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# Boundedness of composition operators on some analytic function spaces *

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**Abstract** In this paper, we investigate the necessary and sufficient conditions for a composition operator $C_\phi$ to be bounded and compact from $\mathcal{B}^\alpha_\omega$ to $Q_{K,\omega}(p,q)$. Moreover, the necessary and sufficient condition for $C_\phi$ from the Dirichlet space $\mathcal{D}$ to the space $Q_{K,\omega}(p,q)$ to be compact is also given in terms of the map $\phi$.

**Key words** $Q_{K,\omega}(p,q)$ spaces, holomorphic functions and weighted $\alpha$-Bloch space.

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## 1 Introduction

Let $\phi$ be an analytic self-map of unit disk $\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $dA(z)$ be the Euclidean area element on $\mathbf{D}$. The composition operator $C_\phi$ induced by $\phi$ is the linear map on the space of all analytic functions on $\mathbf{D}$ given by

$$C_\phi(f) = f \circ \phi.$$

We recall some basic properties of boundedness and compactness of composition operators in Banach spaces of analytic function. In 1993, Shapiro [16] studied the compactness problem for composition operators and classical function theory. Subsequently, this concept was studied for characterization of the compact composition operators on the Bloch space (see, e.g [10]). During the last decades,

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Tjani [18] studied compact composition operators on the Besov spaces and the same problem on $BMOA$ was studied by Bourdon, Cima and Matheson in [4] and Smith and Zhao in [17]. Recently, Li and Wulan in [9] gave a characterization of compact operators on $Q_K$ and $F(p,q,s)$ spaces. Also, Jiang and He [8] studied the boundedness and compactness of composition operators from the Bloch space into the general Besov space.
In this paper, we study compact composition operator on the spaces $Q_{K,\omega}(p,q)$. We define and discuss the properties of these spaces. A particular class of Möbius-invariant function spaces, the so-called $Q_K$ spaces, has attracted a lot of attention in recent years. For $a \in \mathbf{D}$ the Möbius transformation $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \text{for } z \in \mathbf{D}.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|.$$

Note that $\varphi_a(\varphi_a(z)) = z$ and thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \mathbf{D}$ and $0 < r < 1$, the pseudo-hyperbolic disc $\mathbf{D}(a,r)$ is defined by $\mathbf{D}(a,r) = \{z \in \mathbf{D} : |\varphi_a(z)| < r\}$.
Let the Green's function of $\mathbf{D}$ with logarithmic singularity at $a$ be

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}$$

**Definition 1.1.** [11] A family $\mathbb{F}$ of holomorphic functions in $\mathbf{D}$ is analytic normal if every sequence of functions $\{f_n(z)\} \subset \mathbb{F}$ contains a subsequence $\{f_{n_k}(z)\}$ such that the sequence $\{f_{n_k}(z) - f_{n_k}(0)\}$ converges uniformly on the compact subsets of $\mathbf{D}$ to some function in the Euclidean metric.

**Definition 1.2.** [16] Let $f$ be an analytic function in $\mathbf{D}$ and let $0 < p < \infty$. If

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

then $f$ belongs to the Hardy space $H^p$. If $\|f\|_\infty = \sup_{z\in\mathbf{D}} |f(z)| < \infty$, then $f$ belongs to the Hardy space $H^\infty$. Moreover, $f \in H^2$ if and only if

$$\int_{\mathbf{D}} |f'(z)|^2(1-|z|^2)dA(z) < \infty.$$

Two quantities $A_f$ and $B_f$, both depending on an analytic function $f$ on $\mathbf{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant $C$ not depending on $f$ such that for every analytic function $f$ on $\mathbf{D}$ we have:

$$\frac{1}{C}B_f \le A_f \le CB_f.$$

If the quantities $A_f$ and $B_f$, are equivalent, then, in particular, we have $A_f < \infty$ if and only if $B_f < \infty$.
Now, given a reasonable function $\omega : (0,1] \to [0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ (see [5]) is defined as the set of all analytic functions $f$ on $\mathbf{D}$ satisfying

$$(1-|z|)|f'(z)| \le C\omega(1-|z|), \quad z \in \mathbf{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$. Here, the word "reasonable" is a non-mathematical term; it is just intended to mean "not too bad" and the function satisfies some natural conditions. Now, we introduce the following definitions:

