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The twin primes *

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Abstract The primes, including the twin primes and the other prime pairs, are the building-blocks of the integers. Euclid's proof of the infinitude of the primes is generally regarded as elegant. It is a proof by contradiction, or, *reductio ad absurdum*, and it relies on an algorithm which always brings in larger and larger primes, an infinite number of them. However, the proof is also subtle and is misinterpreted by some, with one well-known mathematician even remarking that the algorithm might not work for extremely large numbers. A long unsettled related problem, the twin primes conjecture, has also aroused the interest of many researchers. The author has been conducting research on the twin primes for a long time and had published a paper on them (see, B. Wong, Possible solutions for the "twin" primes conjecture - The infinity of the twin primes, International Mathematical Journal, 3(8), 2003, 873–886). This informative paper presents some important facts on the twin primes which would be of interest to prime number researchers, with some remarks/reasons that point to the infinitude of the twin primes, including a reasoning which is somewhat similar to Euclid's proof of the infinity of the primes; very importantly, two algorithms are developed in Section 5 for sieving out the twin primes from the infinite list of the integers, which would be of interest to cryptographers and even to computer programmers.

Key words indivisible, new primes, twin primes.

2020 Mathematics Subject Classification 11A41, 11A99.

1 Introduction

The twin primes are prime number pairs which differ by 2. In 1919, Viggo Brun (1885-1978) proved that the sum of the reciprocals of the twin primes converges to Brun's constant (see Brun, Bulletin des Sciences Mathématiques (in French), 43: 100 -104, 124 -128) :

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\ldots = 1.9021605\ldots.$$

It is evident that the twin primes thin out as infinity is approached. The problem of whether there is an infinitude of twin primes is an inherently difficult one to solve, as infinity (normally symbolized by: ∞) is a difficult concept and is against common sense. It is impossible to count, calculate or live to infinity,

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perhaps with the exception of God. Infinity is a nebulous idea and appears to be only an abstraction devoid of any actual practical meaning. How do we quantify infinity? How big is infinity? We could either attempt to prove that the twin primes are finite, or, infinite. If the twin primes were finite, how could we prove that a particular pair of twin primes is the largest existing pair of twin primes, and, if they were infinite, how could we prove that there are always larger and larger pairs of them? It is evidently difficult to prove either, with the former appearing more difficult to prove as the odds seem against it. This paper provides some remarks/reasons in support of the latter, i.e., the infinitude of the twin primes. For the interested reader we present a selected reading in the references [1–8].

2 Main results: the infinity of twin primes

The following chain of reasoning points to the possibility of an infinity of the twin primes:-
Let $3, 5, 7, 11, 13, 17, 19, \dots, (n-2), n$ be the list of consecutive primes, wherein n and $(n-2)$ are assumed to be the largest existing twin primes pair, within the infinite list of the primes. Let $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = a$.

Lemma 2.1. *$(a \times \dots \times (n-2) \times n) - 2$, and $(a \times \dots \times (n-2) \times n) - 4$ will never be divisible by any of the consecutive primes in the list: $3, 5, 7, 11, 13, 17, 19, \dots, (n-2), n$, whether they are prime or composite.*

See Section 3 for the proof of the above lemma.
This implies that if $(a \times \dots \times (n-2) \times n) - 2$ and/or $(a \times \dots \times (n-2) \times n) - 4$ are prime, then $(a \times \dots \times (n-2) \times n) - 2 > (a \times \dots \times (n-2) \times n) - 4 > n > n-2$. Further, if $(a \times \dots \times (n-2) \times n) - 2$ and/or $(a \times \dots \times (n-2) \times n) - 4$ are non-prime/composite, then

(a) each prime factor, e.g., y below, of $(a \times \dots \times (n-2) \times n) - 2 > n > n-2$,
(b) each prime factor, e.g., z below, of $(a \times \dots \times (n-2) \times n) - 4 > n > n-2$,

then
$$(a \times \dots \times (n-2) \times n) - 2 = \text{ prime or composite,} \tag{2.1}$$
and
$$(a \times \dots \times (n-2) \times n) - 4 = \text{ prime or composite.} \tag{2.2}$$
The numbers in (2.1) and (2.2) are both twin primes, if both the numbers in (2.1) and (2.2) are prime and further that both the numbers in (2.1) and (2.2) are greater than both n and $n-2$.
Let Y represent the prime factors of $(a \times \dots \times (n-2) \times n) - 2$ if $(a \times \dots \times (n-2) \times n) - 2$ is not prime (i.e., if it is composite), each prime factor may pair up with another prime which differs from it by 2 to form twin primes. Let y be the prime factor in Y. Then y and $y + / - 2$ are twin primes, if $y + / - 2$ is prime and y and $y + / - 2$ both exceed n and $n-2$.
Let Z represent the prime factors of $(a \times \dots \times (n-2) \times n) - 4$ if $(a \times \dots \times (n-2) \times n) - 4$ is not prime (i.e., it is composite), each prime factor may pair up with another prime which differs from it by 2 to form twin primes. Let z be the prime factor in Z. Then z and $z + / - 2$ are twin primes, if $z + / - 2$ is prime and both z and $z + / - 2$ exceed n and $n-2$. Therefore,

$$(a \times \dots \times (n-2) \times n) - 2 > (a \times \dots \times (n-2) \times n) - 4 > y \text{ or } y + / - 2 \text{ or } z \text{ or } z + / - 2 > n > n-2.$$

