

Approximation of modified Favard Szasz-Mirakyan operators of maximum-product type *

Fahri Baruç^{1,†} and Sevilay Kirci Serenbay ²

1,2. Department of Mathematics, Faculty of Arts and Sciences,
Harran University, Şanlıurfa, Turkey.

1. E-mail:  barugfahri@gmail.com , 2. E-mail: sevilaykirci@gmail.com

Abstract Nonlinear positive operators by means of maximum and product were introduced by B. Bede. In this paper, we introduce nonlinear maximum-product type modified Favard Szasz–Mirakyan operators. Our main purpose is to give a theorem on the rate of convergence.

Key words Max-product, modified Favard Szasz–Mirakyan operators, degree of approximation, nonlinear operators.

2020 Mathematics Subject Classification 41A25, 41A36, 47H99.

1 Introduction

The problem of uniform convergence to any continuous function in Korovkin-type approximation theory is that operator sequences are generally linear and positive operator sequences. In linear approximation operators, addition and multiplication, which are algebraic operations in the linear space structure of real numbers, are used. The basic theorems are Weierstrass type uniform approximation theorems and the error rate is usually calculated with the help of the continuity module.

Let f be a function defined on $[0, 1)$. The Favard Szasz–Mirakyan operators S_n applied to f are given by

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}. \quad (1.1)$$

In 1978, Becker et al. [5] investigated the approximation properties of the operators S_n in the exponential weight space and proved main approximation theorems for these operators. In 2002, Ispir and Atakut [7] modified the operator S_n as follows:

$$S_n^*(f; x) = \sum_{k=0}^{\infty} p_k(a_n x) f\left(\frac{k}{b_n}\right), \quad x \in \mathbb{R}_0, \quad (1.2)$$

where

$$p_k(a_n x) = e^{-a_n x} \frac{(a_n x)^k}{k!},$$

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received January 11, 2020 / Revised March 09, 2021 / Accepted April 01, 2021. Online First Published on June 30, 2021 at <https://www.bpasjournals.com/>.

[†]Corresponding author Fahri Baruç, E-mail: barugfahri@gmail.com

$\mathbb{R}_0 = [0, \infty)$, $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers, increasing and unbounded such that

$$\lim_{n \rightarrow \infty} \frac{1}{b_n}, \quad \frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right).$$

When $a_n = b_n = n$ in (1.2), we obtain the original Favard Szasz-Mirakyan operators given in (1.1). It is easily seen that $S_n^*(f; x)$ is a positive linear operator. Ispir and Atakut [7] and Walczak [10] studied some approximation properties of these operators in polynomial weighted spaces of continuous and unbounded functions defined on positive semi-axis. Ispir and Atakut [7] also obtained the order of approximation and investigated the bivariate case of these operators.

Rempulska and Walczak [8] examined the approximation properties of the following modified Favard Szasz-Mirakyan operators,

$$S_n(f; r; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n+r}\right), \quad r > 0, \quad x \in \mathbb{R}_0, \quad (1.3)$$

for function $f \in B_r$, where B_r , $r > 0$, denotes the space of all real-valued functions f defined on $\mathbb{R}_0$ for which $\nu_r f$ is bounded function on $\mathbb{R}_0$, $n \in \mathbb{N}$ (see, Rempulska and Walczak [8]).

Serenbay and Dalmanoğlu [9] defined the following generalized Favard Szasz-Mirakyan operators.

$$S_n^*(f; r; x) := \begin{cases} \sum_{k=0}^{\infty} p_k(a_n x) f\left(\frac{k}{r+b_n}\right), & x > 0, \\ f(0), & x = 0, \end{cases} \quad (1.4)$$

where

$$p_k(a_n x) = e^{-a_n x} \frac{(a_n x)^k}{k!}, \quad (1.5)$$

$\mathbb{R}_0 = [0, \infty)$, $\{a_n\}$ and $\{b_n\}$ are given increasing and unbounded sequences of positive numbers satisfying the conditions

$$(a_n) \leq (b_n), \quad \lim_{n \rightarrow \infty} \frac{1}{r+b_n} = 0, \quad \frac{a_n}{r+b_n} = 1 + o\left(\frac{1}{r+b_n}\right) \quad (1.6)$$

for all $n \in \mathbb{N}$. They proved some approximation theorems in the exponential weighted space of functions by these generalized operators.

