximation to the second derivative u_{xx}, we start with the forward difference

$$\frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x},$$

then

$$\frac{\partial u_x}{\partial x} \approx \frac{u_x(x+\Delta x, t) - u(x,t)}{\Delta x}.$$

We need to approximate the terms in the numerator. It is customary to use a backward difference approximation. This is given by letting $\Delta x \rightarrow -\Delta x$ in the forward difference form,

$$\frac{\partial u}{\partial x} \approx \frac{u(x,t) - u(x-\Delta x, t)}{\Delta x}.$$

Applying this to u_x evaluated at $x = x$ and $x = x + \Delta x$, we have

$$u_x(x,t) \approx \frac{u(x,t) - u(x-\Delta x, t)}{\Delta x},$$

and

$$u_x(x+\Delta x,t)\approx\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}.$$

Inserting these expressions into the approximation for u_{xx}, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial u_x}{\partial x}\\&\approx\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}\\&\approx\frac{\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}}{\Delta x}-\frac{\frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}}{\Delta x}\\&=\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2}.\end{aligned}\tag{2.2}$$

This approximation for u_{xx} is called the central difference approximation of u_{xx}. Combining (2.1) with (2.2) in the heat equation, we have

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}\approx k\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2}.$$

Solving for $u(x,t+\Delta t)$, we find

$$u(x,t+\Delta t)\approx\ u(x,t)+\alpha[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)],\tag{2.3}$$

where $\alpha=k\frac{\Delta t}{(\Delta x)^2}$.
In this equation we have a way to determine the solution at position x and time $t+\Delta t$ given that we know the solution at three positions $x, x+\Delta x$ and $x+2\Delta x$ at time t:

$$u(x,t+\Delta t)\approx\ u(x,t)+\alpha[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)].\tag{2.4}$$

The domain of the solution is $x\in[a,b]$ and $t\geq 0$. We seek approximate values of $u(x,t)$ at specific positions and times. We first divide the interval $[a,b]$ into N subintervals of width $\Delta x=(b-a)/N$. Then, the endpoints of the subintervals are given by

$$x_i=a+i\Delta x,\qquad i=0,1,...,N.$$

Similarly, we take time steps of Δt, at times

$$t_j=j\Delta t,\qquad j=0,1,2,\ldots.$$

This gives a grid of points (x_i,t_j) in the domain leading to an approximate solution to the heat equation $u_{i,j}\approx u(x_i,t_j)$, so (2.4) becomes

$$u_{i,j+1}\approx u_{i,j}+\alpha[u_{i+1,j}-2u_{i,j}+u_{i-1,j}].\tag{2.5}$$

Let us assume that the initial condition is given by

$$u(x,0)=f(x).$$

For example, if we have Dirichlet conditions at $x=a$,

$$u(a,t)=0.$$

The approximation to the derivative gives

$$\left.\frac{\partial u}{\partial t}\right|_{x=a}\approx\frac{u(a+\Delta x,t)-u(a,t)}{\Delta x}=0.$$

Then,

$$u(a+\Delta x,t)-u(a,t)$$

$$u_i^{\text{new}}=u_i^{\text{old}}+\alpha[u_{i+1}-2u_i^{\text{old}}+u_{i-1}^{\text{old}}].$$

3 The uniqueness of solution

In this section, we shall give a uniqueness of solution for one dimensional heat equation with finite length.

Theorem 3.1. *The solution of the problem* (1.1)*, if it exists, is unique.*

Proof. The proposed uniqueness result for the initial and boundary value problem (IBVP) of the heat equation is equivalent to showing that the following IBVP has only a trivial solution,

$$\begin{aligned} &v_t = kv_{xx},\ 0 < x < l,\ t > 0,\\ &v(x,0) = 0,\ 0 \le x \le l,\\ &v(0,t) = v(l,t) = 0,\ t \ge 0. \end{aligned} \tag{3.1}$$

Let $v(x,t)$ be a solution of problem (3.1). Now consider,

$$E(t) = \frac{1}{2k}\int_0^l v^2(x,t)dx.$$

Observe that $E(t)$ is a differentiable function of t, since $v(x,t)$ is twice differentiable. Therefore

