*Bull. Pure Appl. Sci. Sect. E Math. Stat.*
**39E**(1), 31–36 (2020)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.5958/2320-3226.2020.00002.8


BPAS PUBLICATIONS

**Bulletin of Pure and Applied Sciences**
**Section - E - Mathematics & Statistics**

Website : https : //www.bpasjournals.com/

# Power of control chart for zero-truncated Poisson distribution under inspection error *

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**Abstract** In this paper, an effort is made to study the Power Function and Average Run Length (ARL) of Zero-Truncated Poisson Distribution (ZTPD) under inspection error. An investigation is also made to calculate the power function of control chart for variable sample size for ZTPD under inspection error.

**Key words** Inspection Error, Power Function, Average Run Length (ARL), ZTPD.

**2020 Mathematics Subject Classification** 62E99.

## 1 Introduction

The foremost principle of inspection is to take apart products that conform to the measurement. The inspections of raw materials or in process products or the end products are the significant parts of quality assurance. Quality characteristics obtained from inspections are drawn in the control charts in order to scrutinize and control the product procedure. However, the traditional control chart methods presuppose that inspection process has no fault, but, in fact, the inspection error is very complicated to avoid whatsoever using visual or mechanical detection. Kanazuka [6] discussed that if the inspection error is large relative to the process variability then it results in the control chart to perceive a shift in the process level being affected. Walden [8] measured the power of $\bar{X}$, $R$ and $\bar{X}-R$ charts using ARL when inspection error affects the system. Chang and Gan [1] widened Shewhart control chart for scrutinizing the linearity among two measurement gauges. Huwang and Hung [4] considered the effect of inspection error on the control charts for screening multivariate process variability. Yang et al. [10] developed a process model to take into account the measurement error on two dependent processes. Xiaohong and Zhaojun [9] inspected the cause of measurement error on the CUSUM chart for the autoregressive data. Costa and Castagliola [3] observed the consequence of measurement error and autocorrelation on the $X$-chart. Moameni et al. [7] considered the consequence of measurement error on the effectiveness of the fuzzy control chart to identify out-of-control situations.
The power of any test of statistical importance is described as the probability that it will decline a false null hypothesis. Statistical power is affected primarily by the size of the effect and the size of the sample used to identify it. Major effects can be detected effortlessly than minor effects, while large samples recommend greater test sensitivity than small samples. Power analysis can be used to examine the minimum sample size required so that one can be practically prone to identify an effect

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received January 04, 2019 / Revised July 30, 2019 / Accepted August 21, 2019. Online First Published on June 30, 2020 at https://www.bpasjournals.com/.
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of a given size. Furthermore, to calculate among different statistical testing procedures, power analysis is exercised: for example, among a parametric and a nonparametric test of the same hypothesis. The ARL is the average number of points plotted on the chart before an out-of-control condition is signaled. Alternatively ARL can be defined as the average number of points plotted within the limit of control limits of control chart when assessing the process activities for a business operation. On the whole, this tests the precision of business operation over an extended phase of time or large number of performance cycles. It is considered until an unexpected condition in due course occurs that falls significantly outside the average run length.

In this paper we consider the effects of inspection error on the power of control chart and measure the value of average run length for ZTPD. Keeping in mind that the inspection error increases with the decrease in sample size, we also derive the power of control chart for variable sample size under standardized normal variate under inspection error for ZTPD.

## 2 Power of control chart for ZTPD under inspection error

In probability theory, the zero-truncated Poisson distribution (ZTPD) is a definite discrete probability distribution which consists of a set of positive integers. This distribution is also acknowledged as the positive Poisson distribution. It is the conditional probability distribution of a Poisson-distributed random variable, provided that the value of the random variable is not zero. Thus it is impracticable for a ZTP random variable to be zero. Consider for example the random variable of the number of items in a shopper's basket at a supermarket checkout line. Presumably a shopper does not stand in line with nothing to buy, so this observable fact may follow a ZTP distribution. Johnson et al. [5] called the zero-truncated Poisson as positive Poisson random variable and the same is called the conditional Poisson random variable by Cohen [2] and it is a Poisson distribution with parameter $\theta$ and $p\,(0) = 0$. Thus, it is required to scale the other probabilities by a factor of $1/1 - p(0)$ where $p\,(0) = e^{-\theta}$, the original probability that $x = 0$, in order to still have a discrete probability function.