By the above reasoning, the following, which implies that n and $n-2$ are the largest existing twin primes pair, is an oddity:

$$n > n-2 > (a \times \dots \times (n-2) \times n) - 2 > (a \times \dots \times (n-2) \times n) - 4 > y \text{ or } y + / - 2 \text{ or } z \text{ or } z + / - 2.$$

It seems that no number n and $n-2$ in any list of consecutive primes can ever possibly be the largest existing twin primes pair and larger twin primes than them can always be found by applying the same mathematical logic (as is described in Section 3), e.g., by utilizing the evidently effective Algorithm 5.1. That is, a largest existing twin primes pair is an oddity, which suggests that the twin primes are infinite. It is possible to find larger twin primes than n and $n-2$ no matter however large n and $n-2$ are, with the following formulae involving the list of consecutive primes: $(a \times \dots \times (n-2) \times n) - 2$ and $(a \times \dots \times (n-2) \times n) - 4$, which by the nature of their composition are capable of generating new primes/twin primes which will always be larger than n and $n-2$ (see Section 3); this operation is part of Algorithm 5.1. This is an indirect argument, or, argument by contradiction (*reductio ad absurdum*) for the infinity of the twin primes, for our assumption of n and $n-2$ as the largest existing twin primes pair will be contradicted by the discovery of larger twin primes, implying the infinity of the twin primes. Again, by applying the same mathematical logic (as described in Section 3), by way of the evidently effective Algorithm 5.1, and going one step further, we can find that many twin odd integers found between n and $(a \times \ldots \times (n-2) \times n) - 2$, which differ from one another by 2 and are not divisible by any of the primes in the list of consecutive primes: $3, 5, 7, 11, 13, 17, 19, \ldots, n$, will be twin primes larger than n and $n-2$, our assumed the largest existing twin primes pair, which are further or more contradictions of this assumption. In this manner, i.e., by resorting to Algorithm 5.1, by continually adding more and more consecutive primes to the list of consecutive primes: $3, 5, 7, 11, 13, 17, 19, \ldots, n$, i.e., continually extending the value of n, and utilizing the formula for $(a \times \ldots \times (n-2) \times n) - 2$, as well as the formula for $(a \times \ldots \times (n-2) \times n) - 4$, to perform the computations according to Algorithm 5.1, many larger and larger twin primes can be found, all the way to infinity, in parallel with the infinitude of the list of consecutive primes: $3, 5, 7, 11, 13, 17, 19, \ldots$, of which the twin primes are a part together with other primes pairs, wherein the twin primes are not at all likely to be finite (as is evident from Section 4) and can be expected to be infinite. (It is important to note that Algorithm 5.1 would be able to fish out or generate a big quantity of twin primes which are larger than n and $n-2$, our assumed largest existing twin primes pair, giving rise to a large number of contradictions of this assumption, thus implying further the infinitude of the twin primes, as is shown in great detail in Section 5, but is only briefly described above.) A largest existing twin primes pair would now seem to be an impossibility. It can now be concluded that the twin primes are infinite.

3 Proof of Lemma 2.1

Proof. We prove Lemma 2.1 by by using the same mathematical logic that Euclid had used in proving the infinity of the primes. Note that the (only) even prime 2 is omitted from the list of consecutive primes: $3, 5, 7, 11, 13, 17, 19, \ldots, n-2, n$ stated in the paper, wherein n and $n-2$ are assumed to be the largest existing twin primes pair. The list of newly created primes, and twin primes for $n = 5, 7, 11, 13, 17, 19, \ldots$ ($n = 19$ being the maximum limit achievable with a hand-held calculator) is as follows:-

3.1 For $n = 5$, we get the following new primes/new twin primes:

$$(3 \times 5) - 2 = 13 \quad \text{(A)}$$

$$(3 \times 5) - 4 = 11 \quad \text{(B)}$$

3.2 For $n = 7$, we get the following new primes/new twin primes:

$$(3 \times 5 \times 7) - 2 = 103 \quad \text{(A)}$$

$$(3 \times 5 \times 7) - 4 = 101 \quad \text{(B)}$$

3.3 For $n = 11$, we get the following new primes/new twin primes:

$$(3 \times 5 \times 7 \times 11) - 2 = 1,153 \quad \text{(A)}$$

$$(3 \times 5 \times 7 \times 11) - 4 = 1,151 \quad \text{(B)}$$

3.4 For $n = 13$, we get the following new prime and composite number with its prime factors:

$$(3 \times 5 \times 7 \times 11 \times 13) - 2 = 15,013 - \text{Prime Number} \quad \text{(A)}$$

$$(3 \times 5 \times 7 \times 11 \times 13) - 4 = 15,011 - \text{Composite Number} \quad \text{(B)}$$

(Note that $15,011 = 17 \times 883$, with 17 pairing with 19 to form a twin primes pair and 883 pairing with 881 to form another twin primes pair.)