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{k}\int_0^l vv_t dx\\ &= \int_0^l vv_{xx}dx\\ &= vv_x\Big|_0^l - \int_0^l v_x^2 dx. \end{aligned}$$

Since $v(0,t) = v(l,t) = 0$, we have

$$\frac{dE}{dt} = -\int_0^l v_x^2 dx \le 0,$$

i.e., E is a decreasing function of t. Also, by definition $E(t)$ is nonnegative and from the condition $v(x,0) = 0$, we have $E(0) = 0$. Therefore

$$E(t) \equiv 0,\ \forall t > 0 \Rightarrow v(x,t) \equiv 0,\quad 0 \le x \le l, t \ge 0.$$

Hence, the proof is completed.

□

4 The stability results

In this section, we discuss stability of the finite difference scheme for the heat equation.
Consider the following finite difference approximation to the 1-D heat equation:

$$u_n^{k+1} - u_n^k = \frac{\Delta t}{\Delta x^2}(u_{n+1}^k - 2u_n^k + u_{n-1}^k) \quad \text{where} \quad u_n^k \simeq u(x_n, t_k).$$

Let $u_n^k = \phi_k e^{in\Delta x\theta}$ then

$$\begin{aligned} (\phi_{k+1} - \phi_k)e^{in\Delta x\theta} &= \frac{\Delta t}{\Delta x^2}(e^{i\Delta x\theta} - 2 + e^{-i\Delta x\theta})\phi_k e^{in\Delta x\theta}\\ &= \frac{\Delta t}{\Delta x^2}\Big[2\cos(\theta\Delta x) - 2\Big]\phi_k e^{in\Delta x\theta}, \end{aligned}$$

or,

$$\begin{aligned} \phi_{k+1} &= \phi_k - \frac{\Delta t}{\Delta x^2}4\sin^2(\frac{\theta\Delta x}{2})\phi_k\\ &= \Big[1 - \frac{4\Delta t}{\Delta x^2}\sin^2(\frac{\theta\Delta x}{2})\Big]\phi_k. \end{aligned}$$

Now, for stability we require that $|\phi_{k+1}| \leq |\phi_k|$, so that

$$\left|1 - \frac{4\Delta t}{\Delta x^2}\sin^2(\frac{\theta \Delta x}{2})\right| \leq 1$$

$$\Rightarrow -2 \leq -\frac{4\Delta t}{\Delta x^2}\sin^2(\frac{\theta \Delta x}{2}) \leq 0.$$

The right inequality is satisfied automatically, while the left inequality can be re-written in the form:

$$\frac{4\Delta t}{\Delta x^2}\sin^2(\frac{\theta \Delta x}{2}) \leq 2.$$

Since $\sin^2(.) \leq 1$ this condition is satisfied for all θ provided that

$$\Delta t \leq \frac{\Delta x^2}{2}.$$

5 Numerical examples

Example 5.1. We consider the following boundary value problem of one-dimensional heat equation.

$$u_t = u_{xx}. \tag{5.1}$$

subject to

$$u(0,t) = 0 \tag{5.2}$$

$$u(5,t) = 0 \tag{5.3}$$

$$u(x,0) = 25x^2 - x^4 \tag{5.4}$$

where, $0 \leq t \leq 2.5$ and $0 \leq x \leq 5$.

For the solution to the problem, we take the interval of differencing of x as 1, i.e., $h = 1$, and the interval of differencing of t as $\frac{1}{2}$ i.e $k = \frac{1}{2}$, thus $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4,\ x_5 = 5$ and $t_0 = 0,\ t_1 = \frac{1}{2},\ t_2 = 1,\ t_3 = \frac{3}{2},\ t_4 = 2,\ t_5 = \frac{5}{2}$, suppose $u_{rs} = f(x_r, t_s)$ where $x_r = x_0 + rh$ and $t_s = t_0 + sk$, for $(r, s = 0, 1, 2, 3, 4, 5)$.