Since the ZTP is a truncated distribution with the truncation stipulated as $k > 0$, one can derive the probability mass function $g\,(k, \theta)$ as follows:

$$g\,(k, \theta) = \frac{e^{-\theta}\theta^k}{k!(1 - e^{-\theta})}. \tag{2.1}$$

for $k = 1, 2, \ldots$, where $\theta > 0$. The mean and variance of the function given by (2.1) is:

$$M_e = \frac{\theta}{(1 - e^{-\theta})} \tag{2.2}$$

and

$$V_a{}^2 = \frac{\theta\left(1 - e^{-\theta}(1+\theta)\right)}{(1 - e^{-\theta})^2}. \tag{2.3}$$

In exercising zero truncated Poisson distribution under inspection error, a sample is acquired and the number of non-conformities is calculated. Again, the two types of error are (i) falling short to find one or more of the non-conformities in the sample and (ii) proclaiming one or more nonconformities when none exist (a false alarm). Let $r$ be the probability that non-conformity is appropriately noted by the inspector and the number of non-conformities is assumed to follow ZTPD and let $\theta_F$ be the average number of false alarms per part. If $\theta$ is the true value of the first moment of the number of non-conformities per part and $\theta^{'}$ is the average number of non-conformities per part observed by the inspector, then we set:

$$\theta^{'} = r\theta \ + \theta_F \tag{2.4}$$

with both $r$ and $\theta_F$ estimated.

It is assumed that at the time of determining the control limits the process is in a state of statistical control. Thus, the data used for establishing the limits on the control charts comes from a process that is $ZTP(M_e, V_a^2/n)$. When the process shifts, the data is assumed to come from the $ZTP(M_e^{'}, V_a^{'2}/n)$, where $M_e^{'} = \frac{\theta^{'}}{(1-e^{-\theta^{'}})}$ and $V_a^{'2} = \frac{\theta^{'}\left(1-e^{-\theta^{'}}(1+\theta^{'})\right)}{(1-e^{-\theta^{'}})^2}$. Furthermore, in Shewhart control chart the central line signifies hypothesized mean value of the process parameter, the control limits signifies the decisive values of the two-sided test for the null hypothesis acceptance region and every point corresponds to a test value for the given sample. Variable control chart procedures used for comparison of population implies two separate consecutive steps where the general null hypothesis is split into two simple null hypothesis concerning population means and variances:

(i) $H_0 : V_a^2 = V_a^{'2}$ versus $H_0 : V_a^2 \neq V_a^{'2}$,

(ii) $H_0 : M_e = M_e^{'}$ versus $H_0 : M_e \neq M_e^{'}$ .

If the point falls outside the corresponding control limits, the null hypothesis is rejected. Rejection of at least one of the simple null hypothesis leads to the rejection of the general null hypothesis. Thus, a basic assumption made in most traditional applications of control charts is that the observations from the process are independent. When the mean of the observations is being monitored, the mean is assumed to be constant at the target value until a special cause occurs and produces a change in the mean.

In the development of the power of the control chart and $ARL$ for ZTPD given by (2.1), the following suppositions are made and denotations are used:

- the process has ZTPD with mean and variance given by the equation (2.2) and (2.3) respectively;
- the process is in a state of statistical control at the time of determining the control limits and the same measuring instrument is used for later inspection;
- when the process parameter shifts, the data also come from ZTPD with mean $M_e^{'}$ and variance ${V_a}^{'2}$;
- the inspection of items has been taken to ascertain the number of defects per unit.

Under the above assumptions, Shewhart control limits for ZTPD will be:

$$UCL = M_e^{'} + 3V_a^{'} \; ; CL = M_e^{'} \; ; \quad LCL = \; M_e^{'} - 3V_a^{'}. \tag{2.5}$$

If we assume that $X$ is a ZTP variate with mean $M_e^{'}$ and variance ${V_a}^{'2}$ then the power of detecting the change of process parameter for ZTPD is given by:

$$P_{\theta'} = P\left\{X \geq M_e^{'} + 3V_a^{'}\right\} + P\left\{X \leq M_e^{'} - 3V_a^{'}\right\}. \tag{2.6}$$

$$\begin{aligned} P_{\theta'} = P &\left\{ X \geqslant \frac{\theta'}{(1-e^{-\theta'})} + 3\sqrt{\frac{\theta'\left(1-e^{-\theta'}\left(1+\theta'\right)\right)}{(1-e^{-\theta'})^2}} \right\} + \\ P &\left\{ X \leqslant \frac{\theta'}{(1-e^{-\theta'})} - 3\sqrt{\frac{\theta'\left(1-e^{-\theta'}\left(1+\theta'\right)\right)}{(1-e^{-\theta'})^2}} \right\}, \end{aligned} \tag{2.7}$$

$$P_{\theta'} = \left[1 - P\left\{X \leq UCL\right\}\right] + P\left[\left\{X \leq LCL\right\}\right] \;, \tag{2.8}$$

$$P_{\theta'} = \left[1 - \sum_{w=LCL}^{UCL} \frac{e^{-\theta}\theta^k}{w!(1-e^{-\theta})}\right]. \tag{2.9}$$

The calculation and graphical representation of $P_{\theta'}$ for equation (2.9) is shown below in Table 1 (Fig. 1) and Fig. 2 respectively.