3.5 For $n = 17$, we get the following new primes/new twin primes:

$$(3 \times 5 \times 7 \times 11 \times 13 \times 17) - 2 = 255,253 \quad \text{(A)}$$

$$(3 \times 5 \times 7 \times 11 \times 13 \times 17) - 4 = 255,251 \quad \text{(B)}$$

3.6 For $n = 19$, we get the following new prime and composite number with its prime factors:

$$(3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19) - 2 = 4,849,843 - \text{Prime Number} \tag{A}$$

$$(3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19) - 4 = 4,849,841 - \text{Composite Number} \tag{B}$$

(Note that $4,849,841 = 43 \times 112,787$, with 43 pairing with 41 to form a twin primes pair while 112,787 is a stand-alone prime.)

$\vdots$

3.1 Consolidated results from the equations (A) and (B) above

We now summarize our results from the equations (A) and (B) above as follows:

3.1.1a All the equations marked (A) above in the six cases discussed generate six new primes, viz., 13; 103; 1,153; 15,013; 255,253; 4,849,843 respectively and no composite numbers.

3.1.2a Among the equations marked (B) above in the six cases, four of them generate four new primes, viz., 11; 101; 1,151; 255,251 and the remaining two generate two composite numbers, viz., $15,011 = 17 \times 883$, and $4,849,841 = 43 \times 112,787$.

3.1.3a Among the six pairs of equations marked above as (A) and (B), four of them together produce four pairs of new twin primes, viz., 13 and 11; 103 and 101; 1,153 and 1,151; and 255,253 and 255,251.

3.1.4a The prime factors of the numbers in the equations marked (A) and (B) above in the six cases form three pairs of new twin primes with prime partners which differ from them by 2, viz., 19 and 17; 43 and 41; 883 and 881.

3.1.5a All the new twin primes in **3.1.3a** and **3.1.4a** above are larger than n and $n-2$, the assumed largest existing twin primes pair, which provides an indirect argument for the infinitude of the twin primes.

3.2 Why is it impossible for any n and $n - 2$ to be the largest existing twin primes pair?

$\alpha = (3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \ldots \times n) - 2$ and $\beta = (3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \ldots \times n) - 4$ will never be divisible by any of the consecutive prime numbers in the list: $3, 5, 7, 11, 13, 17, 19, \ldots, n$ whether they are prime or composite (non-prime and divisible by prime numbers or prime factors). This means that none of the consecutive prime numbers in the list: $3, 5, 7, 11, 13, 17, 19, \ldots, n$ can ever be the factors of α and β. Thus α and β must be new primes/twin primes larger than all the consecutive prime numbers in the list: $3, 5, 7, 11, 13, 17, 19, \ldots, n$, or, if they were composite (non-prime and divisible by prime numbers or prime factors), their prime factors (and "twin prime" partners which differ from them by 2) must be larger than all the consecutive prime numbers in the list: $3, 5, 7, 11, 13, 17, 19, \ldots, n$. *This is a very important mathematical logic, which needs to be grasped in order to understand the argument.* This all implies that no n and $n - 2$ (if $n - 2$ were also a prime number) in any list of consecutive prime numbers can ever possibly be the largest existing twin primes pair, since all the new primes/twin primes produced or generated by α and β will always be larger than n and $n - 2$. That is, the largest existing twin primes pair is an impossibility, which implies the infinitude of the list of the primes/twin primes.

In other words, by the mathematical logic stated above, which explains why all the new primes/twin primes, which α and β by the nature of their composition are capable of producing or generating, will always be larger than n and $n - 2$, no n and $n - 2$ in any list of consecutive prime numbers: $3, 5, 7, 11, 13, 17, 19, \ldots, n$ can ever possibly be the largest existing twin primes pair, i.e., a largest existing twin primes pair is an impossibility, thus implying the infinitude of the list of the twin primes. This is a very important inference.