The boundary condition (5.2) gives

$$u_{00} = u_{01} = u_{02} = u_{03} = u_{04} = u_{05} = 0,$$

and the boundary condition (5.3) gives

$$u_{50} = u_{51} = u_{52} = u_{53} = u_{54} = u_{55} = 0,$$

while the boundary condition (5.4) gives

$$u_{00} = u_{10} = u_{20} = u_{30} = u_{40} = u_{50} = 0.$$

The other values are obtained from the recurrence relation

$$u_{i,j+1} = u_{i,j} + \lambda(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

for $\lambda = \frac{k}{h^2} = \frac{1}{2}$ which are shown in the Table 1. Since the first and the last column values of Table 1 are zero, therefore, the differences of different orders of $\Delta^{0+j}u_{0j}(i, j = 0, 1, 2, 3, 4, 5)$ are zero. i.e., $\Delta^{0+j}u_{0j} = 0$. Particularly,

$$\Delta^{0+1}u_{00} = \Delta^{0+2}u_{00} = \Delta^{0+3}u_{00} = \Delta^{0+4}u_{00} = \Delta^{0+5}u_{00} = 0, \tag{5.5}$$

$$\Delta^{0+1}u_{50} = \Delta^{0+2}u_{50} = \Delta^{0+3}u_{50} = \Delta^{0+4}u_{50} = \Delta^{0+5}u_{50} = 0. \tag{5.6}$$

By considering the second column values of Table 1, we show the differences $\Delta^{0+j}u_{10}$ in Table 2. From Table 2, we have

$$\Delta^{0+1}u_{10} = 18, \Delta^{0+2}u_{10} = -18, \Delta^{0+3}u_{10} = 15, \Delta^{0+4}u_{10} = -18, \Delta^{0+5}u_{10} = 32.625.$$

By considering the third column values of Table 1, the differences of $\Delta^{0+j}u_{20}$ are obtained and shown in the Table 3. From Table 3, we have

$$\Delta^{0+1}u_{20}=0,\ \Delta^{0+2}u_{20}=-6,\ \Delta^{0+3}u_{20}=-6,\ \Delta^{0+4}u_{20}=29.25,\ \Delta^{0+5}u_{20}=-70.25.$$

Similarly by considering the 4^{th} and 5^{th} columns of Table 1 we obtain

$$\Delta^{0+1}u_{30}=-30,\Delta^{0+2}u_{30}=-6,\Delta^{0+3}u_{30}=31.5,\Delta^{0+4}u_{30}=-64.5,\Delta^{0+5}u_{30}=117,$$

and

$$\Delta^{0+1}u_{40}=-72,\Delta^{0+2}u_{40}=57,\Delta^{0+3}u_{40}=-60,\Delta^{0+4}u_{40}=75.75,\Delta^{0+5}u_{40}=-108.$$

Repeating the above process for the rows of Table 1, we obtain

$$\Delta^{1+0}u_{00}=24,\Delta^{2+0}u_{00}=36,\Delta^{3+0}u_{00}=-36,\Delta^{4+0}u_{00}=-24,\Delta^{5+0}u_{00}=0. \tag{5.7}$$

$$\Delta^{1+0}u_{00}=42,\Delta^{2+0}u_{00}=0,\Delta^{3+0}u_{00}=-12,\Delta^{4+0}u_{00}=-48,\Delta^{5+0}u_{00}=150$$
$$\Delta^{1+0}u_{00}=42,\Delta^{2+0}u_{00}=-6,\Delta^{3+0}u_{00}=-30,\Delta^{4+0}u_{00}=45,\Delta^{5+0}u_{00}=-75$$
$$\Delta^{1+0}u_{00}=39,\Delta^{2+0}u_{00}=-18,\Delta^{3+0}u_{00}=4.5,\Delta^{4+0}u_{00}=-27,\Delta^{5+0}u_{00}=75$$
$$\Delta^{1+0}u_{00}=30,\Delta^{2+0}u_{00}=-6.75,\Delta^{3+0}u_{00}=-20.25,\Delta^{4+0}u_{00}=35.25,\Delta^{5+0}u_{00}=-56.25$$
$$\Delta^{1+0}u_{00}=26.625,\Delta^{2+0}u_{00}=-13.5,\Delta^{3+0}u_{00}=4.125,\Delta^{4+0}u_{00}=-17.25,\Delta^{5+0}u_{00}=46.875.$$

