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A new product of superposition and differentiation operators between Hardy and Zygmund spaces *

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Abstract Our goal in this article is to characterize the boundedness and the compactness of the product of the superposition operator followed by the differentiation operator DS_ϕ from the H^∞ space to the Zygmund space. Moreover, we give the necessary and sufficient conditions for the DS_ϕ operator from the H^∞ space to the Zygmund space to be bounded and compact.

Key words Superposition operator, differentiation operator, compactness, DS_ϕ operator.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. Let ϕ be a complex-valued function in the plane.The superposition operator S_ϕ is defined as follows (see [2]):

Definition 1.1. Let X and Y be two metric spaces of $\mathcal{H}(\mathbb{D})$ and ϕ denote a complex-valued function in the plane $\mathbb{C}$ such that $\phi \circ f \in Y$ whenever $f \in X$, we say that ϕ acts by superposition from X into

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Y and the superposition operator S_ϕ on X is defined by

$$S_\phi(f) = (\phi \circ f), \qquad f \in X. \tag{1.1}$$

Observe that if X contains the linear functions and S_ϕ maps X into Y, then S_ϕ must be an entire function.

The problem of boundedness and compactness of S_ϕ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the attention of many researchers (see, for instance, [1, 5, 12, 13]).
Now, we introduce the definition of the products of superposition operator followed by differentiation operator as follows:

Definition 1.2. The differentiation operator D is defined by $Df = f'$, while the operator DS_ϕ is defined by $DS_\phi(f) = (\phi \circ f)'$.

The Bloch type space is defined as follows (see [10, 13]):

Definition 1.3. An analytic function f is said to belong to the Bloch space $\mathcal{B}$ if

$$\mathcal{B}(f) = \sup_{z\in\mathbb{D}} |1 - |z|^2||f'(z)| < \infty. \tag{1.2}$$

The expression $\mathcal{B}(f)$ defines a seminorm on $\mathcal{B}$ as follows

$$||f||_{\mathcal{B}} = |f(0)| + \mathcal{B}(f).$$

Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0,$$

this space is called the little Bloch space.

The Hardy space can be defined as follows (see [14]):

Definition 1.4. The space H^∞ denotes the space of all bounded analytic functions f on the unit disk $\mathbb{D}$ such that

$$||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty. \tag{1.3}$$

It is well known that with the norm (1.3) the space H^∞ is a Banach space. It is easy to see that the space H^∞ is contained in $\mathcal{B}$ and

$$||f||_B = \sup_{z\in\mathbb{D}} |(1 - |z|^2)||f'(z)| < ||f||_\infty.$$

Definition 1.5. Let $\mathcal{Z}$ denote the space of all $f \in (H(\mathbb{D})) \cap (\mathbb{C}(\bar{\mathbb{D}}))$ such that

$$||f||_{\mathcal{Z}} = \sup_{z\in\mathbb{D}} \frac{|f(e^{i\theta+h}) + f(e^{i\theta-h}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ which denotes the boundary of $\mathbb{D}$ and $h > 0$. By the Zygmund theorem and the closed-graph theorem (see [4, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} |(1 - |z|^2)||f''(z)| < \infty.$$

Moreover, the following asymptotic relation holds:

$$||f||_{\mathcal{Z}} \asymp \sup_{z\in\mathbb{D}} |(1 - |z|^2)||f''(z)| < \infty, \tag{1.4}$$

therefore, $\mathcal{Z}$ is called the Zygmund class. Since the quantities in (1.4) are semi norms, it is natural to add to them the quantity $|f(0)| + |f'(0)|$ to obtain two equivalent norms on the Zygmund class.The Zygmund class with such a defined norm is called the Zygmund space. This norm is also denoted by $\| \,.\, \|_{\mathcal{Z}}$. Some information on Zygmund type spaces on the unit disk and some operators on them can be found in (see [3,7–9]). The little Zygmund space $\mathcal{Z}_0$ was introduced by Li and Stević in (see [6]) in the following natural way:

$$f \in \mathcal{Z}_0 \Leftrightarrow \lim_{|z|\to 1} |(1-|z|^2)|f''(z)| = 0.$$

It is easy to see that $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$ and the set of all polynomials is dense in $\mathcal{Z}_0$.
Now, we introduce the definition of boundedness and compactness of the operator $DS_\phi : H^\infty \to \mathcal{Z}$.

Definition 1.6. The operator $DS_\phi : H^\infty \to \mathcal{Z}$ is said to be bounded, if there is a positive constant C such that $||DS_\phi f||_{\mathcal{Z}} \leq C||f||_\infty$ for all $f \in H^\infty$.

Definition 1.7. The operator $DS_\phi : H^\infty \to \mathcal{Z}$ is said to be compact, if it maps any function in the unit disk in H^∞ onto a pre-compact set in $\mathcal{Z}$.

In this paper we consider the product of the superposition operator followed by the differentiation operator $DS_\phi(f)$. Throughout the paper we denote by the letter C, a positive constant, which may vary at each occurrence but it is independent of the essential variables. The notation a $\preceq b$ means that there is a positive constant C such that $a \preceq Cb$. Also, the notation $a \asymp b$ means that $a \preceq b$ and $b \preceq a$ hold.

2 The boundedness of $DS_\phi : H^\infty \to \mathcal{Z}$

In this section we characterize the operator $DS_\phi : H^\infty \to \mathcal{Z}$. Moreover, we give the conditions which prove the boundedness of the operator DS_ϕ. So, we list up the following two lemmas which are needed to prove our main results.

Lemma 2.1. *(see([10])) Assume that $f \in H^\infty$. Then for each $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z\in\mathbb{D}} (1-\mid z \mid)^n \mid f^{(n)}(z) \mid \leq \| f \|_\infty .$$

The following lemma is introduced in (see [15]).