Regardless of how long the list of the twin primes pairs is, it is possible to find some new twin primes pairs which will always be larger than n and $n-2$, our assumed largest existing twin primes pair – the largest twin primes pair in our assumed finite list of the twin primes pairs, with α and β, which is an indirect argument for the infinity of the twin primes. In fact, by the same principle, many twin odd integers found between n and α, which differ from one another by 2 and are not divisible by any of the primes in the list of consecutive primes: $3, 5, 7, 11, 13, 17, 19, \ldots, n$ will be twin primes pairs larger than n and $n-2$, our assumed largest existing twin primes pair, which is a contradiction to this assumption, hence implying an infinitude of the twin primes. We also refer the reader to Algorithm 5.1. □

4 Anecdotal evidence of the infinity of the twin primes

4.1 Top twin primes discovered in the years 2000, 2001, 2007 and 2009

We now describe the top twin primes discovered in the years 2000, 2001, 2007 and 2009. In the year 2000, the numbers $4648619711505 \times 2^{60000} \pm 1$ (18,075 digits) was the top twin primes pair that was discovered. In the year 2001, it only ranked eighth in the list of top 20 twin primes pairs, with the numbers $318032361 \times 2^{107001} \pm 1$ (32,220 digits) topping the list. In the year 2007, in the list of top 20 twin primes pairs, the numbers $318032361 \times 2^{107001} \pm 1$ (32,220 digits) ranked eighth, while the numbers $4648619711505 \times 2^{60000} \pm 1$ (18,075 digits) were nowhere to be seen. The numbers $2003663613 \times 2^{195000} \pm 1$ (58,711 digits), which were discovered on January 15, 2007, by Eric Vautier (from France) of the Twin Prime Search (TPS) project in collaboration with PrimeGrid (BOINC platform) were at the top of the list. As of August 2009, the numbers $65516468355 \times 2^{333333} \pm 1$ (100,355 digits) were at the top of the list of top 20 twin primes pairs, while $318032361 \times 2^{107001} \pm 1$ (32,220 digits) ranked eleventh, and $2003663613 \times 2^{195000} \pm 1$ (58,711 digits) ranked second in this list.

We can expect larger twin primes than these extremely large twin primes, much larger ones and of course, infinitely larger ones, to be discovered in future. We tabulate below in Table 1 the list of prime pairs for the first 2,500 consecutive primes from 2 to 22,307 ranked according to their frequencies of appearance, which shows a total of 2,498 primes pairs in the list.

Table 1: List of primes.

S.No.	Ranking	Prime pairs	No.of pairs	Percentage
(1)	1	primes pair separated by 6 integers	482	19.29 %
(2)	2	primes pair separated by 4 integers	378	15.13 %
(3)	3	primes pair separated by 2 integers (twin primes)	376	15.05 %
(4)	4	primes pair separated by 12 integers	267	10.68 %
(5)	5	primes pair separated by 10 integers	255	10.20 %
(6)	6	primes pair separated by 8 integers	229	9.16 %
(7)	7	primes pair separated by 14 integers	138	5.52 %
(8)	8	primes pair separated by 18 integers	111	4.44 %

Continued on the next page

Table 1 – Continued from the previous page

S.No.	Ranking	Prime pairs	No. of pairs	Percentage
(9)	9	primes pair separated by 16 integers	80	3.20 %
(10)	10	primes pair separated by 20 integers	47	1.88 %
(11)	11	primes pair separated by 22 integers	46	1.84 %
(12)	12	primes pair separated by 30 integers	24	0.96 %
(13)	13	primes pair separated by 28 integers	19	0.76 %
(14)	14	primes pair separated by 24 integers	16	0.64 %
(15)	15	primes pair separated by 26 integers	10	0.40 %
(16)	16	primes pair separated by 34 integers	9	0.36 %
(17)	17	primes pair separated by 36 integers	5	0.20 %
(18)	18	primes pair separated by 32 integers	2	0.08 %
(19)	18	primes pair separated by 40 integers	2	0.08 %
(20)	19	primes pair separated by 42 integers	1	0.04 %
(21)	19	primes pair separated by 52 integers	1	0.04 %

It is evident from the above list that the primes pairs separated by 6 integers, 4 integers and 2 integers (twin primes), among the 21 classifications of primes pairs separated by from 2 integers to 52 integers (primes pairs separated by 38 integers, 44 integers, 46 integers, 48 integers and 50 integers are not among them, but, they are expected to appear further down in the infinite list of the primes), are the most dominant, important. There is a long list of other primes pairs, besides those shown in the above list, which also play a part as the building-blocks of the infinite list of the integers.
The list of the integers is infinite. The list of the primes is also infinite. The infinite primes are the building-blocks of the infinite integers - the infinite odd integers are all either primes or composites of primes, and, the infinite even integers, except for 2 which is a prime, are all also composites of primes. Therefore, all the primes pairs separated by the integers of various magnitudes, as described above, can never all be finite. If there is any possibility at all for any of these primes pairs to be finite, there

is only the possibility that a number of these primes pairs are finite (but never all of them). However, will it have to be the primes pairs separated by 2 integers or twin primes (which are the subject of our investigation here), which are the only primes pairs, or, one among a number of primes pairs, which are finite? Why question only the infinity of the primes pairs separated by 2 integers, the twin primes? Are not the infinities of the primes pairs separated by 8 integers and more, whose frequencies of appearance are lower, as compared to those of the primes pairs which are separated by 6, 4 and 2 integers respectively, in the above list of primes pairs, more questionable? Why single out only the twin primes? (There are at least 18 other primes pairs, separated by from 8 integers to 52 integers, whose respective infinities should be more suspect, as is evident from the above list of primes pairs, if any infinities should be doubted. Evidently, the primes pairs separated by 2 integers (twin primes) are not at all likely to be finite.)
The above represents anecdotal evidence that the twin primes are infinite, which is a ratification of the reasoning given earlier.