Subsequent studies have obtained linear operators as well as nonlinear maximum-product type operators with the help of the continuity module with less approximation degrees of error. Bede, Coroianu and Gal, (2016) present a broad overview of developments in the field of constructive approximation involving maximum-product type operators. They highlight the maximum product operators as those that enable them to obtain more valuable predictions than those achieved by the conventional approaches in most cases. In Acar et al. [1], Güngör, and Ispir [6], Bede et al. [2, 3] “max-product kind operators” were presented by using maximum in the name of sum in usual linear operators and they gave a Jackson-type error estimate in terms of the modulus of continuity.

In this study, we define the max-product type of the nonlinear modified Favard Szasz-Mirakyan operators as follows:

$$S_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} c_{n,k}(x) f\left(\frac{k}{n+r}\right)}{\bigvee_{k=0}^{\infty} c_{n,k}(x)} \quad (1.7)$$

where $c_{n,k}(x) = \frac{(nx)^k}{k!}$.

We give some auxiliary Lemmas in Subsection 2.1 and Subsection 2.2 is devoted to our main results. We shall prove our approximation theorems in this section.

2 Main Results

2.1 Auxiliary results

Let $f : [0, \infty) \rightarrow \mathbb{R}_+$ be a continuous function. We want to give an error upper limit for the operators

$$S_n^{(M)}(f) : CB_+([0, \infty)) \rightarrow CB_+([0, \infty)).$$

This upper limit is given by (1.7) in terms of the module of continuity. Now we give some general information about nonlinear operators of the max-prod type. Over the set $\mathbb{R}_+$ of positive reals, we will deal with two different algebraic operations on $\mathbb{R}_+$. These operations are “ $\bigvee$ ” (maximum) and “ $\cdot$ ” (product) and then $(\mathbb{R}_+, \bigvee, \cdot)$ is called a max-product algebra. (see, Güngör and Ispir [6]).

Let $I \subset \mathbb{R}$ be a bounded or an unbounded interval and $CB_+(I)$, $f : I \rightarrow \mathbb{R}_+$ denote the space of all continuous and bounded functions and a sequence of approximation operators of maximum product type $L_n : CB_+(I) \rightarrow CB_+(I)$ is defined by

$$L_n(f)(x) := \bigvee_{i=0}^n K_n(x, x_i) \cdot f(x_i),$$

or, by

$$L_n(f)(x) := \bigvee_{i=0}^{\infty} K_n(x, x_i) \cdot f(x_i),$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(\cdot, x_i) \in CB_+(I)$ and $x_i \in I$, for all i . These operators are nonlinear positive operators having pseudolinearity property, i.e., for all $f, g \in CB_+(I)$ and for any $\alpha, \beta \in \mathbb{R}_+$

$$L_n(\alpha f \bigvee \beta g) = \alpha L_n(f)(x) \bigvee \beta L_n(g)(x).$$

Moreover, the max-product operators are positive homogenous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$ (see, Bede et al. [4]). The following auxiliary Lemmas give some properties of the L_n operators.