Lemma 2.2. *Assume that $f \in \mathcal{B}$. Then for each $n \in \mathbb{N}$,*

$$\| f \|_{\mathcal{B}} \asymp \sum_{j=0}^{n-1} \mid f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1-\mid z \mid^2)^n \mid f^{(n)}(z) \mid.$$

Now we introduce the main results of boundedness.

Theorem 2.3. *Let ϕ be an entire function. Then the following statements are equivalent:*
$(a) DS_\phi : H^\infty \to \mathcal{Z}$ is bounded;
$(b) DS_\phi : \mathcal{B} \to \mathcal{Z}$ is bounded;
$(c) DS_\phi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded;
(d) The following conditions are satisfied.

$$\sup_{z\in\mathbb{D}} \frac{\mid \phi^{'''}(f(z)) \mid}{(1-\mid z \mid^2)^2} < \infty, \tag{2.1}$$

$$\sup_{z\in\mathbb{D}} \frac{\mid \phi^{''}(f(z)) \mid}{(1-\mid z \mid^2)^2} < \infty, \tag{2.2}$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mid \phi^{'}(f(z)) \mid}{(1-\mid z \mid^2)^2} < \infty. \tag{2.3}$$

Proof. $(b) \Rightarrow (a)$ and $(b) \Rightarrow (c)$ are obvious. Now, we will prove that $(d) \Rightarrow (a)$ and $(d) \Rightarrow (b)$. Firstly we show that $(d) \Rightarrow (a)$. Let us assume that the conditions (2.1),(2.2) and (2.3) hold. Then, for $z \in \mathbb{D}$ and $f \in H^{\infty}$(or $\mathcal{B}$), by using Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}
\| DS_{\phi} f \|_{\mathcal{Z}} &= \sup_{z\in\mathbb{D}} | (1-\mid z\mid^2)(DS_{\phi} f)^{''}(z) | \\
&= \sup_{z\in\mathbb{D}} (1-\mid z\mid^2) \mid (\phi^{''}(f(z))(f^{'})^2(z) + \phi^{'}(f(z))f^{''}(z))^{'} \mid \\
&= \sup_{z\in\mathbb{D}} (1-\mid z\mid^2) \mid \phi^{'''}(f(z))(f^{'})^3(z) + 3\phi^{''}(f(z))f^{'}(z)f^{''}(z) + \phi^{'}(f(z))f^{'''}(z) \mid \\
&\leq C(1-\mid z\mid^2)\left[\frac{\phi^{'''}(f(z))}{(1-\mid z\mid^2)^3} \| f \|_{\mathcal{B}}^2 + \frac{3\phi^{''}(f(z))}{(1-\mid z\mid^2)^3} \| f \|_{\mathcal{B}} + \frac{\phi^{'}(f(z))}{(1-\mid z\mid^2)^3}\right] \| f \|_{\mathcal{B}} \\
&\leq C\left[\frac{\phi^{'''}(f(z))}{(1-\mid z\mid^2)^2} \| f \|_{\mathcal{B}}^2 + \frac{3\phi^{''}(f(z))}{(1-\mid z\mid^2)^2} \| f \|_{\mathcal{B}} + \frac{\phi^{'}(f(z))}{(1-\mid z\mid^2)^2}\right] \| f \|_{\mathcal{B}} \\
&\leq C(1-\mid z\mid^2)\left[\frac{\phi^{'''}(f(z))}{(1-\mid z\mid^2)^3} \| f \|_{\infty}^2 + \frac{3\phi^{''}(f(z))}{(1-\mid z\mid^2)^3} \| f \|_{\infty} + \frac{\phi^{'}(f(z))}{(1-\mid z\mid^2)^3}\right] \| f \|_{\infty} \\
&\leq C\left[\frac{\phi^{'''}(f(z))}{(1-\mid z\mid^2)^2} \| f \|_{\infty}^2 + \frac{3\phi^{''}(f(z))}{(1-\mid z\mid^2)^2} \| f \|_{\infty} + \frac{\phi^{'}(f(z))}{(1-\mid z\mid^2)^2}\right] \| f \|_{\infty} . \qquad (2.4)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mid (DS_{\phi} f)(0) \mid &= \mid \phi^{'}(f(0))f^{'}(0) \mid \\
&\leq C\phi^{'}(f(0)) \| f \|_{\mathcal{B}} \\
&\leq C\phi^{'}(f(0)) \| f \|_{\infty},
\end{aligned}$$

and

$$\begin{aligned}
\mid (DS_{\phi} f)^{'}(0) \mid &= \mid \phi^{''}(f(0))(f^{'})^2(0) + \phi^{'}(f(0))f^{''}(0) \mid \\
&\leq C[\phi^{''}(f(0)) \| f \|_{\mathcal{B}} + \phi^{'}(f(0))] \| f \|_{\mathcal{B}} \\
&\leq C[\phi^{''}(f(0)) \| f \|_{\infty} + \phi^{'}(f(0))] \| f \|_{\infty} .
\end{aligned}$$

From the fact $|\phi(f(0))| < 1$, by using the conditions (2.1), (2.2) and (2.3) it follows that the operator $DS_{\phi} : H^{\infty}$(or, $B) \to \mathcal{Z}$ is bounded.
Now, we prove that $(a) \Rightarrow (d)$. Assume that $DS_{\phi} : H^{\infty} \to \mathcal{Z}$ is bounded, this means that there exists a constant C such that

$$\| DS_{\phi} f \|_{\mathcal{Z}} \leq C \| f \|_{\infty}, \qquad (2.5)$$

for all $f \in H^{\infty}$. From the above inequality, by taking the function $f(z) = z$,we have

$$\sup_{z\in\mathbb{D}} (1-\mid z\mid^2)\phi^{'''}(f(z)) < C. \qquad (2.6)$$

From the fact that$\| \phi \|_{\infty} \leq 1$, by using (2.6) and by taking the function $f(z) = z^2$, we have that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} 4(1-\mid z\mid^2)[2\phi^{'''}(f(z))z^3 + 3\phi^{''}(f(z))z] \\
\leq\ & C + C\sup_{z\in\mathbb{D}} (1-\mid z\mid^2) \mid \phi^{''}(f(z))z \mid \\
\leq\ & C. \qquad (2.7)
\end{aligned}$$