5 Two algorithms

Now we present two algorithms which will be able to generate or sieve all the twin primes in any range of odd numbers which are all larger than those in the list of known consecutive primes/twin primes; these two important algorithms will provide plenty of numerical evidence that the twin primes are infinite:-

Algorithm 5.1. Consider the following list of products of consecutive primes/twin primes, which should be sufficient for our purpose here:-

(5.1.1) $3 \times 5 = 15$.
(5.1.2) $3 \times 5 \times 7 = 105$.
(5.1.3) $3 \times 5 \times 7 \times 11 = 1,155$.
(5.1.4) $3 \times 5 \times 7 \times 11 \times 13 = 15,015$.
(5.1.5) $3 \times 5 \times 7 \times 11 \times 13 \times 17 = 255,255$.
(5.1.6) $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 4,849,845$.

$$\vdots$$

We would provide below now an example with the items (5.2.1) to (5.2.3) from the above Algorithm 5.1.

Example 5.2.

5.2.1) For $3 \times 5 = 15$, we would find all the consecutive pairs of odd numbers between 5 and 15 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3 and 5 in the list of consecutive primes/twin primes 3×5 whose product is 15.
There is only 1 pair of odd numbers between 5 and 15 which differ from one another by 2 and are not divisible by the consecutive primes/twin primes 3 and 5 in the list of consecutive primes/twin primes 3×5 – they are the twin primes 11 and 13.

5.2.2) Similarly, for $3 \times 5 \times 7 = 105$, we would find all the consecutive pairs of odd numbers between 7 and 105 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3, 5 and 7 in the list of consecutive primes/twin primes $3 \times 5 \times 7$ whose product is 105.
The consecutive pairs of odd numbers between 7 and 105 which differ from one another by 2 and are not divisible by the consecutive primes/twin primes 3, 5 and 7 are the following consecutive twin primes:

(5.2.2a) 11 and 13,
(5.2.2b) 17 and 19,
(5.2.2c) 29 and 31,
(5.2.2d) 41 and 43,

(5.2.2e) 59 and 61,
(5.2.2f) 71 and 73,
(5.2.2g) 101 and 103.

5.2.3) Similarly, in this final case, for $3 \times 5 \times 7 \times 11 = 1,155$, we would find all the consecutive pairs of odd numbers between 11 and 1,155 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3, 5, 7 and 11 in the list of consecutive primes/twin primes $3 \times 5 \times 7 \times 11$ whose product is 1,155.
Many of the consecutive pairs of odd numbers between 11 and 1,155 which differ from one another by 2 and are not divisible by the consecutive primes/twin primes 3, 5, 7 and 11 are twin primes (while the rest are primes larger than 3, 5, 7 and 11 and/or composite numbers whose prime factors are each larger than 3, 5, 7 and 11), some of which are as follows:
(5.2.3a) 17 and 19,
(5.2.3b) 29 and 31,
(5.2.3c) 41 and 43,
(5.2.3d) 59 and 61,
(5.2.3e) 71 and 73,
(5.2.3f) 101 and 103,
(5.2.3g) 107 and 109,
(5.2.3h) 137 and 139,
(5.2.3i) 149 and 151,
(5.2.3j) 179 and 181,
(5.2.3k) etc. to 1,151 and 1,153.
Proceeding in this way, we would also be able to achieve the following:-

5.2.4) For $3 \times 5 \times 7 \times 11 \times 13 = 15,015$, find all the consecutive twin primes between 13 and 15,015.

5.2.5) For $3 \times 5 \times 7 \times 11 \times 13 \times 17 = 255,255$, find all the consecutive twin primes between 17 and 255,255.

5.2.6) For $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 4,849,845$, find all the consecutive twin primes between 19 and 4,849,845.

$$\vdots$$

Now we present our next algorithm below similar to the above Algorithm 5.1.

Algorithm 5.3. Consider the following list of products of consecutive primes/twin primes, which should be sufficient for our purpose here:-

(5.3.1) $3 \times 5 = 15.$
(5.3.2) $3 \times 5 \times 7 = 105.$
(5.3.3) $3 \times 5 \times 7 \times 11 = 1,155.$
(5.3.4) $3 \times 5 \times 7 \times 11 \times 13 = 15,015.$
(5.3.5) $3 \times 5 \times 7 \times 11 \times 13 \times 17 = 255,255.$
(5.3.6) $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 4,849,845.$

$$\vdots$$

We give below the next example based on a different and more complex algorithm, viz., Algorithm 5.3, which we do not recommend as it is a less efficient and more time-consuming algorithm involving more steps (whose mathematical logic is subtle and might be difficult to understand, but it is nevertheless an interesting algorithm):

Example 5.4.