The differences $\Delta^{m+n}u_{00}$ $(m,n=1,2,3,4,5$ and $m+n\leq 5)$ are given by

$$\Delta^{1+1}u_{00}=18,\Delta^{1+2}u_{00}=-18,\Delta^{2+1}u_{00}=-36,\Delta^{3+1}u_{00}=-48,\Delta^{1+3}u_{00}=15,$$

$$\Delta^{2+2}u_{00}=30,\Delta^{1+4}u_{00}=-18,\Delta^{4+1}u_{00}=-24,\Delta^{3+2}u_{00}=-42,\Delta^{2+3}u_{00}=36. \tag{5.8}$$

The formula for interpolating polynomial in two variables up to the 5^{th} difference $(m=5)$ is

$$\begin{aligned}
&u(x,t)\\
=\;& u_{00}+\Big[\frac{(x-x_0)}{h}\Delta^{1+0}u_{00}+\frac{(t-t_0)}{k}\Delta^{0+1}u_{00}\Big]+\frac{1}{2!}\Big[\frac{(x-x_0)(x-x_1)}{h^2}\Delta^{2+0}u_{00}\\
&+\frac{2(x-x_0)(t-t_0)}{hk}\Delta^{1+1}u_{00}+\frac{(t-t_0)(t-t_1)}{k^2}\Delta^{0+2}u_{00}\Big]\\
&+\frac{1}{3!}\Big[\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3}\Delta^{3+0}u_{00}+\frac{3(x-x_0)(x-x_1)(t-t_0)}{h^2k}\Delta^{2+1}u_{00}\\
&+\frac{3(t-t_0)(t-t_1)(t-t_2)}{k^3}\Delta^{0+3}u_{00}\Big]+\frac{1}{4!}\Big[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{h^4}\Delta^{4+0}u_{00}\\
&+\frac{4(x-x_0)(x-x_1)(x-x_2)(t-t_0)}{h^3k}\Delta^{3+1}u_{00}+\frac{6(x-x_0)(x-x_1)(t-t_0)(t-t_1)}{h^2k^2}\Delta^{2+2}u_{00}\\
&+\frac{4(x-x_0)(t-t_0)(t-t_1)(t-t_2)}{hk^3}\Delta^{1+3}u_{00}+\frac{3(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{k^4}\Delta^{0+4}u_{00}\Big]\\
&+\frac{1}{5!}\Big[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{h^5}\Delta^{5+0}u_{00}\\
&+\frac{5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(t-t_0)}{h^4k}\Delta^{4+1}u_{00}\\
&+\frac{10(x-x_0)(x-x_1)(x-x_2)(t-t_0)(t-t_1)}{h^3k^2}\Delta^{3+2}u_{00}\\
&+\frac{10(x-x_0)(x-x_1)(t-t_0)(t-t_1)(t-t_2)}{h^2k^3}\Delta^{2+3}u_{00}\\
&+\frac{3(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{hk^4}\Delta^{0+4}u_{00}\\
&+\frac{3(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{k^5}\Delta^{0+5}u_{00}\Big].
\end{aligned} \tag{5.9}$$

Table 1: Function Table

t / x	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.5	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0

Table 2: Difference Table 1.

u_{1i}	$\Delta^{0+1}u_{1i}$	$\Delta^{0+2}u_{1i}$	$\Delta^{0+3}u_{1i}$	$\Delta^{0+4}u_{1i}$	$\Delta^{0+5}u_{1i}$
24					
→	18				
42	→	−18			
→	0	→	15		
42	→	−3	→	−18	
→	−3	→	−3	→	32.625
39	→	−6	→	14.625	
→	−9	→	11.625		
30	→	5.625			
→	−3.375				
26.625					