## 3 Average run length (ARL)

It is the expected value of the run length distribution. For any Shewhart control chart, the ARL is ARL $= \left[P_{\theta'}\right]^{-1}$ where $P_{\theta'}$ is the probability that a single point exceeds the control limits. The probability of not detecting this shift on the first subsequent sample or $\beta$ risk (see, Montgomery [11]) is:

$$\beta = P\left\{X \geq UCL/\theta' \;\right\} + P\left\{X \leq LCL/\theta' \;\right\}. \tag{3.1}$$

It is related to the Power curve as follows:

$$\text{ARL} = \left[1-\beta\right]^{-1} \text{ and } \; \text{Power}(P_{\theta'}) = 1 - \beta, \tag{3.2}$$

| Power | | | | |
|---|---|---|---|---|
| $\theta$ | $(r,\theta_F)$ = (0.8,0) | $(r,\theta_F)$ = (1,0) | $(r,\theta_F)$ = (0.8,2) | $(r,\theta_F)$ = (1,2) |
| 2 | 0.6280 | 0.6565 | 0.7503 | 0.7687 |
| 4.5 | 0.7503 | 0.7890 | 0.8251 | 0.8477 |
| 7 | 0.8251 | 0.8581 | 0.8689 | 0.8890 |
| 9.5 | 0.8689 | 0.8948 | 0.8959 | 0.9132 |
| 12 | 0.8959 | 0.9169 | 0.9140 | 0.9298 |
| 14.5 | 0.9140 | 0.9327 | 0.9274 | 0.9440 |
| 17 | 0.9274 | 0.9468 | 0.9389 | 0.9583 |
| 19.5 | 0.9389 | 0.9611 | 0.9502 | 0.9722 |
| 22 | 0.9502 | 0.9747 | 0.9617 | 0.9837 |

Fig. 1: **Table 1.** Power of control chart for ZTPD.

where,

$$\beta = P\left\{X \geq \frac{\theta'}{(1-e^{-\theta'})} + 3\left[\frac{\theta'\left(1-e^{-\theta'}(1+\theta')\right)}{(1-e^{-\theta'})^2}\right]^{-1}\right\}. \tag{3.3}$$

The values of ARL obtained by using (3.3) and its diagrammatical representation are shown in Table 2 (Fig. 3) and Fig. 4 respectively.

## 4 Numerical illustration

For the purpose of numerical illustration, we will consider four cases as:

$$(r, \theta_F) = (1, 0), (1, 2), (0.8, 0), (0.8, 2).$$

The first case corresponds to sampling without inspection error while the other three represent different error rates. Also, to calculate the power function given by the equation (2.9), we consider a process with a targeted value or currently operating value of $\theta = 12$, so that $UCL$ and $LCL$ are given by (2.5) as:

$$UCL = 12 + 3\sqrt{12} = 22.3923 = 22$$
$$CL = 12$$
$$LCL = 12 - 3\sqrt{12} = 1.6076 = 2$$

The Table-1 (Fig. 1) and Fig. 2 illustrate the power function and power curve corresponding to the above four cases. From Table-1 (Fig. 1) one can see that $(1, 2)$ results in a shifting of the power function curve to the left. This occurs because $r = 1$ means that none of the non-conformities were missed while $\theta_F = 2$, means that there will be, on an average, two false alarms per part and this will shift the power curve two units to the left of the true power curve. An error rate of $(0.8, 0)$ results in a shift of the power curve to the right. A value of $r = 0.8$ means that the inspector is finding only 80% of the non-conformities. An error rate of $(0.8, 2)$ shifts the power curve two units to the left of the power curve for $(0.8, 0)$. The power curve for $(0.8, 2)$ is closer to the true curve because the average of two false alarms per part partially compensates for the inspector finding only 80% of the non-conformities. On the whole, while coming across the visual comparison through Fig. 2 reveals that inspection errors result in power curves considerably different from that obtained under error free inspection.