Lemma 2.1. (Bede et al. [4]) Let $I \subset \mathbb{R}$ be a bounded or an unbounded interval,

$$CB_+(I) = \{f : I \rightarrow \mathbb{R}_+; f \text{ is continuous and bounded on } I\},$$

and $L_n : CB_+ \rightarrow CB_+, n \in \mathbb{N}$ be a sequence of operators satisfying the following properties:

(i) *Monotonicity*

$$f, g \in CB_+ \text{ satisfy } f \leq g \text{ then } L_n(f) \leq L_n(g) \text{ for all } n \in \mathbb{N};$$

(ii) *Subadditivity*

$$L_n(f + g) \leq L_n(f) + L_n(g) \text{ for all } f, g \in CB_+(I).$$

Then for all $f, g \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|)(x).$$

Remark 2.2. (Bede et al. [4])

2.2.1. The max-product for the Favard Szasz-Mirakyan operators defined by (1.7) verify the conditions (i) and (ii) of Lemma 2.1. In fact, instead of the condition (i) of Lemma 2.1 it satisfies the stronger condition

$$S_n(f \bigvee g)(x) = S_n(f)(x) \bigvee S_n(g)(x), f, g \in CB_+(I).$$

Indeed, by taking in the above equality $f \leq g$, $f, g \in CB_+(I)$, it easily follows that

$$L_n(f)(x) \leq L_n(g)(x).$$

2.2.2. Furthermore, the Favard Szasz-Mirakyan operators of the max-product type is positive homogenous, that is, $S_n(\lambda f) = \lambda S_n(f)$ for all $\lambda \geq 0$.

Corollary 2.3. (Bede et al. [4]) Let $L_n : CB_+ \rightarrow CB_+, n \in \mathbb{N}$ be a sequence of operators satisfying the conditions (i) and (ii) of Lemma 2.1 and in addition L_n be a positive homogenous operator. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|f(x) - L_n(f)(x)| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x) \right] W_1(f; \delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$ for all $t \in I$, $x \in I$.

$$W_1(f; \delta)_I = \max_{\substack{x, y \in I \\ |x - y| \leq \delta}} |f(x) - f(y)|$$

is the first modulus of continuity. If I is unbounded then we suppose that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \cup \{+\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

Corollary 2.4. (Bede et al. [4]) Suppose that in addition to the conditions in Corollary 2.3, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|f(x) - L_n(f)(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] W_1(f; \delta)_I.$$

Definition 2.5. Let $f: [0, \infty) \rightarrow R_+$ be a continuous function in $0 \leq x < \infty$, $r > 0$; $\lim_{n \rightarrow \infty} \frac{1}{n+r} = 0$ the maximum-product type modified Favard Szasz-Mirakyan operators, with an increasing and unlimited series of positive real numbers, are defined as

$$S_n^{(M)}(f)(x) = \frac{V_{k=0}^{\infty} S_{n,k}(x) f\left(\frac{k}{n+r}\right)}{V_{k=0}^{\infty} S_{n,k}(x)} \quad (2.1)$$

where $S_{n,k}(x) = \frac{(nx)^k}{k!}$.

For a continuous and limited function $f: [0, \infty) \rightarrow R_+$, $S_n^{(M)}(f)(x)$ is positive and continuous over the interval $[0, \infty)$. Also $S_n^{(M)}(f)(0) - f(0) = 0$ for each n .

Definition 2.6. For each $k, j \in \{0, 1, 2, \dots\}$ and $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$

$$M_{k,n,j}(x) = \frac{S_{n,k}(x) \left| \frac{k}{n+r} - x \right|}{S_{n,j}(x)}, \quad m_{k,n,j}(x) = \frac{S_{n,k}(x)}{S_{n,j}(x)}$$

where $S_{n,k}(x) = \frac{(nx)^k}{k!}$.

Definition 2.7. For each $k, j \in \{0, 1, 2, \dots\}$ and $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$, if $k \geq j+1$

$$M_{k,n,j}(x) = \frac{S_{n,k}(x) \left(\frac{k}{n+r} - x \right)}{S_{n,j}(x)}$$

and if $k \leq j-1$

$$M_{k,n,j}(x) = \frac{S_{n,k}(x) \left(x - \frac{k}{n+r} \right)}{S_{n,j}(x)}$$

where $S_{n,k}(x) = \frac{(nx)^k}{k!}$.