Similarly, from the fact that $\| \phi \|_{\infty} \leq 1$, by taking the function $f(z) = z^3$, and by using (2.6), (2.7) we have that

$$\sup_{z\in\mathbb{D}} (1-\mid z\mid^2)[27\phi^{'''}(f(z))z^6 + 54\phi^{''}(f(z))z^3 + 6\phi^{'}(f(z))] < C. \qquad (2.8)$$

For a fixed $w \in \mathbb{D}$, we consider the following test functions

$$f_w(z) = \frac{(1-\mid w\mid^2)}{1-\overline{w}z} - \frac{(1-\mid w\mid^2)^2}{(1-\overline{w}z)^2} + \frac{\frac{1}{3}(1-\mid w\mid^2)^3}{(1-\overline{w}z)^3}. \tag{2.9}$$

By the triangle inequality, we have

$$\begin{aligned}\mid f_w(z)\mid &\leq \frac{|1|(1-\mid w\mid^2)}{1-|wz|} + \frac{\mid -1|(1-\mid w\mid^2)^2}{(1-|wz|)^2} + \frac{|\frac{1}{3}|(1-\mid w\mid^2)^3}{(1-|wz|)^3}.\\ &\leq \frac{(1-\mid w\mid^2)}{1-|w|} + \frac{|(1-\mid w\mid^2)^2}{(1-|w|)^2} + \frac{\frac{1}{3}(1-\mid w\mid^2)^3}{(1-|w|)^3}.\\ &\leq \frac{26}{3}.\end{aligned}$$

So, it is clear that for all $f_w \in H^\infty$,

$$\sup_{w\in\mathbb{D}} \parallel f_w \parallel_\infty \leq \frac{26}{3}. \tag{2.10}$$

Then

$$f_w^{'}(z) = (\frac{(1-\mid w\mid^2)}{(1-\overline{w}z)^2} - \frac{2(1-\mid w\mid^2)^2}{(1-\overline{w}z)^3} + \frac{(1-\mid w\mid^2)^3}{(1-\overline{w}z)^4})\overline{w} \tag{2.11}$$

$$f_w^{''}(z) = (\frac{2(1-\mid w\mid^2)}{(1-\overline{w}z)^3} - \frac{6(1-\mid w\mid^2)^2}{(1-\overline{w}z)^4} + \frac{4(1-\mid w\mid^2)^3}{(1-\overline{w}z)^5})\overline{w}^2,$$

$$f_w^{'''}(z) = (\frac{6(1-\mid w\mid^2)}{(1-\overline{w}z)^4} - \frac{24(1-\mid w\mid^2)^2}{(1-\overline{w}z)^5} + \frac{20(1-\mid w\mid^2)^3}{(1-\overline{w}z)^6})\overline{w}^3.$$

and

$$f_w^{'}(w) = 0, f_w^{''}(w) = 0,$$

and

$$f_w^{'''}(w) = \frac{2\overline{w}^3}{(1-\mid w\mid^2)^3}.$$

Thus for $w \in \mathbb{D}$, we have

$$\begin{aligned}\frac{26}{3}\parallel DS_\phi \parallel &\geq \parallel DS_\phi f_w \parallel_{\mathcal{Z}}\\ &\geq (1-\mid w\mid^2)\mid \phi^{'''}(f(w))(f_w^{'})^3(w) + 3\phi^{''}(f(w))f_w^{'}(w)f_w^{''}(w)\\ &+ \phi^{'}(f_w(w))f_w^{'''}(w)\mid\\ &\geq (1-\mid w\mid^2)\left|\frac{2\phi^{'}(f(w))\overline{w}^3}{(1-\mid w\mid^2)^3}\right|\\ &\geq \left|\frac{2\phi^{'}(f(w))\overline{w}^3}{(1-\mid w\mid^2)^2}\right|.\end{aligned} \tag{2.12}$$

For $\delta \in (0,1)$, by using (2.12) and (2.8) we have

$$\begin{aligned}\sup_{w\in\mathbb{D}}\left|\frac{\phi^{'}(f(w))}{(1-\mid w\mid^2)^2}\right| &\leq \sup_{|w|>\delta}\mid\frac{\phi^{'}(f(w))}{(1-\mid w\mid^2)^2}\mid + \sup_{|w|\leq\delta}\mid\frac{\phi^{'}(f(w))}{(1-\mid w\mid^2)^2}\mid\\ &\leq \frac{1}{\delta^3}\sup_{|w|>\delta}\mid\frac{\phi^{'}(f(w))\overline{w}^3}{(1-\mid w\mid^2)^2}\mid + \frac{1}{(1-\delta^2)^2}\sup_{|w|\leq\delta}\mid\phi^{'}(f(w))\mid\\ &\leq C.\end{aligned} \tag{2.13}$$

It follows that, the condition (2.3) holds, as desired.
Next, we prove the condition (2.2). To see this, for a fixed $w \in \mathbb{D}$, put

$$g_w(z) = \frac{-2(1-\mid w\mid^2)}{1-\overline{w}z} + \frac{3(1-\mid w\mid^2)^2}{(1-\overline{w}z)^2} - \frac{(1-\mid w\mid^2)^3}{(1-\overline{w}z)^3}. \tag{2.14}$$