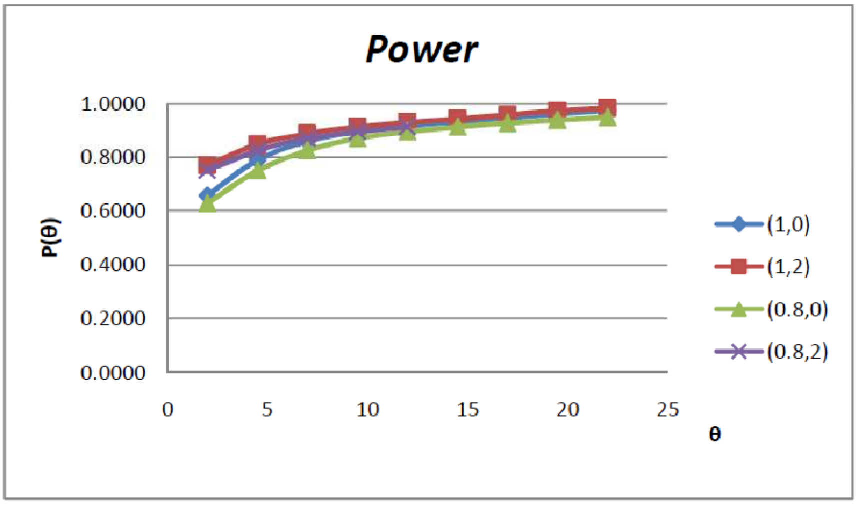


Fig. 2: Power of control chart for ZTPD.

| *ARL* | | | | |
|---|---|---|---|---|
| $\theta$ | $(r, \theta_F) = (0.8{,}0)$ | $(r, \theta_F) = (1{,}0)$ | $(r, \theta_F) = (0.8{,}2)$ | $(r, \theta_F) = (1{,}2)$ |
| 2 | 1.5924 | 1.5232 | 1.3328 | 1.3010 |
| 4.5 | 1.3328 | 1.2674 | 1.2119 | 1.1797 |
| 7 | 1.2119 | 1.1654 | 1.1509 | 1.1248 |
| 9.5 | 1.1509 | 1.1175 | 1.1162 | 1.0950 |
| 12 | 1.1162 | 1.0906 | 1.0941 | 1.0755 |
| 14.5 | 1.0941 | 1.0722 | 1.0783 | 1.0594 |
| 17 | 1.0783 | 1.0562 | 1.0651 | 1.0436 |
| 19.5 | 1.0651 | 1.0404 | 1.0524 | 1.0286 |
| 22 | 1.0524 | 1.0259 | 1.0398 | 1.0166 |

Fig. 3: **Table 2.** Values of ARL for ZTPD control chart.

From Table 2 (Fig. 3) it can be easily interpreted that, an error rate of $(0.8, 0)$ results in a shifts of the ARL curve to the right of the true curve i.e., it is highly affected by the inspection error. An error rate of $(0.8, 2)$ shifts the ARL curve to the left of the true curve $(1, 0)$. This occurs because inspector is finding only 80% of the non-conformities.
Overall, coming across the visual comparison through Fig. 4 reveals that the inspection errors result in ARL curves significantly dissimilar from that obtained under error free inspection.

## 5 Conclusion

In this paper we see that both types of errors i.e. deteriorating to note non-conformity and noting one where none exists, critically affect the power curves and ARL curves. The inspection error rates that are inevitable in industry seriously affect the power curve of a ZTPD control chart especially, when the center line and control limits are based on a target value, the process can very easily be moderated in-control when, in fact, it is not. When the control chart is based upon data obtained under inspection error, the Power and ARL curve is again indistinct, but not nearly so seriously.

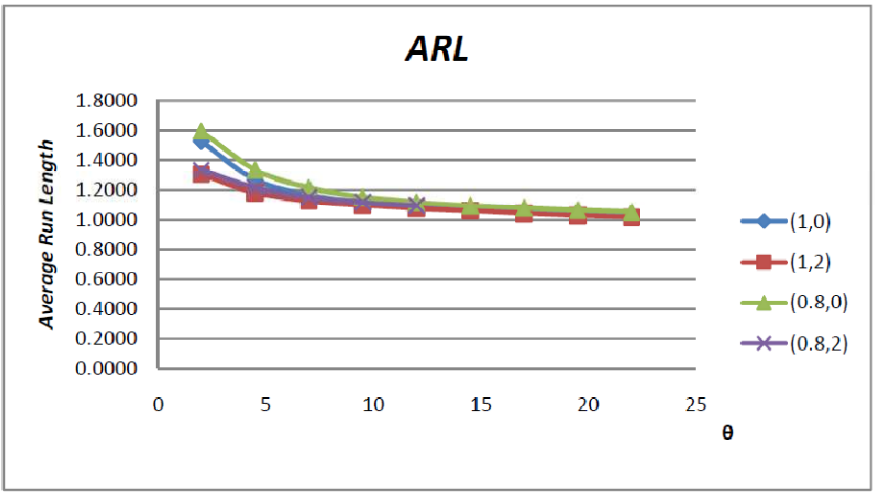


Fig. 4: ARL for ZTPD control chart.

**Acknowledgments** The authors would like to thank the referees and the Editor-in-Chief for their helpful comments for the improvement of the quality of the paper.

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