5.4.1) For $3 \times 5 = 15$, we would first find all the consecutive pairs of even numbers between 5 and 15 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3 and 5 in the list of consecutive primes/twin primes 3×5. Then we deduct each of these consecutive pairs of even numbers which are not divisible by any of the consecutive primes/twin primes 3 and 5 from the product of these consecutive primes/twin primes 3×5 which is 15. The results would each be one pair of twin primes (which are each larger than 3 and 5), one prime (which is larger than 3 and 5) and one composite of primes (whose prime factors are each larger than 3 and 5), or, 2 composites of primes (whose prime factors are each larger than 3 and 5). In this way, we would be able to find all the consecutive twin primes between 5 and 15. There are no pairs of even numbers between 5 and 15 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3 and 5 in the list of consecutive primes/twin primes 3×5 – the exception is the smallest pair of the even numbers 2 and 4, which are not divisible by any of the consecutive primes/twin primes 3 and 5 in the list of consecutive primes/twin primes 3×5.

The following is the result after we deduct this pair of even numbers 2 and 4 which are not divisible by any of the consecutive primes/twin primes 3 and 5 from the product of these consecutive primes/twin primes 3×5 which is 15:

(5.4.1a) 15 – 2 and 15 – 4, i.e., 13 and 11 (twin primes).

5.4.2) Similarly, for $3 \times 5 \times 7 = 105$, we would first find all the consecutive pairs of even numbers between 7 and 105 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3, 5 and 7 in the list of consecutive primes/twin primes $3 \times 5 \times 7$, which are as follows (the smallest pair of even numbers 2 and 4 is the exception and they are not divisible by any of the consecutive primes/twin primes 3, 5 and 7 in the list of consecutive primes/twin primes $3 \times 5 \times 7$):

(5.4.2a) 2 and 4,

(5.4.2b) 32 and 34,

(5.4.2c) 44 and 46,

(5.4.2d) 62 and 64,

(5.4.2e) 74 and 76,

(5.4.2f) 86 and 88,

(5.4.2g) 92 and 94.

Then we deduct each of those consecutive pairs of even numbers which are not divisible by any of the consecutive primes/twin primes 3, 5 and 7 from the product of these consecutive primes/twin primes $3 \times 5 \times 7$ which is 105. The results would each be one pair of twin primes (which are each larger than 3, 5 and 7), one prime (which is larger than 3, 5 and 7) and a composite of primes (whose prime factors are each larger than 3, 5 and 7), or, two composites of primes (whose prime factors are each larger than 3, 5 and 7). In this way, we would be able to find all the consecutive twin primes between 7 and 105 which are as follows:

(5.4.2.1a) 105 – 2 and 105 – 4, i.e., 103 and 101 (twin primes),

(5.4.2.1b) 105 – 32 and 105 – 34, i.e., 73 and 71 (twin primes),

(5.4.2.1c) 105 – 44 and 105 – 46, i.e., 61 and 59 (twin primes),

(5.4.2.1d) 105 – 62 and 105 – 64, i.e., 43 and 41 (twin primes),

(5.4.2.1e) 105 – 74 and 105 – 76, i.e., 31 and 29 (twin primes),

(5.4.2.1f) 105 – 86 and 105 – 88, i.e., 19 and 17 (twin primes),

(5.4.2.1g) 105 – 92 and 105 – 94, i.e., 13 and 11 (twin primes).

5.4.3) Similarly, in this final case, for $3 \times 5 \times 7 \times 11 = 1,155$, we would first find all the consecutive pairs of even numbers between 11 and 1,155 which differ from one another by 2 and are not divisible by any of the consecutive primes/twin primes 3, 5, 7 and 11 in the list of consecutive primes/twin primes $3 \times 5 \times 7 \times 11$, some of which are as follows (the smallest pair of even numbers 2 and 4 is the exception and they are not divisible by any of the consecutive primes/twin primes 3, 5, 7 and 11 in the list of consecutive primes/twin primes $3 \times 5 \times 7 \times 11$):

(5.4.3a) 2 and 4,

(5.4.3b) 32 and 34,
(5.4.3c) 62 and 64,
(5.4.3d) 74 and 76,
(5.4.3e) 92 and 94,
(5.4.3f) 116 and 118,
(5.4.3g) 122 and 124,
(5.4.3h) 134 and 136,
(5.4.3i) etc. to 1,136 and 1,138.

