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On a solution of fuzzy non-linear programming problem*

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Abstract: An alternative way of solving FNLPP is provided in this paper. Here the problem is first converted to a crisp multi-objective non-linear programming problem. Then a compromise solution is obtained using Zimmermann's fuzzy technique. In the process the non-linear functions are converted to piecewise linear functions and thus the problem is converted to linear optimization problem using separable convex programming technique and then simplex method in adopted to find the solution.

Keywords: Fuzzy Non-linear programming, Zimmermann's fuzzy technique, Trapezoidal fuzzy number, Triangular Fuzzy number, Crisp multi-objective function.
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1. Introduction

As we know a basic linear programming problem deals with a single linear objective function, subject to a set of linear constrains in which the coefficients are known with certainty. LPP having more than one conflicting objective problems are called multi objective LPP.

Concept of fuzzy decision making was first proposed by Bellman and Zadeh [8]. Authors like L. Compose et. al., I. Takeshi et. al. and G. Zhang et. al. [1, 4, 5] focused their attention to solve fuzzy linear programming FLPPs. An application of fuzzy Optimisation techniques to linear programming with multiple objectives has been given by Zimmerman [2]. P. A. Thakre et. al. [6] provided a method to solve FLPP where both the coefficient matrix of the constraint are fuzzy in nature. R.B. Dash et. al. [7] extended the idea to FNLPP.

In this paper we propose a general MONLPP model in fuzzy environment where some objectives are to be maximised and minimized. We have used triangular fuzzy numbers for representing the parameters of the problems with the help of Zimmermann's fuzzy technique.

2. Preliminaries

Fuzzy number:

A number $\tilde{a}$ whose membership function in general is defined as

$$\mu_a(x) = \begin{cases} \mu_a^L(x) & a_1 \le x \le a_2 \\ 1 & a_2 \le x \le a_3 \\ \mu_a^R(x) & a_3 \le x \le a_4 \\ 0 & Otherwise \end{cases} \tag{2.1}$$

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where $\mu_a^L: [a_{1,}, a_2] \rightarrow [0,1]$

and $\mu_a^R: [a_3, a_4] \rightarrow [0,1]$

are strictly monotonic and continuous mappings is called a fuzzy number.

Trapezoidal fuzzy number:

A fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$

associated with the membership function

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & for \quad x < a_1 \\ \frac{x-a_1}{a_2-a_1} & for\, a_1 \leq x < a_2 \\ 1 & for\, a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} & for\, a_3 < x \leq a_4 \\ 0 & for \quad x > a_4 \end{cases} \tag{2.2}$$

is called Trapezoidal Fuzzy Number.

Triangular Fuzzy Number:

A fuzzy number $\tilde{a} = (a_1, a_2, a_3)$

is said to be triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & x = a_2 \\ \frac{x-a_3}{a_2-a_3}, & a_2 \leq x \leq a_3 \\ 0, & Otherwise \end{cases} \tag{2.3}$$

Theorem-1: For any two triangular fuzzy numbers

$\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$, $a \leq b$ if and only if

$a_1 \leq b_1$, $a_1 - a_2 \leq b_1 - b_2$ and $a_1 + a_3 \leq b_1 + b_3$.

Proof: Obvious.

3. Fuzzy Non-linear Programming Problem

We consider the following fuzzy non-linear programming problem (FNLPP) in which the cost of the decision variables as well as the co-efficient matrix of the constraints are fuzzy in nature.

$$f(x) = \langle \tilde{c}, x \rangle = Max\tilde{Z} = \sum_{j=1}^{n} \tilde{C}_j\, x_j^{\alpha j} \tag{3.1}$$

subject to the constraints

$$\sum_{j=1}^{n} \tilde{a}_{ij}\, x_j \leq \tilde{b}_i, \quad 1 \leq i \leq m, \quad x_j \geq 0$$

where $\tilde{C}_j = (\alpha_j, \beta_j, \gamma_j, \delta_j)$, $j = 1, 2, \dots, n$.