Now, it is easy to prove that for $g_w \in H^\infty$

$$\sup_{w\in\mathbb{D}} \parallel g_w \parallel_\infty \leq 24.$$

Then

$$g_w^{'}(z) = (\frac{-2(1-\mid w\mid^2)}{(1-\overline{w}z)^2} + \frac{6(1-\mid w\mid^2)^2}{(1-\overline{w}z)^3} - \frac{3(1-\mid w\mid^2)^3}{(1-\overline{w}z)^4})\overline{w}, \tag{2.15}$$

and

$$g_w^{''}(z) = (\frac{-4(1-\mid w\mid^2)}{(1-\overline{w}z)^3} + \frac{18(1-\mid w\mid^2)^2}{(1-\overline{w}z)^4} - \frac{12(1-\mid w\mid^2)^3}{(1-\overline{w}z)^5})\overline{w}^2,$$
$$g_w^{'''}(z) = (\frac{-12(1-\mid w\mid^2)}{(1-\overline{w}z)^4} + \frac{72(1-\mid w\mid^2)^2}{(1-\overline{w}z)^5} - \frac{60(1-\mid w\mid^2)^3}{(1-\overline{w}z)^6})\overline{w}^3.$$

Now,

$$g_w^{'}(w) = \frac{\overline{w}}{(1-\mid w\mid^2)},$$
$$g_w^{''}(w) = \frac{2\,\overline{w}^2}{(1-\mid w\mid^2)^2},$$

and

$$g_w^{'''}(w) = 0.$$

Thus for $w \in \mathbb{D}$, we have

$$\begin{aligned}
24 \parallel DS_\phi \parallel & \\
&\geq \parallel DS_\phi g_w \parallel_{\mathcal{Z}} \\
&\geq (1-\mid w\mid^2)\mid \phi^{'''}(g(w))(g_w^{'})^3(w) + 3\phi^{''}(g(w))g_w^{'}(w)g_w^{''}(w) \\
&+ \phi^{'}(g_w(w))g_w^{'''}(w)\mid \\
&\geq (1-\mid w\mid^2)\left|\frac{\phi^{'''}(g(w))\overline{w}^3}{(1-\mid w\mid^2)^3}\right| \\
&+ (1-\mid w\mid^2)\left|\frac{6\,\phi^{''}(g(w))\overline{w}^3}{(1-\mid w\mid^2)^3}\right| \\
&\geq \left|\frac{\phi^{'''}(g(w))\overline{w}^3}{(1-\mid w\mid^2)^2}\right| + \left|\frac{6\,\phi^{''}(g(w))\overline{w}^3}{(1-\mid w\mid^2)^2}\right|.
\end{aligned} \tag{2.16}$$

For $\delta \in (0,1)$, by using (2.16) and (2.7) we have

$$\begin{aligned}
&\sup_{w\in\mathbb{D}}\left|\frac{\phi^{''}(g(w))}{(1-\mid w\mid^2)^2}\right| + \sup_{w\in\mathbb{D}}\left|\frac{\phi^{'''}(f(w))}{(1-\mid w\mid^2)^2}\right| \\
\leq\ & \sup_{|w|>\delta}\left|\frac{\phi^{''}(g(w))}{(1-\mid w\mid^2)^2}\right| + \sup_{|w|\leq\delta}\left|\frac{\phi^{''}(g(w))}{(1-\mid w\mid^2)^2}\right| \\
+\ & \sup_{|w|>\delta}\left|\frac{\phi^{'''}(g(w))}{(1-\mid w\mid^2)^2}\right| + \sup_{|w|\leq\delta}\left|\frac{\phi^{'''}(g(w))}{(1-\mid w\mid^2)^2}\right| \\
\leq\ & \frac{1}{\delta^3}\sup_{|w|>\delta}\left|\frac{\phi^{''}(g(w))\overline{w}^3}{(1-\mid w\mid^2)^2}\right| + \frac{1}{(1-\delta^2)^2}\sup_{|w|\leq\delta}\mid \phi^{''}(g(w))\mid \\
+\ & \frac{1}{\delta^3}\sup_{|w|>\delta}\mid\frac{\phi^{'''}(g(w))\overline{w}^3}{(1-\mid w\mid^2)^2}\mid + \frac{1}{(1-\delta^2)^2}\sup_{|w|\leq\delta}\mid \phi^{'''}(g(w))\mid \\
\leq\ & C.
\end{aligned} \tag{2.17}$$

It follows that we get the condition (2.2), as desired.
Now, we will prove the condition (2.1), for $w \in \mathbb{D}$, put

$$h_w(z) = \frac{12(1-\mid w\mid^2)}{1-\overline{w}z} - \frac{8(1-\mid w\mid^2)^2}{(1-\overline{w}z)^2} + \frac{2(1-\mid w\mid^2)^3}{(1-\overline{w}z)^3}. \tag{2.18}$$

So, it is clear that for all $h_w \in H^\infty$ and

$$\sup_{w\in\mathbb{D}} \parallel h_w \parallel_\infty \leq 72.$$

Then

$$h_w^{'}(z) = (\frac{12(1-\mid w\mid^2)}{(1-\overline{w}z)^2} - \frac{16(1-\mid w\mid^2)^2}{(1-\overline{w}z)^3} + \frac{6(1-\mid w\mid^2)^3}{(1-\overline{w}z)^4})\overline{w}, \tag{2.19}$$

$$h_w^{''}(z) = (\frac{24(1-\mid w\mid^2)}{(1-\overline{w}z)^3} - \frac{48(1-\mid w\mid^2)^2}{(1-\overline{w}z)^4} + \frac{24(1-\mid w\mid^2)^3}{(1-\overline{w}z)^5})\overline{w}^2,$$

$$h_w^{'''}(z) = (\frac{72(1-\mid w\mid^2)}{(1-\overline{w}z)^4} - \frac{192(1-\mid w\mid^2)^2}{(1-\overline{w}z)^5} + \frac{120(1-\mid w\mid^2)^3}{(1-\overline{w}z)^6})\overline{w}^3.$$

and

$$h_w^{'}(w) = \frac{2\overline{w}}{1-\mid w\mid^2}, h_w^{''}(w) = 0,\ and\ h_w^{'''}(w) = 0.$$