Next we deduct each of these consecutive pairs of even numbers which are not divisible by any of the consecutive primes/twin primes 3, 5, 7 and 11 from the product of these consecutive primes/twin primes $3 \times 5 \times 7 \times 11$ which is 1,155. The results would each be a pair of twin primes (which are each larger than 3, 5, 7 and 11), a prime (which is larger than 3, 5, 7 and 11) and a composite of primes (whose prime factors are each larger than 3, 5, 7 and 11), or, two composites of primes (whose prime factors are each larger than 3, 5, 7 and 11). In this way, we would be able to find all the consecutive twin primes between 11 and 1,155, some of which are as follows:

(5.4.3.1a) 1,155 – 2 and 1,155 – 4, i.e., 1,153 and 1,151 (twin primes),

(5.4.3.1b) 1,155 – 32 and 1,155 – 34, i.e., 1,123 (prime) and 1,121 ($= 19 \times 59$),
(It may be seen here that 1121 is the composite of primes which are each larger than 3, 5, 7 and 11.)

(5.4.3.1c) 1,155 – 62 and 1,155 – 64, i.e., 1,093 and 1,091 (twin primes),

(5.4.3.1d) 1,155 – 74 and 1,155 – 76, i.e., 1,081 ($= 23 \times 47$) and 1,079 ($= 13 \times 83$),
(It may be observed here that both the numbers 1,081 and 1,079 are composites of primes which are each larger than 3,5,7 and 11.)

(5.4.3.1e) 1,155 – 92 and 1,155 – 94, i.e., 1,063 and 1,061 (twin primes),

(5.4.3.1f) 1,155 – 116 and 1,155 – 118, i.e., 1,039 (prime) and 1,037 ($= 17 \times 61$),
(It may be observed here that the number 1,037 is a composite of primes which are each larger than 3,5,7 and 11.)

(5.4.3.1g) 1,155 – 122 and 1,155 – 124, i.e., 1,033 and 1,031 (twin primes),

(5.4.3.1h) 1,155 – 134 and 1,155 – 136, i.e., 1,021 and 1,019 (twin primes),

(5.4.3.1i) etc. to 1,155 – 1,136 and 1,155 – 1,138, i.e., 19 and 17 (twin primes).

In a similar manner, we can also achieve the following:-

5.4.4) For $3 \times 5 \times 7 \times 11 \times 13 = 15,015$, to find all the consecutive twin primes between 13 and 15,015.

5.4.5) For $3 \times 5 \times 7 \times 11 \times 13 \times 17 = 255,255$, to find all the consecutive twin primes between 17 and 255,255.

5.4.6) For $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 4,849,845$, find all the consecutive twin primes between 19 and 4,849,845.

$$\vdots$$

By utilizing any of the above Algorithms 5.1 and 5.3 (preferably the evidently more efficient Algorithm 5.1), we will be able to find many twin primes which are all larger than those in any chosen list of consecutive primes/twin primes, i.e., we will be able to generate many larger and larger twin primes with any of the above algorithms.
It would evidently be difficult to accept an argument supporting the twin primes conjecture without having to confirm or check the validity of the logic by computing a sufficiently long list of twin primes, even to the extent of looking out for counter-examples. Hence, the great importance of the above Algorithms 5.1 and 5.3.

6 Further remarks on the twin primes

We note a very important intrinsic characteristic of the primes. Like all the houses in a neighborhood or in a location which are separated from each other by the number of houses between them, the primes are also separated from each other by the number of integers separating them. The closest will of course be the prime neighbors separated by 2 integers (i.e., twin primes), followed next in proximity by the prime neighbors separated respectively by 4, 6, 8, 10, 12, ... and so on integers as shown in Section 4. The twin primes are actually comparable to 2 closest neighbors living just next door to one another. There will always be 2 closest next-door neighbors, neighbors living 2 doors away, neighbors living respectively, 3, 4, 5, 6, ... doors away, and so on, by greater and greater intervals, in any neighborhood or residential area. There will always be different intervals separating all the houses in a neighborhood or location. Similarly, in the infinite list of the primes, there will always be different intervals separating all the primes, ranging from the smallest interval of 2 integers (in the case of the twin primes) to 4, 6, 8, 10, 12, ... integers, and more and more integers, etc., which is an intrinsic characteristic of the primes. In other words, there will always be intervals of various magnitudes or sizes (i.e., intervals of various numbers of integers) between, separating, all the primes in the infinite list of the primes, and, each of these intervals of various magnitudes or sizes can be expected to be infinite as the list of the primes is infinite. The twin primes, which we are examining here, are not at all likely to be finite (as is evident from Section 4), and should be infinite; in fact, to say that the twin primes are finite is like saying that next-door neighbors who are closest are rare and limited, which, in our opinion, is absurd.