$$\tilde{a}_{ij} = ({a_{ij}}^1, {a_{ij}}^2, {a_{ij}}^3), \quad \tilde{b}_{ij} = ({b_i}^1, {b_i}^2, {b_i}^3)$$

The above problem can be rewritten as

$$f(x) = max \sum_{j=1}^{n} \tilde{c}_j\, x_j^{\alpha j}$$

such that

$$\sum_{j=1}^{n} ({a_{ij}}^1, {a_{ij}}^2, {a_{ij}}^3)\, x_j \leq ({b_i}^1, {b_i}^2, {b_i}^3), \quad 1 \leq i \leq m, \; 1 \leq j \leq n$$

Using theorem-1, the problem is further reduced to

$$f(x) = max \sum \tilde{C}_j\, {x_j}^{\alpha j} = max \sum (\alpha_j, \beta_j, \gamma_{j,} \delta_j)\, {x_j}^{\alpha j}$$

such that

$$\sum_{j=1}^{n} a_{ij}^{1}\, x_j \le b_i$$
$$\sum_{j=1}^{n} \left(a_{ij}^{1} - a_{ij}^{2}\right) x_j \le b_i^{1} - b_i^{2}$$
$$\sum_{j=1}^{n} (a_{ij}^{1} + a_{ij}^{3})\, x_j \le b_i^{1} + b_i^{3}\ ,\ x_j \ge 0 \qquad (3.2)$$

Definition:

A point $x^* \in X$ is said to be an optimal solution to a FLPP/FNLPP if $\langle \widetilde{C}, x^* \rangle \ge \langle \widetilde{C}, x \rangle$ for all $x \in X$.

4. Reduction to Multi-objective Non-linear Programming Problem

The fuzzy objective function of (3.2) breads four crisp objective functions such as

$$f_1(x) = \sum \alpha_J\, x_j^{\alpha j}$$
$$f_2(x) = \sum \beta_J\, x_j^{\alpha j}$$
$$f_3(x) = \sum \gamma_J\, x_j^{\alpha j}$$

and $$f_4(x) = \sum \delta_J\, x_j^{\alpha j}$$

Ultimately the FNLPP (3.2) reduces to a crisp multi-objective linear programming problem as follows:

$$\underset{x \in X}{Max} \{f_1(x),\ f_2(x),\ f_3(x),\ f_4(x)\} \qquad (4.1)$$

subject to the constraints of (3.2), where f_i: $R^n - R$.

This crisp multi-objective function could have been solved reducing to weighted single objective function such as

$$Max\ \{w_1 f_1 + w_2 f_2 + w_3 f_3 + w_4 f_4\}$$

subject to the constraints of (3.2).

But in this paper, an alternative approach has been adopted by solving the mutli-objective crisp problem (4.1) using Zimmerman's fuzzy technique as stated below.

5. Zimmerman's Fuzzy Programming Technique

To solve the MOLPP

$$Maximize\ Z_r = \sum_j c_{rj}\, x_j\ ,\ \ r = 1, 2, \dots, q$$

subject to $\sum_j a_{ij}\, x_j \le b\ ,\quad i\ = 1, 2, \dots, m$ (5.1)

$$x_j \ge 0.$$

We use fuzzy programming technique suggested by Zimmermann. The method is presented briefly in the following steps.

Step-1

Solve the multi-objective linear programming problem by considering one objective at a time and ignoring all others. Repeat the process q times for q different objective functions. Let $X^1, X^2, \dots, X^q$ be the ideal solutions for respective functions.

