Bull. Pure Appl. Sci. Sect. E Math. Stat.
39E(2), 206–211 (2020)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2020.00021.1

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

The parameter estimation of inverse Gaussian distribution under different loss functions *

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Abstract In this paper the Inverse Gaussian distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter are derived under squared error, precautionary entropy, K-loss, and Al-Bayyati's loss functions by using the quasi and inverted gamma priors.

Key words Inverse Gaussian distribution, Bayesian method, quasi and inverted gamma priors, squared error, precautionary, entropy, K-loss, Al-Bayyati's loss functions.

2020 Mathematics Subject Classification 62E99, 62F15.

1 Introduction

The Inverse Gaussian (IG) distribution is a long-tailed positively skewed distribution. Its shape is similar to that of the Normal distribution. This distribution was originally derived as the first passage of time distribution of Brownian motion with positive drift. In reliability and life-span applications it is primarily useful when there is substantial skewness. Pandey and Rao [1] estimated the parameter of IG distribution using Linex loss function. The probability density function of the Inverse Gaussian distribution (Chhikara and Folks [2]) is given by

$$f\left(x;\theta\right)=\left(2\pi\theta\right)^{-1/2}x^{-3/2}\mathrm{e}^{-(x-\mu)^2/2\mu^2\theta x}\quad ;\ x\geq 0. \tag{1.1}$$

The joint density function or likelihood function of (1.1) is given by

$$f\left(\underline{x};\theta\right)=\left(2\pi\right)^{-n/2}\theta^{-n/2}\left(\prod_{i=1}^{n}x_i^{-3/2}\right)\mathrm{e}^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}. \tag{1.2}$$

The log likelihood function is given by

$$\log f\left(\underline{x};\theta\right)=-\frac{n}{2}\log 2\pi-\frac{n}{2}\log\theta+\log\left(\prod_{i=1}^{n}x_i^{-3/2}\right)-\frac{1}{2\mu^2\theta}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}. \tag{1.3}$$

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received January 23, 2019 / Revised August 04, 2020 / Accepted September 01, 2020. Online First Published on December 26, 2020 at https://www.bpasjournals.com/.
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Differentiating (1.3) with respect to θ and equating to zero, we get the maximum likelihood estimator (MLE) of θ which is given as

$$\hat{\theta} = \frac{1}{n\mu^2} \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{x_i}. \tag{1.4}$$

2 The Bayesian method of estimation

The Bayesian inference procedures are developed generally under the squared error loss function

$$L\left(\hat{\theta}, \theta\right) = \left(\hat{\theta} - \theta\right)^2. \tag{2.1}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E\left(\theta\right). \tag{2.2}$$

Zellner [3], Basu and Ebrahimi [4] have recognized the inappropriateness of using symmetric loss function. Norstrom [5] introduced the precautionary loss function, which is given as

$$L\left(\hat{\theta}, \theta\right) = \frac{\left(\hat{\theta} - \theta\right)^2}{\hat{\theta}}. \tag{2.3}$$

The Bayes estimator under the precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2}. \tag{2.4}$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case, Calabria and Pulcini [6] point out that a useful asymmetric loss function is the entropy loss

$$L\left(\delta\right) \propto \left[\delta^p - p\ \log_e\left(\delta\right) - 1\right]$$

where, $\delta = \frac{\hat{\theta}}{\theta}$ and whose minimum occurs at $\hat{\theta} = \theta$. Also, the loss function $L\left(\delta\right)$ is used in Dey et al. [7] and Dey and Liu [8], in the original form having $p = 1$. Thus $L\left(\delta\right)$ can written be as

$$L\left(\delta\right) = b\left[\delta - \log_e\left(\delta\right) - 1\right];\ b > 0. \tag{2.5}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation

$$\hat{\theta}_E = \left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{2.6}$$

Wasan [9] proposed the K-loss function which is given as

$$L\left(\hat{\theta}, \theta\right) = \frac{\left(\hat{\theta} - \theta\right)^2}{\hat{\theta}\theta}. \tag{2.7}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K = \left[\frac{E\left(\theta\right)}{E\left(1/\theta\right)}\right]^{\frac{1}{2}}. \tag{2.8}$$

Al-Bayyati [10] introduced a new loss function using Weibull distribution which is given as

$$L\left(\hat{\theta}, \theta\right) = \theta^c\left(\hat{\theta} - \theta\right)^2. \tag{2.9}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{\text{Al}}$ and is obtained as

$$\hat{\theta}_{\text{Al}} = \frac{E\left(\theta^{c+1}\right)}{E\left(\theta^c\right)}. \tag{2.10}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.
(i) Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \; ; \theta > 0, \; d \geq 0, \tag{2.11}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.
(ii) Inverted gamma prior: Generally, the inverted gamma density is used as prior distribution of the parameter θ and is given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} \; ; \theta > 0. \tag{2.12}$$