Thus for $w \in \mathbb{D}$, we have

$$\begin{aligned}
72 \parallel DS_\phi \parallel &\geq \parallel DS_\phi h_w \parallel_{\mathcal{Z}} \\
&\geq (1-\mid w\mid^2) \mid \phi^{'''}(h(w))(h_w^{'})^3(w) + 3\phi^{''}(h(w))h_w^{'}(w)h_w^{''}(w) \\
&+ \phi^{'}(h_w(w))h_w^{'''}(w) \mid \\
&\geq (1-\mid w\mid^2) \mid \frac{8\,\phi^{'''}(h(w))\overline{w}^3}{(1-\mid w\mid^2)^3} \mid \\
&\geq \mid \frac{8\,\phi^{'''}(h(w))\overline{w}^3}{(1-\mid w\mid^2)^2} \mid .
\end{aligned} \tag{2.20}$$

From (2.6) and (2.20), which is similar to (2.13), we get the condition (2.1).

$(c) \Rightarrow (d)$. For $w, z \in \mathbb{D}$ and by using (2.11), (2.15) and (2.19), we have

$$\mid f_w^{'}(z) \mid \leq \frac{C_1}{1-\mid w\mid},$$

$$\mid g_w^{'}(z) \mid \leq \frac{C_2}{1-\mid w\mid}$$

and

$$\mid h_w^{'}(z) \mid \leq \frac{C_3}{1-\mid w\mid},$$

where C_1, C_2 and C_3 are constants, since our test functions $f_w(w), g_w(w)$ and $h_w(w)$ also belong to the little Bloch space $\mathcal{B}_0$. Similarly we can prove that $(a) \Rightarrow (d)$, thus finishing the proof of Theorem 2.3. □

Theorem 2.4. *Let ϕ be an entire function. Then $DS_\phi : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded if and only if $DS_\phi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded and*

$$\lim_{|z|\to 1}(1-\mid z\mid^2)(\mid \phi^{'''}(f(z)) \mid + \mid \phi^{''}(f(z)) \mid + \mid \phi^{'}(f(z)) \mid) = 0. \tag{2.21}$$

Proof. Firstly, we assume that $DS_\phi : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded, then $DS_\phi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded. If we take the functions given by $f(z) = z, f(z) = z^2$ and $f(z) = z^3$, then from the boundedness of $DS_\phi : \mathcal{B}_0 \to \mathcal{Z}_0$, we see that the condition of (2.21) follows.
Secondly, we assume that $DS_\phi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded and (2.21) holds. Since for each polynomial p we can obtain that

$$\begin{aligned} & \mid (1- \mid z \mid^2)(DS_\phi p)^{''}(z) \mid \\ = \; & (1- \mid z \mid^2) \mid \phi^{'''}(p(z))(p^{'})^3(z) + 3\phi^{''}(p(z))p^{'}(z)p^{''}(z) + \phi^{'}(p(z))p^{'''} \mid \\ \leq \; & (1- \mid z \mid^2)(\mid \phi^{'''}(p(z)) \mid \| (p^{'})^3 \|_\infty + 3 \mid \phi^{''}(p(z)) \mid \| p^{'} \|_\infty \| p^{''} \|_\infty) \\ + \; & (1- \mid z \mid^2)(\mid \phi^{'}(p(z)) \mid \| p^{'''} \|_\infty). \end{aligned} \tag{2.22}$$

Hence, for each polynomial p, $DS_\phi p \in \mathcal{Z}_0$ and by using the condition (2.21) we see that the set of all polynomials is dense in $\mathcal{B}_0$, thus for every $f \in \mathcal{B}_0$, there is a sequence of polynomials $\{p_k\}$ such that $\| p_k - f \|_\mathcal{B} \to 0$ as $k \to \infty$.
In view of Theorem 2.3, the operator $DS_\phi : \mathcal{B} \to \mathcal{Z}$ is bounded, so we have

$$\| DS_\phi p_k - DS_\phi f \|_\mathcal{Z} \leq \| DS_\phi \| . \| p_k - f \|_\mathcal{B} \to 0 \text{ as } k \to \infty.$$

Hence $DS_\phi(B_0) \subset \mathcal{Z}_0$, since $\mathcal{Z}_0$ is a closed subset of $\mathcal{Z}$. That ends the proof of Theorem 2.4. □

3 The compactness of $DS_\phi : H^\infty \to \mathcal{Z}$

In this section, we characterize the compactness of the operator $DS_\phi : H^\infty \to \mathcal{Z}$. Moreover, we give the conditions which prove the compactness of the operator DS_ϕ. For this purpose we list the following two lemmas which are needed to prove our main results:

Lemma 3.1. *Let ϕ be an entire function. Let $X = H^\infty$, $\mathcal{B}$ or $\mathcal{B}_0$. Then $DS_\phi : X \to \mathcal{Z}$ is compact if and only if $DS_\phi : X \to \mathcal{Z}$ is bounded and for any bounded sequence $\{f_n\}$ in X which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\| DS_\phi f_n \|_\mathcal{Z} \to 0$ as $n \to \infty$.*

The following second lemma was introduced and proved in [6] which is similar to the corresponding lemma in [11].