Step-2

Using all the above ideal q solutions in Step -1 construct a pay –off matrix of size q by q. Then from the pay-off matrix find the lower bound (L_r) and upper bound (U_r) for the objective function z_r as:

$$L_r \le Z_r \le U_r,\ \ r = 1, 2, \dots, q$$

Step-3
Define fuzzy linear membership function $\mu_{z_r}(x)$ for the r[th] objective function z_r, $r = 1, 2, \dots, q$ as

$$\mu_{Z_r}(x) = \begin{cases} 0 & if \quad Z_r \le L_r \\ \dfrac{Z_r - L_r}{U_r - L_r} & if \quad L_r < Z_r < U_r \\ 1 & if \quad Z_r \ge U_r \end{cases}$$

Step-4
Using the above membership functions, we formulate a crisp model by introducing an augmented variable λ as:
Min: λ
Subject to
$\sum C_{r_j x_j} + (U_r - L_r)\lambda \ge U_r , \qquad r = 1, 2, \dots, q$
$\sum a_{ij} x_j \le bi, \qquad i = 1, 2, \dots, m$
$\lambda \ge 0, x_j \ge 0, \qquad j = 1, 2, \dots, n$
Step-5
Solve crisp model to find the optimal compromise solution. Evaluate the value of objective functions at the obtained compromise solution.

6. Numerical Example

We illustrate the method by a numerical example given below.
Solve the following FLNPP
Max $f(x_1, x_2) = \tilde{c}_1(-x_1^2 + x_1)\tilde{c}_2(x_2)$
subject to the constraints
$(3, 2, 1)x_1 + (6, 4, 1)x_2 \le (13, 5, 2)$
$(4, 1, 2)x_1 + (6, 5, 1)x_2 \le (7, 4, 2)$
Max $f(x_1, x_2) = \tilde{c}_1 x_1^2 + \tilde{c}_1 x_1 + \tilde{c}_1 x_2$
where $\tilde{c}_1 = (7, 10, 14, 25)$
$\tilde{c}_2 = (20, 25, 35, 40)$
subject to
$3x_1 + 6x_2 \le 13$
$4x_1 + 6x_2 \le 7$
$x_1 + 2x_2 \le 8$
$3x_1 + x_2 \le 3$
$4x_1 + 7x_2 \le 15$
$6x_1 + 10x_2 \le 9$
$x_1, x_2 \ge 0.$ (6.1)

Now the fuzzy non-linear programming problem (6.1) reduces to
Max.$(-7x_1^2 + 7x_1 + 20x_2, -10x_1^2 + 10x_1 + 25x_2, -14x_1^2 + 14x_1 + 35x_2, -25x_1^2 + 25x_1 + 40x_2)$
subject to
$3x_1 + 6x_2 \le 13$
$4x_1 + 6x_2 \le 7$
$x_1 + 2x_2 \le 8$
$3x_1 + x_2 \le 3$
$4x_1 + 7x_2 \le 15$
$6x_1 + 10x_2 \le 9$
$x_1, x_2 \ge 0.$ (6.2)

The problem (6.2) can be written as
$Max. z_1 = -7x_1^2 + 7x_1 + 20x_2$
$Max. z_2 = -10x_1^2 + 10x_1 + 25x_2$
$Max. z_3 = -14x_1^2 + 14x_1 + 35x_2$
$Max. z_4 = -25x_1^2 + 25x_1 + 40x_2$
subject to
$3x_1 + 6x_2 \leq 13$
$4x_1 + 6x_2 \leq 7$
$x_1 + 2x_2 \leq 8$
$3x_1 + x_2 \leq 3$
$4x_1 + 7x_2 \leq 15$
$6x_1 + 10x_2 \leq 9$
$x_1, \quad x_2 \geq 0.$ (6.3)

The crisp MONLPP is decomposed as four non-linear programming problems corresponding to each objective function and subject to same set of constraints.

Now using Zimmerman's method we find the lower bound (L.B) and Upper bound (U.B.) of the objective functions Z_1, Z_2, Z_3, Z_4 as in the following table

Functions	1st	2nd	3rd	4th
Z_1^1	18.68 L_1	23.47	32.87	38.44 U_1
Z_2^1	19.24 L_2	24.2	38.88	39.81 U_2
Z_3^1	18.2 L_3	22.75	31.85	36.4 U_3
Z_4^1	19.93 L_4	25.05	39.87	41.04 U_4

As per step-4 of Zimmerman's method, we finally solve:
Minimize: λ
such that
$3x_1 + 6x_2 + 19.76\,\lambda \geq 38.44$
$4x_1 + 6x_2 + 20.57\,\lambda \geq 39.81$
$x_1 + 2x_2 + 18.2\,\lambda \geq 36.4$
$3x_1 + x_2 + 21.11\,\lambda \geq 41.04$
$4x_1 + 7x_2 \leq 15$
$6x_1 + 10x_2 \leq 9$
$x_1, x_2 \geq 0.$

Solving this LPP by simplex method, we get the optimal solution $(x_1{}^*, \ x_2{}^*)$ as follows. In the process we have used as in the case of separable programming.