3 Posterior density under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (1.2), is given by

$$f(\theta/\underline{x}) = \frac{(2\pi)^{-n/2}\theta^{-n/2}\left(\prod_{i=1}^n x_i^{-3/2}\right) e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}} \theta^{-d}}{\int_0^\infty (2\pi)^{-n/2}\theta^{-n/2}\left(\prod_{i=1}^n x_i^{-3/2}\right) e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}} \theta^{-d} d\theta}$$

$$= \frac{\theta^{-\left(\frac{n}{2}+d\right)} e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}}}{\int_0^\infty \theta^{-\left(\frac{n}{2}+d\right)} e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}} d\theta}$$

$$= \frac{\left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^{\frac{n}{2}+d-1}}{\Gamma\left(\frac{n}{2}+d-1\right)} \theta^{-\left(\frac{n}{2}+d\right)} e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}} \tag{3.1}$$

Theorem 3.1. *On using (3.1), we have*

$$E(\theta^c) = \frac{\Gamma\left(\frac{n}{2}+d-c-1\right)}{\Gamma\left(\frac{n}{2}+d-1\right)} \left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^c. \tag{3.2}$$

Proof. By definition,

$$E(\theta^c) = \int \theta^c f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^{\frac{n}{2}+d-1}}{\Gamma\left(\frac{n}{2}+d-1\right)} \int_0^\infty \theta^{-\left(\frac{n}{2}+d-c\right)} e^{-\frac{1}{2\theta\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}} d\theta$$

$$= \frac{\left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^{\frac{n}{2}+d-1}}{\Gamma\left(\frac{n}{2}+d-1\right)} \frac{\Gamma\left(\frac{n}{2}+d-c-1\right)}{\left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^{\frac{n}{2}+d-c-1}}$$

$$= \frac{\Gamma\left(\frac{n}{2}+d-c-1\right)}{\Gamma\left(\frac{n}{2}+d-1\right)} \left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^c.$$

From (3.2), for $c = 1$, we have

$$E(\theta) = \frac{\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}}{\frac{n}{2}+d-2}. \tag{3.3}$$

From (3.2), for $c = 2$, we have

$$E(\theta^2) = \frac{\left(\frac{1}{2\mu^2}\sum_{i=1}^n \frac{(x_i-\mu)^2}{x_i}\right)^2}{\left(\frac{n}{2}+d-2\right)\left(\frac{n}{2}+d-3\right)}. \tag{3.4}$$

From (3.2), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{\frac{n}{2}+d-1}{\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}. \tag{3.5}$$

From (3.2), for $c\to c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma\left(\frac{n}{2}+d-c-2\right)}{\Gamma\left(\frac{n}{2}+d-1\right)}\left(\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)^{c+1} \tag{3.6}$$

□

4 Bayes Estimators under $g_1(\theta)$

From (2.2), on using (3.3), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=\frac{\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}{\frac{n}{2}+d-2} \tag{4.1}$$

From (2.4), on using (3.4), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_P=\left[\left(\frac{n}{2}+d-2\right)\left(\frac{n}{2}+d-3\right)\right]^{-\frac{1}{2}}\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i} \tag{4.2}$$

From (2.6), on using (3.5), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E=\frac{\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}{\frac{n}{2}+d-1} \tag{4.3}$$

From (2.8), on using (3.3) and (3.5), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K=\left[\left(\frac{n}{2}+d-2\right)\left(\frac{n}{2}+d-1\right)\right]^{-\frac{1}{2}}\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i} \tag{4.4}$$

From (2.10), on using (3.2) and (3.6), the Bayes estimator of θ under Al-Bayyati's loss function is given by

$$\hat{\theta}_{\mathrm{Al}}=\frac{\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}{\frac{n}{2}+d-c-2}. \tag{4.5}$$

5 Posterior density under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using (1.2), is obtained as

$$f(\theta/\underline{x})=\frac{(2\pi)^{-n/2}\theta^{-n/2}\left(\prod_{i=1}^{n}x_i^{-3/2}\right)e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}}{\int_0^\infty(2\pi)^{-n/2}\theta^{-n/2}\left(\prod_{i=1}^{n}x_i^{-3/2}\right)e^{-\frac{1}{2\mu^2\theta}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}d\theta}$$

$$=\frac{\theta^{-\left(\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)}}{\int_0^\infty\theta^{-\left(\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)}d\theta}$$

$$=\frac{\theta^{-\left(\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)}}{\Gamma\left(\frac{n}{2}+\alpha\right)\Big/\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)^{\frac{n}{2}+\alpha}}$$

$$=\frac{\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)^{\frac{n}{2}+\alpha}}{\Gamma\left(\frac{n}{2}+\alpha\right)}\theta^{-\left(\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right)} \tag{5.1}$$