Lemma 2.8. For all $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$ and $j = 0, 1, \dots$, we have,

$$\bigvee_{k=0}^{\infty} S_{n,k}(x) = S_{n,j}(x)$$

where, $S_{n,k}(x) = \frac{(nx)^k}{k!}$.

Proof. Firstly, we show that for fixed $n \in \mathbb{N}$ and $0 \leq k$, we have $0 \leq S_{n,k+1}(x) \leq S_{n,k}(x)$ if and only if $x \in \left[0, \frac{k+1}{n+r}\right]$.

Indeed, writing the the above inequality explicitly, we have

$$0 \leq \frac{(nx)^{k+1}}{(k+1)!} \leq \frac{(nx)^k}{k!}.$$

If $x = 0$, this inequality is true. For $x > 0$, after simplifications it becomes

$$0 \leq \frac{nx}{(k+1)} \leq 1$$

then

$$0 \leq x \leq \frac{k+1}{n+r} < \frac{k+1}{n}.$$

By taking $k = 0, 1, 2, \dots$ in the inequality just proved above, we get

$$S_{n,1}(x) \leq S_{n,0}(x) \Rightarrow x \in \left[0, \frac{1}{n+r}\right],$$

$$S_{n,2}(x) \leq S_{n,1}(x) \Rightarrow x \in \left[0, \frac{2}{n+r}\right],$$

$$S_{n,3}(x) \leq S_{n,2}(x) \Rightarrow x \in \left[0, \frac{3}{n+r}\right],$$

so,

$$S_{n,k+1}(x) \leq S_{n,k}(x) \Rightarrow x \in \left[0, \frac{k+1}{n+r}\right].$$

From the above inequalities, we obtain that

$$\text{if } x \in \left[0, \frac{1}{n+r}\right], \text{ then } S_{n,k}(x) \leq S_{n,0}(x) \text{ for all } k = 0, 1, 2, \dots,$$

$$\text{if } x \in \left[\frac{1}{n+r}, \frac{2}{n+r}\right], \text{ then } S_{n,k}(x) \leq S_{n,1}(x) \text{ for all } k = 0, 1, 2, \dots,$$

$$\text{if } x \in \left[\frac{2}{n+r}, \frac{3}{n+r}\right], \text{ then } S_{n,k}(x) \leq S_{n,2}(x) \text{ for all } k = 0, 1, 2, \dots,$$

and so on, in general,

$$\text{if } x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right], \text{ then } S_{n,k}(x) \leq S_{n,j}(x) \text{ for all } k = 0, 1, 2, \dots,$$

which proves the lemma. $\square$

Lemma 2.9. For each $k, j \in \{0, 1, 2, \dots\}$ and $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$, we have

$$m_{k,n,j}(x) \leq 1.$$

Proof. There are two situations: either $k \geq j$ or $k \leq j$.

Case I Let $k \geq j$. Since the function $t(x) = \frac{1}{x}$ and as the function is nonincreasing on $\left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$ it follows that

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{(k+1)(n+r)}{n} \cdot \frac{1}{x} \geq \frac{k+1}{n+r} \cdot \frac{n+r}{j+1} = \frac{k+1}{j+1} \geq 1$$

which implies that

$$m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq m_{j+2,n,j}(x) \geq \dots$$

Case II Let $k \leq j$. Since the function $k(x) = x$ and since the function is nondecreasing on $\left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$ it follows that

$$\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{n}{(n+r)k} \cdot x \geq \frac{n}{(n+r)k} \cdot \frac{j \cdot (n+r)}{n} \geq 1$$

which implies that

$$m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq m_{j-2,n,j}(x) \geq \dots \geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, the proof of the lemma is complete. $\square$

Lemma 2.10. Let $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$. In this case