$(x_1{}^*, \ x_2{}^*) = (0.003, \ 0.8982)$

$$\begin{aligned} Max. f(x_1{}^*, \ x_2{}^*) &= -\tilde{c}_1 x_1^{*2} + \tilde{c}_1 x_1^* + \tilde{c}_2 x_2^* \\ &= \tilde{c}_1 \{x_1^* - x_1{}^{*2}\} + \tilde{c}_2\, x_2^* \\ &= (7, 10, 14, 25)\,\{0.003 - (0.003)^2\} + (20, \ 25, \ 35, \ 40)(0.8982) \\ &= (7, 10, 14, 25)\,(0.002991) + (20, \ 25, \ 35, \ 40)(0.8982) \\ &= (17.9849, \ 22.4849, \ 31.4788, \ 36.0027) \end{aligned}$$

$$\mu f(x_1^*, \ x_2^*)(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \leq c \\ \frac{d-x}{d-c} & c < x \leq d \\ 0 & x > d \end{cases}$$

Or $\mu f(0.003,0.8982)(x) = \begin{cases} 0 & x < 17.9849 \\ \frac{x-17.9849}{4.5} & 17.9849 < x \leq 22.4849 \\ 1 & 22.4849 < x \leq 31.4788 \\ \frac{36.0027-x}{4.5239} & 31.4788 < x \leq 36.0027 \\ 0 & x > 36.0027 \end{cases}$

7. Observation

We observe that the solution of this problem is very close to that of the weighted method adopted in the paper [8].

8. Conclusion

This paper provides an alternative method of solution of Fuzzy Nonlinear programming problem. This method is effective only on separable convex non-linear programming problems having imprecise co-efficient.

In future, non-linear fuzzy quadratic programming problems can be derived using Beal's and Wolf's methods to the crisp non-linear programming problems.

References

[1] G. Zhang, Y. H. Wu, M. Remias, J. Lu (2003): Formulation of Fuzzy Linear Programming Problems as Four-objective Constrained Optimization Problems, *Applied Mathematics and Computation*, Vol. 139, 383-399.

[2] H. J. Zimmerman (1978): Fuzzy Programming and Linear Programming with Several Objective Functions, *Fuzzy Sets and Systems*, Vol. 1(1), 45-55.

[3] H. Tanaka, H. Ichihashi, K. Asai (1991): A Formulation of Fuzy Linear Programming Problem Based on Comparison of Fuzzy Numbers, *Control and Cybernetics*, Vol. 3(3), 185-194.

[4] I. Takeshi, I. Hiroaki, N. Teruaki (1991): A Model of Crop Planning under Uncertainty in Agricultural Management, Int. J. of Production Economics, Vol.1, 159-171.

[5] L. Carnpose, J. L. Verdegay (1989): Linear Programming Problems and Ranking of Fuzzy Members, *Fuzzy Sets and Systems*, Vol. 32(1), 1-11.

[6] P. A. Thakre, D. S. Shelar, S. P. Thakre (2009): Solving Fuzzy Linear Programming Problem as Multi Objective Linear Programming Problem, *Proceedings of the World Congress on Engineering*, London, Vol.2, 978-988.

[7] R.B. Dash, P.D.P Dash: Solving Fuzzy Non-Linear Programming Problem as Multi Objective Non-Linear Programming Problem, *Karpagam IJAM*, Vol. 2, Oct 11- March 12, 324-331.

[8] R. E. Bellman, L.A. Zadeh (1970): Decision Making in a Fuzzy Environment, Management *Sciences*, Vol. 17, 141-164.