7 Conclusion and concluding remarks

The list of the twin primes appears to be infinite, which is ratified by the strong anecdotal evidence provided in Section 4. Very importantly, Algorithm 5.1 provides the means for fishing out the twin primes from the infinite list of the integers; a strong argument supporting the infinity of the twin primes without any means or algorithm for finding them would be less strong than an argument backed by some algorithm for finding the twin primes - such an argument strongly backed by the algorithm would be a constructive argument. This algorithm would be able to sieve out many, many twin primes from the list of the integers, to infinity. Finally, it should be noted that as "closest next-door neighbors" among the primes, the infinity of the twin primes seems evident, a point which is explained in Section 6. There are 376 pairs of twin primes (752 primes) found within the 2,500 consecutive primes from 2 to 22,307 – this means that 30.08%, which is sizable, of the 2,500, not a small quantity, consecutive primes are twin primes. 3, 5 and 7 are the only "triple" primes found. There is no regularity in pattern in the appearance of the twin primes, except that the intervals between consecutive twin primes vary greatly by from 4 integers to 370 integers – the intervals between the consecutive twin primes increase and decrease, and, then increase and decrease again, by turns, giving rise to a graph that is characterized by many peaks, i.e., the curve is rough and nonlinear, making its description (hence, forecast of the twin primes) by differential equations practically impossible.
The argument used here to support the twin primes' infinity is the indirect (*reductio ad absurdum*) method, which was used by Euclid and other mathematicians after him. Logically, one or two examples of "contradiction" should be a sufficient evidence of infinity, for it does not make sense to have a need for an infinite number of cases of "contradiction", as our argument would then have to be infinitely and impossibly long, an absurdity. This method of argument is "argument by implication" as a result of "contradiction" - which is a "short-cut" and smart way in indicating infinity, instead of "showing infinity by counting to infinity", which is ludicrous, and, impossible. Hence, one or two cases of "contradiction" should be sufficient for implying that there would be an infinitude of twin primes, which of course also tacitly implies that there would be an infinitude of the number of cases of such "contradiction". (Euclid evidently had this logical point in mind when he formulated the indirect (*reductio ad absurdum*) argument for the infinity of the primes.) This method of argument is cleverly used by a number of mathematicians, even by the great German mathematician, David Hilbert, who used an indirect method (the "*reductio ad absurdum*" argument) to support Gordan's theorem without having to show an actual "construction", an argument which was accepted by his peers.
The paper presents two Algorithms 5.1 and 5.3 for generating or sieving many of the twin primes in any range of odd numbers – by utilizing any of these two algorithms (preferably the evidently more efficient Algorithm 5.1), we can find many twin primes which are all larger than those in any chosen list of consecutive primes, i.e., we can generate many larger and larger twin primes. This is indeed significant. There is evidently some deep meaning in the ease with which the twin primes turn up, as is shown in this paper. It is thus evident that the twin primes are an inherent characteristic of the infinite

prime numbers (as well as odd numbers), a characteristic which could be regarded as "self-similar" or "fractal". A twin primes pair is in effect any pair of odd numbers which differ from one another by 2 and are indivisible by any number except itself, the negative of itself, +1 and -1 (i.e., the pair of odd numbers are prime numbers). Any consecutive odd numbers or odd numbers that differ from one another by 2 are therefore potential prime numbers, as well as potential twin primes, and, the likelihood of them being prime is infinite (vide Euclid's proof and Dirichlet's Theorem), i.e., the primes will always be found amongst them and will be there all the way to infinity (the primes being evidently the "atoms" or building-blocks of all the whole numbers or integers, i.e., all the odd numbers and even numbers – every odd number or integer is either a prime number or a composite of prime numbers (i.e., the integer has prime factors), and, every even number is the sum of two prime numbers (vide the Goldbach conjecture which, it appears, practically all mathematicians believe to be true), as well as the product of prime numbers (composite)); hence, the likelihood of them being twin primes is infinite as well (the twin primes being an inherent property of the infinite prime numbers - as well as odd numbers - the twin primes can in fact be likened to next-door neighbors, which is a common, expected thing (see Section 6).

So far, there has not been any indication or confirmation that the number of twin primes is finite and the so-called largest existing pair of twin primes has not been found and confirmed (which of course would be impossible to find and confirm if the twin primes were infinite). On the other hand, practically everyone could intuit that the number of twin primes is infinite.

Due to the evident effectiveness of the two Algorithms 5.1 and 5.3 in bringing out larger and larger twin primes, the above argument for the infinitude of the twin primes is not only an indirect argument or argument by contradiction (*reductio ad absurdum*), importantly, it is also a constructive argument. It should be noted that the characteristic of a mountain or infinite volume of sand is reflected in the characteristic of some grains of sand found there so that studying the characteristic of some grains of sand found there is enough for deducing the characteristic of the mountain or infinite volume of sand, to ascertain the quality of a batch of products it is only necessary to inspect some carefully selected samples from that batch of products and not every one of the products and to carry out a population census, i.e., to find out the characteristics of a population, it is only necessary to carry out a survey on some carefully selected respondents and not the whole population. With the aid of any of these two Algorithms 5.1 and 5.3 (preferably the evidently more efficient Algorithm 5.1), in a like manner, by the same principle, we can carry out a study of a carefully selected list of integers and their associated primes and twin primes and deduce by induction whether the twin primes would always turn up, appear infinitely, in the list which is itself infinite - this act is similar to extrapolation!

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