(i) if $k \in \{j+1, j+2, \dots\}$ such that $k - \sqrt{k+1} \geq j$ then

$$M_{k,n,j}(x) \geq M_{k+1,n,j}(x).$$

(ii) if $k \in \{1, 2, \dots, j-1\}$ such that $k + \sqrt{k} \leq j$ then

$$M_{k,n,j}(x) \geq M_{k-1,n,j}(x).$$

Proof. Firstly

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} = \frac{(n+r)(k+1)}{n} \cdot \frac{1}{x} \cdot \frac{\frac{k}{n+r} - x}{\frac{k+1}{n+r} - x}$$

since the function $\theta_{n,k}(x) = \frac{1}{x} \cdot \frac{\frac{k}{n+r} - x}{\frac{k+1}{n+r} - x}$ is decreasing, for all $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$ therefore,

$$\theta_{n,k}(x) \geq \theta_{n,k}\left(\frac{j+1}{n+r}\right) = \frac{n}{(n+r)(j+1)} \cdot \frac{k-j-1}{k-j}.$$

Since the condition $k - \sqrt{k+1} \geq j$ in the hypothesis is

$$(k+1) \cdot (k-j-1) \geq (j+1)(k-j)$$

we obtain

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq \frac{(n+r)(k+1)}{n} \cdot \frac{n}{(j+1)(n+r)} \cdot \frac{k-j-1}{k-j} \geq 1.$$

Similarly,

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} = \frac{n}{k(n+r)} \cdot x \cdot \frac{x - \frac{k}{n+r}}{x - \frac{k+1}{n+r}}.$$

Since the function $\varphi_{n,k}(x) = x \cdot \frac{x - \frac{k}{n+r}}{x - \frac{k+1}{n+r}}$ is increasing, for all $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$,

$$\varphi_{n,k}(x) \geq \varphi_{n,k}\left(\frac{j}{n+r}\right) = \frac{j(n+r)}{n} \cdot \frac{j-k}{j-k+1}.$$

Then, since the condition $k + \sqrt{k} \leq j$, we get

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \frac{n}{k(n+r)} \cdot \frac{j(n+r)}{n} \cdot \frac{j-k}{j-k+1} \geq 1,$$

which proves the lemma. $\square$

2.2 Approximation theorem

For estimating the degree of approximation of a function $f \in CB_+(I)$ we use the Shisha-Mond Theorem given for nonlinear max-product type operators in Bede et al. [2].

Theorem 2.11. If $f : [0, \infty) \rightarrow \mathbb{R}_0^+$ be a uniform, continuous and bounded function and $S_n^{(M)}(f)(x)$ be the modified Favard Szasz-Mirakyan operator of the max-product type defined in (2.1), then the following pointwise estimate holds.

$$\left| S_n^{(M)}(f)(x) - (f)(x) \right| \leq 8W_1 \left(f : \sqrt{\frac{x}{n+r}} \right), \quad \forall n \in \mathbb{N}, x \in [0, \infty),$$

where,

$$W_1(f : \delta) = \sup \{ |f(x) - f(y)| : x, y \in [0, \infty), |x - y| \leq \delta \}.$$

Proof. Since $S_n^{(M)}(e_0)(x) = 1$ and using the modified Favard Szasz-Mirakyan Theorem, we have

$$\left| S_n^{(M)}(f)(x) - (f)(x) \right| \leq \left(1 + \frac{1}{\delta_n} \delta_n^{(M)} \varphi_x(x) \right) W_1(f, \delta_n) \quad (2.2)$$

where $(\varphi_x)(t) = |t - x|$. Hence, it is sufficient to estimate the following term

$$E_n(x) := S_n^{(M)}(\varphi_x)(x) = \frac{\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left| \frac{k}{n+r} - x \right|}{\sum_{k=0}^{\infty} \frac{(nx)^k}{k!}}, \quad x \in [0, \infty).$$

Let $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r} \right]$ and $j \in \{0, 1, 2, \dots\}$ be arbitrarily fixed. We get

$$E_n(x) = \max_{k=0,1,\dots} \left\{ M_{k,n,j}(x), \quad x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r} \right] \right\}.$$

Here $j = 0$ implies,

$$M_{k,n,0}(x) = \frac{(nx)^k}{k!} \left| \frac{k}{n+r} - x \right|, \quad k \geq 0.$$