Substituting the values of $\Delta^{m+n}u_{00}$ from (5.5), (5.7) and (5.8) in (5.9) and simplifying we obtain

$$\begin{aligned} u(x,t) = \; & 25x^2 - x^4 + 36xt - 36x(x-1)t - 36xt(t-\frac{1}{2})t + 8x(x-1)(x-2)t \\ & +30x(x-1)(t-\frac{1}{2}) + 20xt(t-\frac{1}{2})(t-1) - 2x(x-1)(x-2)(x-3)t \\ & -14x(x-1)(x-2)(t-\frac{1}{2}) + 24x(x-1)t(t-\frac{1}{2})(t-1) \\ & -12xt(t-\frac{1}{2})(t-1)(t-\frac{3}{2}). \end{aligned} \tag{5.10}$$

The required interpolating polynomial $u(x,t)$ which is an approximate solution to the heat equation (5.1) is given by (5.10).

The numerical solution of Example 5.1 using the FDM is depicted in Fig. 1.

Example 5.2. To show the stability and instability of the finite difference method and the accuracy of the numerical solution, we consider the following boundary value problem of the one-dimensional heat equation

$$u_t = u_{xx}, \tag{5.11}$$

subject to

$$u(1,t) = 0, \qquad u(0,t) = 0, \quad 0 \le t \le 1, \qquad u(x,0) = -x^2 + x, \quad 0 \le x < 1.$$

The numerical solution of Example 5.2 using the FDM is depicted in Fig. 2.

Table 3: Difference Table 2.

u_{2i}	$\Delta^{0+1}u_{2i}$	$\Delta^{0+2}u_{2i}$	$\Delta^{0+3}u_{2i}$	$\Delta^{0+4}u_{2i}$	$\Delta^{0+5}u_{2i}$
84					
$\rightarrow$	0				
84	$\rightarrow$	-6			
$\rightarrow$	-6	$\rightarrow$	-6		
78	$\rightarrow$	-12	$\rightarrow$	29.25	
$\rightarrow$	-18	$\rightarrow$	23.25	$\rightarrow$	-70.25
60	$\rightarrow$	11.25	$\rightarrow$	-14.25	
$\rightarrow$	-6.75	$\rightarrow$	-18		
53.25	$\rightarrow$	-6.75			
$\rightarrow$	-13.5				
39.75					

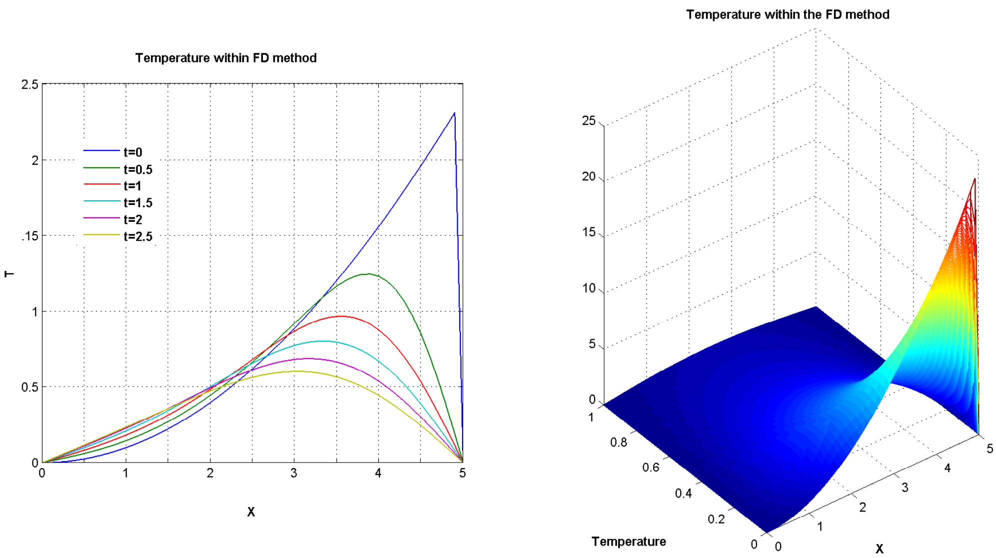

Fig. 1: Numerical solution using FDM for Example (5.1).