Lemma 3.2. *A closed set Kin $\mathcal{Z}_0$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z| \to 1} \sup_{f \in K} (1- \mid z \mid^2) \mid f^{''}(z) \mid = 0.$$

Now, we begin with the necessary and sufficient conditions for the compactness of $DS_\phi : H^\infty \to \mathcal{Z}$.

Theorem 3.3. *Let ϕ be an entire function. Then the following statements are equivalent:*
$(a) DS_\phi : H^\infty \to \mathcal{Z}$ is compact;
$(b) DS_\phi : \mathcal{B} \to \mathcal{Z}$ is compact;
$(c) DS_\phi : \mathcal{B}_0 \to \mathcal{Z}$ is compact;
$(d) DS_\phi : \mathcal{B} \to \mathcal{Z}$ is bounded and

$$\lim_{|\phi(z)| \to 1} \frac{\mid \phi^{'''}(f(z)) \mid}{(1- \mid z \mid^2)^2} = 0, \tag{3.1}$$

$$\lim_{|\phi(z)| \to 1} \frac{\mid \phi^{''}(f(z)) \mid}{(1- \mid z \mid^2)^2} = 0, \tag{3.2}$$

and

$$\lim_{|\phi(z)| \to 1} \frac{\mid \phi^{'}(f(z)) \mid}{(1- \mid z \mid^2)^2} = 0. \tag{3.3}$$

Proof. Since $\mathcal{B}_0 \subset \mathcal{B}$, the implication $(b) \Rightarrow (c)$ is obvious. Now, we will prove that $(d) \Rightarrow (a)$ and $(d) \Rightarrow (b)$. Firstly we prove that $(d) \Rightarrow (a)$. For any bounded sequence $\{f_n\}$ in H^∞ (or $\mathcal{B}$) with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, by using Lemma 3.1, it is enough to prove that

$$\| DS_\phi f_n \|_{\mathcal{Z}} \to 0 \text{ as } n \to \infty.$$

Suppose $\| f_n \|_\infty \leq 1$ (or, $\| f_n \|_{\mathcal{B}} \leq 1$). From (3.1), (3.2) and (3.3), given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |\phi(z)| < 1$, we have

$$(1- | z |^2)(\frac{\phi^{'''}(f(z))}{(1- | z |^2)^3} + \frac{\phi^{''}(f(z))}{(1- | z |^2)^3} + \frac{\phi^{'}(f(z))}{(1- | z |^2)^3}) < \epsilon. \tag{3.4}$$

By the boundedness of $DS_\phi : H^\infty \to \mathcal{Z}$ and Theorem 2.3, we see that (2.1), (2.2) and (2.3) are satisfied. Since $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, the Cauchy's estimate gives that $f_n^{'}$, $f_n^{''}$ and $f_n^{'''}$ converge to 0 uniformly on the compact subsets of $\mathbb{D}$. Hence there is an $N_0 \in \mathbb{N}$ such that for $n > N_0$

$$\begin{aligned}
& | \phi^{'}(f(0))(f_n^{'})(0) | + | \phi^{''}(f(0))(f_n^{'})^2(0) | + | \phi^{'}(f(0))(f_n^{''})(0) | \\
+ & \sup_{|z|\leq\delta} (1- | z |^2) | \phi^{'''}(f(z))(f_n^{'})^3(z) + 3\phi^{''}(f(z))f_n^{'}(z)f_n^{''}(z) + \phi^{'}(f(z))f_n^{'''} | \\
\leq & \; C\epsilon.
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5) we have

$$\begin{aligned}
& \| DS_\phi f_n \|_{\mathcal{Z}} \\
\leq & \; | (DS_\phi f_n)(0) | + | (DS_\phi f_n)^{'}(0) | + \sup_{z\in\mathbb{D}}(1- | z |^2) | (DS_\phi f_n)^{''}(z) | \\
\leq & \; | \phi^{'}(f(0))(f_n^{'})(0) | + | \phi^{''}(f(0))(f_n^{'})^2(0) | + | \phi^{'}(f(0))(f_n^{''})(0) | \\
+ & \sup_{z\in\mathbb{D}} (1- | z |^2) | \phi^{'''}(f(z))(f_n^{'})^3(z) + 3\phi^{''}(f(z))f_n^{'}(z)f_n^{''}(z) + \phi^{'}(f(z))f_n^{'''}(z) | \\
\leq & \; | \phi^{'}(f(0))(f_n^{'})(0) | + | \phi^{''}(f(0))(f_n^{'})^2(0) | + | \phi^{'}(f(0))(f_n^{''})(0) | \\
+ & \sup_{|z|\leq\delta} (1- | z |^2) | \phi^{'''}(f(z))(f_n^{'})^3(z) + 3\phi^{''}(f(z))f_n^{'}(z)f_n^{''}(z) + \phi^{'}(f(z))f_n^{'''}(z) | \\
+ & \sup_{\delta<|z|<1} (1- | z |^2) | \phi^{'''}(f(z))(f_n^{'})^3(z) + 3\phi^{''}(f(z))f_n^{'}(z)f_n^{''}(z) + \phi^{'}(f(z))f_n^{'''}(z) | \\
\leq & \; C\epsilon + C \sup_{\delta<|z|<1} (1- | z |^2)(\frac{\phi^{'''}(f(z))}{(1- | z |^2)^3} + \frac{\phi^{''}(f(z))}{(1- | z |^2)^3} + \frac{\phi^{'}(f(z))}{(1- | z |^2)^3}) \\
\leq & \; 2C\epsilon.
\end{aligned} \tag{3.6}$$

when $n > N_0$. It follows that the operators $DS_\phi : H^\infty$ (or, $\mathcal{B}$) $\to Z$ is compact. Now, we shall prove the second direction that $(a) \Rightarrow (d)$. We assume that $DS_\phi : H^\infty \to Z$ is compact. Then it is clear that $DS_\phi : H^\infty \to Z$ is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\phi(z_k)| \to 1$ as $k \to \infty$. If such a sequence does not exist then (3.1) - (3.3) automatically hold. We use the following test functions

$$f_k(z) = \frac{(1- | z_k |^2)}{1 - \overline{z_k} z} - \frac{(1- | z_k |^2)^2}{(1 - \overline{z_k} z)^2} + \frac{\frac{1}{3}(1- | z_k |^2)^3}{(1 - \overline{z_k} z)^3}. \tag{3.7}$$