**Definition 1.3.** For a given reasonable function $\omega : (0,1] \to [0,\infty)$ and for $0 < \alpha < \infty$, an analytic function $f$ on $\mathbf{D}$ is said to belong to the $\alpha-$weighted Bloch space $\mathcal{B}_\omega^\alpha$ if

$$\|f\|_{\mathcal{B}_\omega^\alpha} = \sup_{z\in\mathbf{D}} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| < \infty.$$

**Definition 1.4.** For a given reasonable function $\omega : (0,1] \to [0,\infty)$ and for $0 < \alpha < \infty$, an analytic function $f$ on $\mathbf{D}$ is said to belong to the little weighted Bloch space $\mathcal{B}^{\alpha}_{\omega,0}$ if

$$\|f\|_{\mathcal{B}^{\alpha}_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}|f'(z)| = 0.$$

Throughout this paper and for some techniques we consider the case of $\omega \not\equiv 0$. Now, we introduce the following definition (see [14]):

**Definition 1.5.** For a nondecreasing function $K : [0,\infty) \to [0,\infty), 0 < p < \infty,\ -2 < q < \infty$ and for a given reasonable function $\omega : (0,1] \to (0,\infty)$, an analytic function $f$ in $\mathbf{D}$ is said to belong to the space $Q_{K,\omega}(p,q)$ if

$$\|f\|^p_{K,\omega,p,q} = \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\, dA(z) < \infty.$$

We assume throughout the paper that

$$\int_0^1 (1-r^2)^{-2} K(\log\frac{1}{r}) r dr < \infty. \tag{1.1}$$

The authors of [14] collected the following immediate relations of $Q_{K,\omega}(p,q)$ and $Q_{K,\omega,0}(p,q)$ as follows:

**Theorem 1.6.** *Let $0 < p < \infty,\ -2 < q < \infty$. Then, for each non-decreasing function $K : [0,\infty) \to [0,\infty)$ and for a given reasonable non-decreasing function $\omega : (0,1] \to (0,\infty)$ with $\omega(kt) \approx \omega(t),\ \ k > 0$, we have that*
*(i)* $Q_{K,\omega}(p,q) \subset \mathcal{B}_{\omega}^{\frac{q+2}{p}}$ *and*
*(ii)* $Q_{K,\omega}(p,q) = \mathcal{B}_{\omega}^{\frac{q+2}{p}}$ , *iff*

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\, dr < \infty.$$

The following lemma is useful for our study (see [14]).

**Lemma 1.7.** *Let $K : [0,\infty) \to [0,\infty)$, $0 < p < \infty,\ -2 < q < \infty$ and $\omega : (0,1] \to (0,\infty)$. Then*
*(i)* $f \in \mathcal{B}_{\omega}^{\frac{q+2}{p}}$ *if and only if there exists $R \in (0,1)$ such that*

$$\sup_{a\in\Delta} \int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\, dA(z) < \infty,$$

*(ii)* $f \in \mathcal{B}_{\omega,0}^{\frac{q+2}{p}}$ *if and only if there exists $R \in (0,1)$ such that*

$$\lim_{|a|\to 1^-} \int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\, dA(z) = 0.$$

The following lemma is proved by Ohno et al., in [12]:

**Lemma 1.8.** *Assume that $\alpha > 0$. A closed set $\eta$ in $\mathcal{B}^{\alpha}_0$ is compact if and only if it is bounded and satisfies* $\lim\limits_{|z|\to 1^-} \sup\limits_{f\in\eta}(1-|z|^2)^{\alpha}|f'(z)| = 0.$

## 2 Holomorphic $Q_{K,\omega}(p,q)$ spaces

In this section we study the weighted $Q_{K,\omega}(p,q)$ spaces with the help of the weighted $\alpha-$Bloch spaces $\mathcal{B}^{\alpha}_{\omega}$. Our results will be needed to study the composition operators between $Q_{K,\omega}(p,q)$ and the weighted $\alpha-$ Bloch spaces.
To study the composition operators on $Q_{K,\omega}(p,q)$ spaces, we prove the following result:

**Theorem 2.1.** *Let $0<\alpha<\infty$, $0<r<1$, $0<p<\infty$, $-2<q<\infty, \omega:(0,1]\to[0,\infty)$ and $K:(0,\infty)\to(0,\infty)$. Also, let $f$ be an analytic function on $\mathbf{D}$. Then the following quantities are equivalent:*

*(A)* $\|f\|^p_{\mathcal{B}^\alpha_\omega}<\infty$.