Table 4: Exact solution for Example (5.2).

t/x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.000	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0
0.001	0	0.0880	0.1580	0.2080	0.2380	0.2480	0.2380	0.2080	0.1580	0.0880	0
0.002	0	0.0861	0.1560	0.2060	0.2360	0.2460	0.2360	0.2060	0.1560	0.0861	0
0.003	0	0.0845	0.1540	0.2040	0.2340	0.2440	0.2340	0.2040	0.1540	0.0845	0
0.004	0	0.0829	0.1520	0.2020	0.2320	0.2420	0.2320	0.2020	0.1520	0.0829	0
0.005	0	0.0815	0.1501	0.2000	0.2300	0.2400	0.2300	0.2000	0.1501	0.0815	0
0.006	0	0.0802	0.1482	0.1980	0.2280	0.2380	0.2280	0.1980	0.1482	0.0802	0
0.007	0	0.0789	0.1464	0.1960	0.2260	0.2360	0.2260	0.1960	0.1464	0.0789	0
0.008	0	0.0778	0.1446	0.1941	0.2240	0.2340	0.2240	0.1941	0.1446	0.0778	0
0.009	0	0.0767	0.1428	0.1921	0.2220	0.2320	0.2220	0.1921	0.1428	0.0767	0

Table 5: Solution using the finite difference method for Example (5.2).

t/x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.000	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0
0.001	0	0.0880	0.1580	0.2080	0.2380	0.2480	0.2380	0.2080	0.1580	0.0880	0
0.002	0	0.0862	0.1560	0.2060	0.2360	0.2460	0.2360	0.2060	0.1560	0.0862	0
0.003	0	0.0846	0.1540	0.2040	0.2340	0.2440	0.2340	0.2040	0.1540	0.0846	0
0.004	0	0.0830	0.1521	0.2020	0.2320	0.2420	0.2320	0.2020	0.1521	0.0830	0
0.005	0	0.0816	0.1502	0.2000	0.2300	0.2400	0.2300	0.2000	0.1502	0.0816	0
0.006	0	0.0803	0.1483	0.1980	0.2280	0.2380	0.2280	0.1980	0.1483	0.0803	0
0.007	0	0.0791	0.1465	0.1960	0.2260	0.2360	0.2260	0.1960	0.1465	0.0791	0
0.008	0	0.0779	0.1447	0.1941	0.2240	0.2340	0.2240	0.1941	0.1447	0.0779	0
0.009	0	0.0768	0.1430	0.1921	0.2220	0.2320	0.2220	0.1921	0.1430	0.0768	0

Exact Solution
Using the separation of variables method the exact solution of (5.11) is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{-4}{n^3\pi^3}\right)(-1^n - 1)\sin(n\pi x)e^{(-n^2\pi^2 t)}.$$

The values of u at (x_i, t_j), $i = 1, 2, ..., 11$ and $j = 1, 2, ..., 10$ are given in Table 4 for $(k = 0.001)$ for Example 5.2, while the solution of Example 5.2 using the FDM is shown in Table 5.

6 Comparisons of solutions

We compare the solutions using the finite difference method and the exact method using the separation of variables at $t = 0.08$, $t = 0.09$ as shown in Figs. 3(a) and 3(b). We observe from Figs. 3(a) and 3(b) that the numerical solutions using the finite difference method approximate to the exact solutions, that is, u decreases as t increases.

7 Conclusions

The numerical resolution of PDEs remains a challenging problem. In this paper, the uniqueness of the solution to the heat equation problem is demonstrated. In order to find the numerical solution to the