If $k = 0$, then we have

$$M_{0,n,0}(x) = x = \sqrt{x} \sqrt{x} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n}},$$

where so if

$$M_{k,n,0}(x) \leq \frac{(nx)^k}{k!} \cdot \frac{k}{n+r} = \frac{n^k \cdot \sqrt{x} \cdot x^{k-\frac{1}{2}}}{(k-1)!(n+r)}.$$

If $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r} \right]$ then

$$x^{k-\frac{1}{2}} < \left(\frac{j+1}{n+r} \right)^{k-\frac{1}{2}}$$

where $j = 0$ so if

$$x^{k-\frac{1}{2}} < \left(\frac{1}{n+r} \right)^{k-\frac{1}{2}}$$

then

$$M_{k,n,0}(x) \leq \sqrt{x} \left(\frac{n}{n+r} \right)^{k-\frac{1}{2}} \cdot \frac{1}{(k+1)!} \cdot \frac{1}{\sqrt{n+r}},$$

or,

$$M_{k,n,0}(x) \leq \frac{\sqrt{x}}{\sqrt{n+r}} \leq \frac{\sqrt{x}}{\sqrt{r}},$$

so, $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r} \right]$ for $M_{k,n,j}(x) \leq \frac{4\sqrt{x}}{\sqrt{n}}$ where $k = 0, 1, 2, \dots$

We shall show that $E_n(x) \leq \frac{4\sqrt{x}}{\sqrt{n}}$ and $\delta_n = \frac{4\sqrt{x}}{\sqrt{n}}$ for every $x \in [0, \infty)$, $n \in \mathbb{N}$. We shall prove this for the following three cases:

viz., **Case 1:** when $k = j$, **Case 2:** when $k \geq j + 1$ and **Case 3:** when $k \leq j - 1$.

Case 1: If $k = j$, then $M_{j,n,j}(x) = \left| \frac{j}{n+r} - x \right|$. Since $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r} \right]$ then $M_{j,n,j}(x) \leq \frac{1}{n+r}$.

Since $j \geq 1$ and $x \geq \frac{1}{n+r}$, thus $\frac{1}{n+r} = \frac{1}{\sqrt{n+r}} \cdot \frac{1}{\sqrt{n+r}} \leq \frac{\sqrt{x}}{\sqrt{n+r}}$.

Case 2a: Let $k - \sqrt{k+1} < j$, then

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x) \left(\frac{k}{n+r} - x \right) \leq \frac{k}{n+r} - x \leq \frac{k}{n+r} - \frac{j}{n+r} \\ &\leq \frac{k}{n+r} - \frac{k - \sqrt{k+1}}{n+r} = \frac{\sqrt{k+1}}{n+r}. \end{aligned}$$

But we have $k \leq 3j$. When $k > 3j$ we have a contradiction. Since the function $g(x) = x - \sqrt{x+1}$ is non-decreasing, it can be written that

$$j > k - \sqrt{k+1} \geq 3j - \sqrt{3j+1},$$

which gives the contradiction $j > 3j - \sqrt{3j+1}$, consequently,

$$M_{k,n,j}(x) \leq \frac{\sqrt{k+1}}{n+r} \leq \frac{\sqrt{3j+1}}{n+r} \leq \frac{2\sqrt{j}}{n+r} \leq \frac{2\sqrt{x}}{n+r}$$

is obtained because $\sqrt{x} \geq \frac{\sqrt{j}}{\sqrt{n+r}}$.