It is easy to prove that

$$\sup_{z\in\mathbb{D}} \| f_k \|_\infty \leq \frac{26}{3},$$

$$f_k^{'}(z_k) = 0,$$

$$f_k^{''}(z_k) = 0,$$

and

$$f_k^{'''}(z_k) = \frac{2 \, \overline{z_k}^3}{(1- | z_k |^2)^3}.$$

It is clear that f_k converges to 0 uniformly on the compact subsets of $\mathbb{D}$, using (2.13) and the compactness of $DS_\phi : H^\infty \to Z$ we get

$$\begin{aligned} & (1-\mid z_k \mid^2)\left|\frac{2\phi^{'}(f(z_k))\overline{z_k}^3}{(1-\mid z_k \mid^2)^3}\right| \\ = \;& (1-\mid z_k \mid^2)\left|\phi^{'''}(f(z_k))(f_k^{'})^3(z_k)+3\phi^{''}(f(z_k))f_k^{'}(z_k)f_k^{''}(z_k)+\phi^{'}(f(z_k))f_k^{'''}\right| \\ \leq \;& \parallel DS_\phi f_k \parallel_{\mathcal{Z}} \to 0 \quad \text{as} \quad k\to\infty. \end{aligned} \tag{3.8}$$

From this and $|\phi(z_k)| \to 1$, it follows that

$$\lim_{k\to\infty}\frac{\phi^{'}(f(z_k))}{(1-\mid z_k \mid^2)^2}=0,$$

thus, we get (3.3).
In order to prove (3.2), we use the following test functions

$$g_k(z) = -\frac{2(1-\mid z_k \mid^2)}{1-\overline{z_k}z}+\frac{3(1-\mid z_k \mid^2)^2}{(1-\overline{z_k}z)^2}-\frac{(1-\mid z_k \mid^2)^3}{(1-\overline{z_k}z)^3}. \tag{3.9}$$

It is easy to prove that

$$\sup_{z\in\mathbb{D}}\parallel g_k \parallel_\infty \leq 24,$$

$$g_k^{'}(z_k)=\frac{\overline{z_k}}{(1-\mid z_k \mid^2)},$$

$$g_k^{''}(z_k)=\frac{2\,\overline{z_k}^2}{(1-\mid z_k \mid^2)^2},$$

and

$$g_k^{'''}(z_k)=0.$$

Since g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, the compactness of $DS_\phi : H^\infty \to Z$ implies that

$$\lim_{k\to\infty}\parallel DS_\phi g_k \parallel_{\mathcal{Z}}=0.$$

From (2.16) we obtain

$$(1-\mid z_k \mid^2)\mid\frac{\phi^{'''}(f(z_k))\overline{z_k}^3}{(1-\mid z_k \mid^2)^3}\mid+(1-\mid z_k \mid^2)\left|\frac{6\phi^{''}(f(z_k))\overline{z_k}^3}{(1-\mid z_k \mid^2)^3}\right| \leq C\parallel DS_\phi g_k \parallel_{\mathcal{Z}} \to 0$$

as $k\to\infty$.
Thus,

$$\lim_{k\to\infty}\frac{\phi^{''}(g(z_k))}{(1-\mid z_k \mid^2)^2}=0,$$

$$\lim_{k\to\infty}\frac{\phi^{'''}(g(z_k))}{(1-\mid z_k \mid^2)^2}=0.$$

So, (3.2) holds. Finally we take the following test function

$$h_k(z)=\frac{12(1-\mid z_k \mid^2)}{1-\overline{z_k}z}-\frac{8(1-\mid z_k \mid^2)^2}{(1-\overline{z_k}z)^2}+\frac{2(1-\mid z_k \mid^2)^3}{(1-\overline{z_k}z)^3}. \tag{3.10}$$

It is clear that

$$\sup_{z\in\mathbb{D}}\parallel h_k \parallel_\infty \leq\; 72,$$

where,

$$h_k^{'}(z_k)=\frac{2\,\overline{z_k}}{1-\mid z_k \mid^2},$$

$$h_k^{''}(z_k)=0,$$

$$h_k^{'''}(z_k) = 0,$$

and h_k converges to 0 uniformly on the compact subsets of $\mathbb{D}$, the compactness of $DS_\phi : H^\infty \to Z$ implies that

$$\lim_{k\to\infty} \| DS_\phi h_k \|_{\mathcal{Z}} = 0.$$

From this and (2.20) we obtain (3.1), the implication thus follows.