Theorem 5.1. *On using* (5.1), *we have*

$$E\left(\theta^{c}\right)=\frac{\Gamma\left(\frac{n}{2}+\alpha-c\right)}{\Gamma\left(\frac{n}{2}+\alpha\right)}\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{\left(x_{i}-\mu\right)^{2}}{x_{i}}\right)^{c}. \tag{5.2}$$

Proof. By definition,

$$E\left(\theta^{c}\right)=\int\theta^{c}f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}\right)^{\frac{n}{2}+\alpha}}{\Gamma\left(\frac{n}{2}+\alpha\right)}\int_{0}^{\infty}\theta^{-\left(\frac{n}{2}+\alpha+1-c\right)}\,e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}\right)}d\theta$$

$$=\frac{\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}\right)^{\frac{n}{2}+\alpha}}{\Gamma\left(\frac{n}{2}+\alpha\right)}\frac{\Gamma\left(\frac{n}{2}+\alpha-c\right)}{\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}\right)^{\frac{n}{2}+\alpha-c}}$$

$$=\frac{\Gamma\left(\frac{n}{2}+\alpha-c\right)}{\Gamma\left(\frac{n}{2}+\alpha\right)}\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{\left(x_{i}-\mu\right)^{2}}{x_{i}}\right)^{c}.$$

From (5.2), for $c=1$, we have

$$E\left(\theta\right)=\frac{\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}}{\frac{n}{2}+\alpha-1} \tag{5.3}$$

From (5.2), for $c=2$, we have

$$E\left(\theta^{2}\right)=\frac{\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}\right)^{2}}{\left(\frac{n}{2}+\alpha-1\right)\left(\frac{n}{2}+\alpha-2\right)} \tag{5.4}$$

From (5.2), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{\frac{n}{2}+\alpha}{\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}} \tag{5.5}$$

From (5.2), for $c\to c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma\left(\frac{n}{2}+\alpha-c-1\right)}{\Gamma\left(\frac{n}{2}+\alpha\right)}\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{\left(x_{i}-\mu\right)^{2}}{x_{i}}\right)^{c+1} \tag{5.6}$$

□

6 Bayes Estimators under $g_2(\theta)$

From (2.2), on using (5.3), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_{S}=\frac{\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}}{\frac{n}{2}+\alpha-1} \tag{6.1}$$

From (2.4), on using (5.4), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_{P}=\left[\left(\frac{n}{2}+\alpha-1\right)\left(\frac{n}{2}+\alpha-2\right)\right]^{-\frac{1}{2}}\left(\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{\left(x_{i}-\mu\right)^{2}}{x_{i}}\right) \tag{6.2}$$

From (2.6), on using (5.5), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_{E}=\frac{\beta+\frac{1}{2\mu^{2}}\sum_{i=1}^{n}\frac{(x_{i}-\mu)^{2}}{x_{i}}}{\frac{n}{2}+\alpha} \tag{6.3}$$

From (2.8), on using (5.3) and (5.5), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[\left(\frac{n}{2}+\alpha-1\right)\left(\frac{n}{2}+\alpha\right)\right]^{-\frac{1}{2}}\left(\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right) \tag{6.4}$$

From (2.10), on using (5.2) and (5.6), the Bayes estimator of θ under Al-Bayyati's loss function is given by

$$\hat{\theta}_{\mathrm{Al}} = \frac{\beta+\frac{1}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}}{\frac{n}{2}+\alpha-c-1}. \tag{6.5}$$

7 Conclusion

In this paper, we obtained a number of estimators of parameter of the Inverse Gaussian distribution. In (1.4) we obtained the maximum likelihood estimator of the parameter, while in (4.1)–(4.5) we obtained the Bayes estimators under different loss functions using quasi prior. In (6.1)–(6.4) and finally in (6.5) we obtained the Bayes estimators under different loss functions using the inverted gamma prior. From the above equations it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

Acknowledgments The authors express their wholehearted thanks to the referees for their useful remarks in improving the manuscript and to the Editor-in-Chief for his valuable suggestions on the original draft of this manuscript resulting in an improved presentation of the same.

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