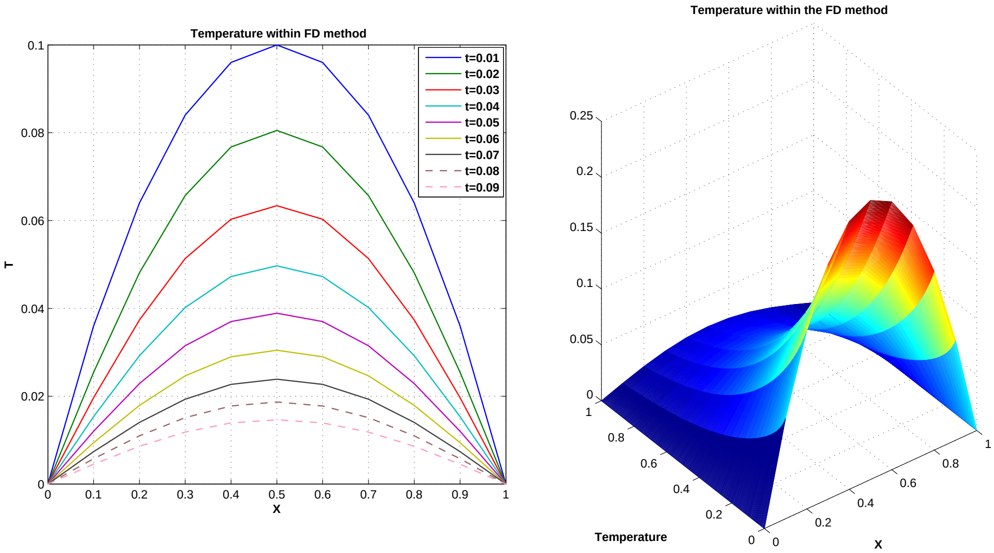

Fig. 2: Numerical solution using FDM for Example (5.2).

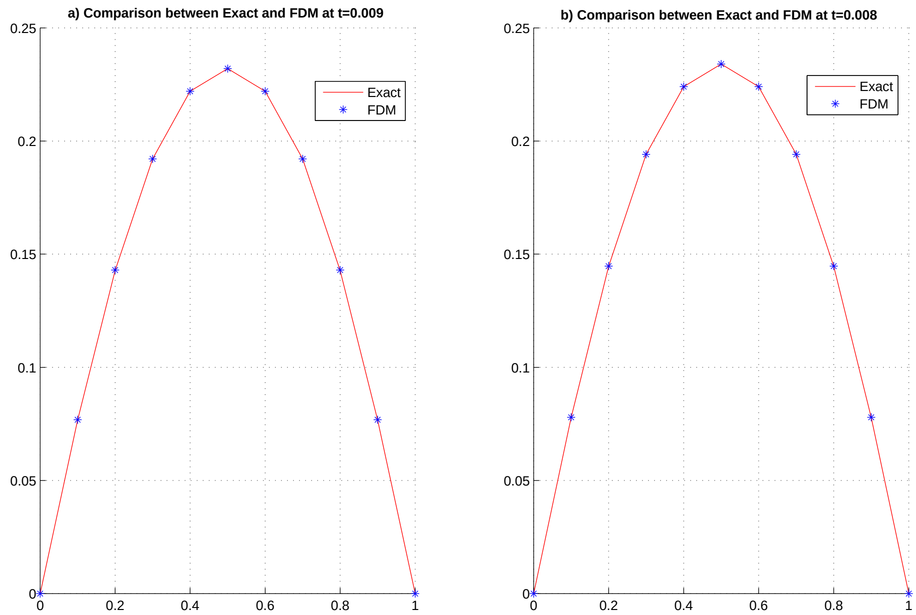

Fig. 3: Comparison of the numerical and the exact solutions.

problem, we used the finite difference method. We employed the method of variation of constants to find the exact solution to the problem. Finally, we implemented numerical simulations in Matlab to find that the numerical solution approaches towards the exact solution. In future we plan to investigate the theoretical convergence of this model and we will also try to solve the two and three-dimensional heat problems.

Acknowledgments The authors would like to thank the referees and the Editor-in-Chief of this journal for their valuable suggestions and comments which have led to an improvement in this paper.

References

[1] Ames, W.F. (1992). Numerical Methods for Partial Differential Equations, Academic Press, Inc., Boston, third edition.

[2] Aslak T. and Ragnar, W. (2005). Introduction to partial differential equations: a computational approach, Springer Verlag, New York.