$(c) \Rightarrow (d)$. Since our test functions $f_k(z), g_k(z)$ and $h_k(z)$ also belong to the little Bloch space, following a similar way as in the proof of $(a) \Rightarrow (d)$, we can obtain the desired result. This concludes the proof of this theorem. □

Next we prove the following theorem:

Theorem 3.4. *Let ϕ be an entire function. Then the following statements are equivalent:*
$(a) DS_\phi : H^\infty \to \mathcal{Z}$ is compact;
$(b) DS_\phi : \mathcal{B} \to \mathcal{Z}_0$ is compact;
$(c) DS_\phi : \mathcal{B}_0 \to \mathcal{Z}_0$ is compact;
(d) The following conditions are satisfied

$$\lim_{|\phi(z)|\to 1} \frac{| \phi^{'''}(f(z)) |}{(1-| z |^2)^2} = 0, \tag{3.11}$$

$$\lim_{|\phi(z)|\to 1} \frac{| \phi^{''}(f(z)) |}{(1-| z |^2)^2} = 0, \tag{3.12}$$

and

$$\lim_{|\phi(z)|\to 1} \frac{| \phi^{'}(f(z)) |}{(1-| z |^2)^2} = 0. \tag{3.13}$$

Proof. The proof of $(b) \Rightarrow (c)$ is obvious. Now, we prove that $(d) \Rightarrow (a)$ and (b). By taking the supremum in inequality (2.4) over all $f \in H^\infty$(or, $f \in \mathcal{B}$) such that $\| f \|_\infty \leq 1$ (correspondingly, $\| f \|_{\mathcal{B}} \leq 1$), and letting $|z| \to 1$, yields

$$\lim_{|z|\to 1} \sup_{\|f\|_\infty \leq 1} (1-| z |^2) \,|\, (DS_\phi f)^{''}(z) \,|= 0,$$

or,

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{B}} \leq 1} (1-| z |^2) \,|\, (DS_\phi f)^{''}(z) \,|= 0.$$

Therefore, by Lemma 3.2, we see that the operator $DS_\phi : H^\infty(\text{or}, \mathcal{B}) \to \mathcal{Z}_0$ is compact.
(a) or $(c) \Rightarrow (d)$. Now we assume that $DS_\phi : H^\infty(\text{or}, \mathcal{B}_0) \to \mathcal{Z}_0$ is compact. Thus, $DS_\phi : H^\infty(\text{or}, \mathcal{B}_0) \to \mathcal{Z}_0$ is bounded, and by choosing the function $f(z) = z$, it follows that

$$\lim_{|z|\to 1} (1-| z |^2) \,|\, (\phi(f(z)))^{'''} \,|= 0. \tag{3.14}$$

By choosing the function $f(z) = z^2$, we get

$$\lim_{|z|\to 1} (1-| z |^2) \,|\, 2(\phi(f(z)))^{'''} z^3 + 3(\phi(f(z)))^{''} z \,|= 0. \tag{3.15}$$

From (3.14), (3.15) and the fact that $\| \phi \|_\infty \leq 1$, we get

$$\lim_{|z|\to 1} (1-| z |^2) \,|\, (\phi(f(z)))^{''} \,|= 0. \tag{3.16}$$

By choosing the function $f(z) = z^3$, we have

$$\lim_{|z|\to 1}(1- \mid z \mid^2) \mid 27(\phi(f(z)))^{'''} z^6 + 18(\phi(f(z)))^{''} z^3 + (\phi(f(z)))^{'} \mid = 0. \tag{3.17}$$

From (3.14), (3.16), (3.17) and the fact that $\| \phi \|_\infty \leq 1$, we get

$$\lim_{|z|\to 1}(1- \mid z \mid^2) \mid (\phi(f(z)))^{'} \mid = 0. \tag{3.18}$$

By (2.12), (2.16), (2.20) and $DS_\phi f_w, DS_\phi g_w, DS_\phi h_w \in \mathcal{Z}_0$ we know that

$$\lim_{|\phi(z)|\to 1} \frac{\mid \phi^{'''}(f(z)) \mid}{(1- \mid z \mid^2)^2} = 0, \tag{3.19}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mid \phi^{''}(f(z)) \mid}{(1- \mid z \mid^2)^2} = 0, \tag{3.20}$$

and

$$\lim_{|\phi(z)|\to 1} \frac{\mid \phi^{'}(f(z)) \mid}{(1- \mid z \mid^2)^2} = 0. \tag{3.21}$$

We have thus proved that (3.14) and (3.19) imply (3.11). The proofs of (3.12) and (3.13) follow by the same procedure, thus we omit them. From (3.19), it follows that for every $\epsilon > 0$, there exists a $\delta \in (0, 1)$ such that

$$\frac{\mid \phi^{'''}(f(z)) \mid}{(1- \mid z \mid^2)^2} < \epsilon, \tag{3.22}$$

when $\delta < |z| < 1$. Using (3.14) we see that there exists a $\tau \in (0, 1)$ such that

$$\mid \phi^{'''}(f(z)) \mid < \epsilon(1 - \delta^2)^2, \tag{3.23}$$

when $\tau <\mid z \mid< 1$.
Then, when $\tau <\mid z \mid< 1$ and $\delta < |z| < 1$ by (3.22) we get

$$\frac{\mid \phi^{'''}(f(z)) \mid}{(1- \mid z \mid^2)^2} < \epsilon. \tag{3.24}$$

On the other side, when $\tau <\mid z \mid< 1$ and $|z| \leq \delta$ by (3.23) we get

$$\frac{\mid \phi^{'''}(f(z)) \mid}{(1- \mid z \mid^2)^2} \leq \frac{\mid \phi^{'''}(f(z)) \mid}{(1 - \delta^2)^2} < \epsilon. \tag{3.25}$$

□

4 Conclusion

In this paper we characterized the boundedness and the compactness of the new product operator DS_ϕ, the so-called product of the superposition operator followed by the differentiation operator from H^∞ to Zygmund space. Moreover, we proved that the properties of boundedness and compactness still hold for this operator from H^∞ to Zygmund spaces and we gave the conditions which guarantee that the product operator DS_ϕ is bounded and compact.

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