*(B) For $0<\alpha<\infty$ and $0<p<\infty$, we have*

$$\sup_{a\in\mathbf{D}}\frac{1}{\left|\mathbf{D}(a,r)\right|^{1-\frac{p\alpha}{2}}}\int\int_{\mathbf{D}(a,r)}|f'(z)|^p\left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)<\infty.$$

*(C) For $0<\alpha<\infty$ and $0<p<\infty$, we have*

$$\sup_{a\in\mathbf{D}}\int\int_{\mathbf{D}(a,r)}|f'(z)|^p\Big(1-|z|\Big)^{p\alpha-2}\left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)<\infty.$$

*(D) For $0<\alpha<\infty$ ,$0<p<\infty$ and $-2<q<\infty$, we have*

$$\sup_{a\in\mathbf{D}}\int\int_{\mathbf{D}(a,r)}|f'(z)|^p(1-|z|)^{p\alpha-2}K(1-|\varphi_a(z)|)\left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)<\infty.$$

*(E) For $0<\alpha<\infty$ and $0<p<\infty$, we have*

$$\sup_{a\in\mathbf{D}}\int\int_{\mathbf{D}(a,r)}|f'(z)|^p\left(\log\frac{1}{|z|}\right)^{p\alpha}|\varphi_a'(z)|^2\left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)<\infty.$$

*(F)*

$$\sup_{a\in\mathbf{D}}\int\int_{\mathbf{D}(a,r)}|f'(z)|^p\,K\big(g(z,a)\big)\big(1-|z|\big)^{p\alpha-2}\left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)<\infty.$$

**Proof.** Let $0<\alpha<\infty$, $0<r<1$, $0<p<\infty$ and $K:(0,\infty)\to[0,\infty)$. Because for every analytic function $g$ on $\mathbf{D}$, $|g|^p$ is a subharmonic function we have

$$\big|g(0)\big|^p\le\frac{1}{\pi r^2}\int\int_{\mathbf{D}(0,r)}\big|g(w)\big|^p\,dA(w).$$

Set $g=f'\circ\varphi_a$, we obtain that

$$\begin{aligned}\big|f'(a)\big|^p&\le\frac{1}{\pi r^2}\int\int_{\mathbf{D}(0,r)}\big|f'\circ\varphi_a(w)\big|^p dA(w)\\&=\frac{1}{\pi r^2}\int\int_{\mathbf{D}(a,r)}\big|f'(z)\big|^p\frac{(1-|\varphi_a(z)|^2)^2}{(1-|z|^2)^2}dA(z).\end{aligned}$$

Since,

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2}=|\varphi_a'(z)|\ ,\quad\text{where}\ \frac{1-|\varphi_a(z)|^2}{1-|z|^2}\le\frac{4}{1-|a|^2}\quad a,z\in\mathbf{D}$$

(see [24]). Then, we obtain that

$$\big|f'(a)\big|^p\ \le\frac{16}{\pi r^2(1-|a|^2)^2}\int\int_{\mathbf{D}(a,r)}\big|f'(z)\big|^p dA(z).$$

Therefore, by $(1-|a|^2)^2\sim(1-|z|^2)^2\sim|\mathbf{D}(a,r)|$, for $z\in\mathbf{D}(a,r)$, we deduce that

$$|f'(a)|^p\frac{(1-|a|)^{p\alpha}}{\omega^p(1-|z|)}\ \le\frac{16(1-|a|)^{p\alpha}}{\pi r^2(1-|a|^2)^2\omega^p(1-|z|)}\int\int_{\mathbf{D}(a,r)}|f'(z)|^p dA(z).$$

Since $(1-|a|)^2 \sim (1-|a|^2)^2$, then

$$\begin{aligned}
|f'(a)|^p \frac{(1-|a|)^{p\alpha}}{\omega^p(1-|z|)} &\leq \frac{16}{\pi r^2(1-|a|)^{2-p\alpha}\omega^p(1-|z|)} \int\int_{\mathbf{D}(a,r)} |f'(z)|^p dA(z) \\
&\leq \frac{16\lambda}{\pi r^2 \big|\mathbf{D}(a,r)\big|^{1-\frac{p\alpha}{2}}} \int\int_{\mathbf{D}} |f'(z)|^p \Big(\frac{1}{\omega(1-|z|)}\Big)^p dA(z) \\
&= \frac{M(r)}{\big|\mathbf{D}(a,r)\big|^{1-\frac{p\alpha}{2}}} \int\int_{\mathbf{D}} |f'(z)|^p \Big(\frac{1}{\omega(1-|z|)}\Big)^p dA(z),
\end{aligned}$$