[3] Atshan, S. and Hamoud, A. (2018). Approximate solutions of fourth-order fractional integrodifferential equations, *Acta Universitatis Apulensis*, 55(3), 49–61.

[4] Beghr, H. and Harutyunyan, G. (2006). Robin boundary value problem for the Poisson equation, *J. Anal. Appl.*, 4(3), 201–213.

[5] Beghr, H., Saule, B. and Harutjunjan, G. (2017). Remark on Robin problem for Poisson equation, *Complex Variables and Elliptic Equations - An International Journal*, 62(10), 1589–1599. https://doi.org/10.1080/17476933.2017.1303052

[6] Burden, R.L. and Faires, J.D. (1997). Numerical Analysis, sixth edition, Brooks/Cole Publishing Co., New York.

[7] Bourchra, B., Guillaume, C. and Jacques, L. (2015). Penalization of Robin boundary condition, *Applied Numerical Mathematics*, 96(2015), 134–152. https://doi.org/10.1016/j.apnum.2015.06.001

[8] Cho, C.H. and Okamoto, H. (2020). Finite difference schemes for an axisymmetric nonlinear heat equation with blow-up, *Electronic Transactions on Numerical Analysis*, 52(2), 391–415.

[9] Cooper, J. (1998). Introduction to Partial Differential Equations with Matlab, Birkhäuser, Boston.

[10] Fletcher, C.A. (1988). Computational Techniques for Fluid Dynamics, Springer-Verlag, Berlin.

[11] Ghadle, K.P., Hamoud, A.A. and Issa, M.S.B. (2018). A comparative study of variational iteration and Adomian decomposition techniques for solving Volterra integro-differential equations, *International Journal of Mathematics Trends and Technology*, 64(2), 16–21.

[12] Golub, G. and Ortega, J.M. (1993). Scientific Computing: An Introduction with Parallel Computing, Academic Press, Inc., Boston.

[13] Hamoud, A. and Ghadle, K. (2018). Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations, *J. Indian Math. Soc.*, 85(1-2), 52–69.

[14] Hamoud, A. and Ghadle, K. (2017). The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations, *The Korean Journal of Mathematics*, 25(3), 323–334.

[15] Hamoud, A.A., Mohammed, N.M. and Ghadle, K.P. (2019). Solving FIDEs by using semi-analytical techniques, *Communications in Advanced Mathematical Sciences*, 2(3), 192–198.

[16] Hinze, M., Pinnau, R., Ulbrich, M. and Ulbrich, S. (2009). Mathematical modelling: Theory and Applications, Springer. https://doi.org/10.1007/978-1-4020-8839-1

[17] Hoffman, J.D. (1992). Numerical Methods for Engineers and Scientists, McGraw- Hill, New York.

[18] Isaacson, E. and Keller, H. (1994). Analysis of Numerical Methods, Dover, New York.

[19] John, C.S. (2004). Finite Difference Schemes and Partial Differential Equations, second edition, SIAM, Philadelphia. https://doi.org/10.1137/1.9780898717938

[20] Kreyszig, E. (1993). Advanced Engineering Mathematics, seventh edition, Wiley, New York.

[21] Lu, X., Tervola, P. and Viljanen, M. (2005). An efficient analytical solution to the transient heat conduction in one dimensional hollow composite cylinder, *J. Phy. A. Math. Gen.* 38(47), 10145–10155.

[22] Morton, K.W. and Mayers, D.F. (1994). Numerical Solution of Partial Differential Equations: An Introduction, Cambridge University Press, Cambridge, England.

[23] Monte, F. (2000). Transient heat conduction in one-dimensional composite slab: a natural analytic approach, *International Journal of Heat and Mass Transfer*, 43(19), 3607–3619.

[24] Peter, K. and Lutz, A. (2000). Numerical methods for elliptic and parabolic partial differential equations, second edition, Springer, New York. https://doi.org/10.1007/b97419

[25] Thomas, J.W. (1995). Numerical Partial Diffential Equations: Finite Difference Methods, Springer, New York. https://doi.org/10.1007/978-1-4899-7278