Case 2b: Let $k - \sqrt{k+1} \geq j$. Since the function $g(x) = x - \sqrt{x+1}$ does not decrease on $[0, \infty)$, it has a maximum value of $\bar{k} \in \{0, 1, 2, \dots\}$ such that $\bar{k} - \sqrt{\bar{k}+1} < j$. In this case, $k_1 - \sqrt{k_1+1} \geq j$ and

$$\begin{aligned} M_{\bar{k}+1,n,j}(x) &= m_{\bar{k}+1,n,j}(x) \left(\frac{(\bar{k}+1)}{n+r} - x \right) \leq \frac{\bar{k}+1}{n+r} - x \\ &\leq \frac{\bar{k}+1}{n+r} - \frac{j}{n+r} \leq \frac{\bar{k}+1}{n+r} - \frac{\bar{k} - \sqrt{\bar{k}+1}}{n+r} \\ &= \frac{\sqrt{\bar{k}+1} + 1}{n+r} \leq \frac{3\sqrt{x}}{\sqrt{n+r}} \leq \frac{3\sqrt{x}}{\sqrt{n}} \end{aligned}$$

is found for $k_1 = \bar{k} + 1$.

Where $\bar{k} - \sqrt{\bar{k}+1} < j$, it requires that $\bar{k} < 3j$, the final inequality is obtained. In addition, since the function g is non-decreasing, $k_1 \geq j+1$ and $g(j) < j$, we get

$$M_{\bar{k}+1,n,j}(x) \geq M_{\bar{k}+2,n,j}(x) \geq \dots$$

so $M_{k,n,j}(x) \leq \frac{3\sqrt{x}}{\sqrt{n+r}}$ for all $k \in \{\bar{k}+1, \bar{k}+2, \dots\}$. Then $M_{k,n,j}(x) \leq 3\sqrt{\frac{x}{n+r}}$ is obtained from these two sub-cases.

Case 3a: Let $k + \sqrt{k} \geq j$. We have,

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x) \left(x - \frac{k}{n+r} \right) \leq \frac{j+1}{n+r} - \frac{k}{n+r} \\ &\leq \frac{k + \sqrt{k} + 1}{n+r} - \frac{k}{n+r} = \frac{\sqrt{k} + 1}{n+r} \\ &\leq \frac{\sqrt{j-2} + 1}{n+r} = \frac{1}{\sqrt{n+r}} \cdot \frac{\sqrt{j-2} + 1}{\sqrt{n+r}} \\ &\leq \frac{2\sqrt{x}}{\sqrt{n+r}} \leq \frac{2\sqrt{x}}{\sqrt{n}} \end{aligned}$$

where

$$\frac{\sqrt{j-2} + 1}{\sqrt{n+r}} \leq \frac{2\sqrt{j}}{\sqrt{n+r}} \leq 2\sqrt{x}.$$

Case 3b: Let $k + \sqrt{k} \leq j$. Let $\bar{k} \in \{0, 1, 2, \dots\}$, be the minimum value such that $\bar{k} + \sqrt{\bar{k}} > j$. In this case $k_2 = \bar{k} - 1$ satisfies the inequality $k_2 + \sqrt{k_2} \leq j$ and we get

$$\begin{aligned} M_{\bar{k}-1,n,j}(x) &= m_{\bar{k}-1,n,j}(x) \left(x - \frac{\bar{k}-1}{n+r} \right) \leq \frac{j+1}{n+r} - \frac{\bar{k}-1}{n+r} \\ &\leq \frac{\bar{k} + \sqrt{\bar{k}} + 1}{n+r} - \frac{\bar{k}-1}{n+r} = \frac{\sqrt{\bar{k}} + 2}{n+r} \\ &\leq \frac{4\sqrt{x}}{\sqrt{n+r}} \leq \frac{4\sqrt{x}}{\sqrt{n}}, \end{aligned}$$

which on using with

$$\bar{k} - 1 = k_2 \leq k_2 + \sqrt{k_2} \leq j$$

for the final inequality, we obtain the expressions $\bar{k} \leq j + 1$ and $\sqrt{\bar{k}} + 2 \leq \sqrt{j+1} + 2 \leq 4\sqrt{j}$. Also in this case, since $j \geq 1$, $k_2 \leq j - 2$, we get

$$M_{\bar{k}-1,n,j}(x) \geq M_{\bar{k}-2,n,j}(x) \geq \cdots \geq M_{0,n,j}(x).$$

We thus obtain $M_{k,n,j}(x) \leq 4\sqrt{\frac{x}{n+r}}$ for any $k \leq j - 1$ and $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$. Then $M_{k,n,j}(x) \leq 4\sqrt{\frac{x}{n+r}}$ is obtained from these two sub-cases. Of all the values obtained above $M_{k,n,j}(x) \leq 4\sqrt{\frac{x}{n+r}}$ for every $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$. $E_n(x) \leq 4\sqrt{\frac{x}{n+r}}$ gives every $x \in \left[\frac{j}{n+r}, \frac{j+1}{n+r}\right]$, $n \in \mathbb{N}$, $\delta_n = 4\sqrt{\frac{x}{n+r}}$. Collecting all the above estimates we get

$$\left| S_n^{(M)}(f)(x) - (f)(x) \right| \leq 8W_1 \left(f : \sqrt{\frac{x}{n+r}} \right), \quad \forall n \in \mathbb{N}, x \in [0, \infty),$$

which completes the proof. $\square$

3 Conclusion

In this paper as the most important result, by defining the nonlinear modified Favard Szasz-Mirakyan operator, a more valuable error rate is calculated with the help of the continuity module and the approximation degree is also calculated.

Acknowledgments The authors are highly thankful to the referees and the Editor-in-Chief for their constructive criticism of the original version of this paper which, they believe, has improved the final version of this paper.

References

- [1] Acar, E., Karahan, D. and Serenbay, S.K. (2020). Approximation for the Bernstein operator of max-product kind in symmetric range, *Khayyam J. Math.*, 6(2), 257–273.
- [2] Bede, B., Coroianu, L. and Gal, S.G. (2010). Approximation and shape preserving properties of the nonlinear Favard Szasz–Mirakyan operator of max-product kind, *Filomat*, 24(3), 55–72.
- [3] Bede, B., Coroianu, L. and Gal, S.G. (2011). Approximation by truncated Favard Szasz–Mirakyan operator of max-product kind, *Demonstratio Mathematica*, 44(1), 105–122.
- [4] Bede, B., Coroianu, L. and Gal, S.G. (2016). Approximation by Max-Product Type Operators, Springer International Publishing, Switzerland.
- [5] Becker, M., Kucharski, D. and Nessel, R.J. (1978). Global approximation theorems for the Szász Mirakyan operators in exponential weight spaces, *Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1977)*, Birkhäuser Verlag, Basel, Internat. Series of Num. Math., 40, 319–333.
- [6] Güngör, Y. and Ispir, N. (2016). Quantitative estimates for generalized Szasz operators of max-product kind, *Results Math.*, 70(3–4), 447–456.
- [7] Ispir, N. and Atakut, C. (2002). Approximation by modified Favard Szasz–Mirakyan operators on weighted spaces, *Proc. Indian Acad. Sci. Math. Sci.*, 112(4), 571–578.
- [8] Rempulska, L. and Walczak, Z. (2001). Approximation properties of certain modified Favard Szasz-Mirakyan operators, *Le Matematiche (Catania)*, 55(1), 121–132.
- [9] Serenbay, S.K. and Dalmanoğlu, Ö (2017). Rate of convergence for generalized Szasz-Mirakyan operators in exponential weighted space, *Appl. Appl. Math.*, 12(2), 884–897.
- [10] Walczak, Z. (2002). On approximation by modified Favard Szasz-Mirakyan operators, *Glasnik Matematicki*, 37(2), 303–319.

Appendix: Abbreviations of symbols used

$\mathbb{N}$ The set of Natural numbers.

$\mathbb{R}$ The set of real numbers.

$\mathbb{R}_0^+$ The set of nonnegative real numbers.

$\bigvee$ The max operation.

$S_n(f; x)$ The linear Favard Szasz-Mirakyan operator.

$W(f; \delta)$ The continuity modulus of function f .

$S_n^{(M)}(f)(x)$ The maximum-product type modified Favard Szasz-Mirakyan operator.