where $\lambda$ is a positive constant and $M(r) = \frac{16\lambda}{\pi r^2}$ is a constant depending on $r$. Thus the quantity (A) is less than or equal to a constant times the quantity (B).
From $|\mathbf{D}(a,r)| \sim (1-|z|^2)^2$ for all $z \in \mathbf{D}(a,r)$, it is obvious that $(B) \sim (C)$. By $1-|\varphi_a(z)|^2 > 1-r^2$ and $1-|\varphi_a(z)| > 1-r$ for $z \in \mathbf{D}(a,r)$, we thus obtain

$$\begin{aligned}
&\int\int_{\mathbf{D}(a,r)} |f'(z)|^p (1-|z|)^{p\alpha-2} \Big(\frac{1}{\omega(1-|z|)}\Big)^p dA(z) \\
&= \int\int_{\mathbf{D}(a,r)} |f'(z)|^p (1-|z|)^{p\alpha-2} \left(\frac{1}{\omega(1-|z|)}\right)^p \frac{K(1-|\varphi_a(z)|^2)}{K(1-|\varphi_a(z)|^2)} dA(z) \\
&\leq \frac{1}{K(1-r^2)} \int\int_{\mathbf{D}(a,r)} |f'(z)|^p (1-|z|)^{p\alpha-2} \left(\frac{1}{\omega(1-|z|)}\right)^p K(1-|\varphi_a(z)|^2) dA(z).
\end{aligned}$$

Hence, the quantity (C) is less than or equal to a constant times the qauntity (D). By $1-|\varphi_a(z)|^2 \leq 2g(z,a)$ for all $z, a \in \mathbf{D}$, we obtain that the quantity (D) is less than or equal to a constant times of the quantity (F).
From the following inequality

$$\begin{aligned}
&\int\int_{\mathbf{D}(a,r)} \big|f'(z)\big|^p (1-|z|)^{p\alpha-2} \Big(\frac{1}{\omega(1-|z|)}\Big)^p K(g(z,a))\, dA(z) \\
&= \int\int_{\mathbf{D}(a,r)} \big|f'(\varphi_a(w))\big|^p (1-|\varphi_a(w)|)^{p\alpha} \Big(\frac{1}{\omega(1-|z|)}\Big)^p K\big(\log\frac{1}{|w|}\big) \frac{dA(w)}{(1-|w|^2)^2} \\
&\leq \|f\|^p_{\mathcal{B}^\alpha_\omega} \int\int_{\mathbf{D}(a,r)} K\big(\log\frac{1}{|w|}\big) \frac{dA(w)}{(1-|w|^2)^2}\ ,
\end{aligned}$$

where

$$C(K,2) = \int\int_{\mathbf{D}(a,r)} K(\log\frac{1}{|w|})(1-|w|^2)^{-2} dA(w) < \infty,$$

then we deduce that the quantity (E) is less than or equal to a constant times (A).
Now, from the inequality $1-|z|^2 \leq 2\log\frac{1}{|z|}$ for every $z \in \mathbf{D}$, putting $K(1-|\varphi_a(z)|) = (1-|\varphi_a(z)|)^2$ in $(D)$, we see that the quantity $(D)$ is less than or equal to the quantity $(E)$. Finally, let

$$\begin{aligned}
I(a) &= \int\int_{\mathbf{D}(a,r)} \big|f'(z)\big|^p \Big(\log\frac{1}{|z|}\Big)^{p\alpha} |\varphi'_a(z)|^2 \Big(\frac{1}{\omega(1-|z|)}\Big)^p dA(z) \\
&= \Big(\int\int_{\mathbf{D}_{\frac{1}{4}}} + \int\int_{\mathbf{D}\setminus\mathbf{D}_{\frac{1}{4}}}\Big) \big|f'(z)\big|^p \Big(\log\frac{1}{|z|}\Big)^{p\alpha} |\varphi'_a(z)|^2 \Big(\frac{1}{\omega(1-|z|)}\Big)^p dA(z) \\
&= I_1(a) + I_2(a),
\end{aligned}$$

where for $z \in \mathbf{D}_{\frac{1}{4}} = \{z : |z| < \frac{1}{4}\}$, $|\varphi_a'(z)|^2 = \frac{(1-|a|^2)}{|1-\bar{a}z|^4} \leq \frac{1}{(1-|z|)^4} \leq (\frac{4}{3})^4$, then we obtain

$$\begin{aligned}
I_1(a) &= \int\int_{\mathbf{D}_{\frac{1}{4}}} |f'(z)|^p \left(\log\frac{1}{|z|}\right)^{p\alpha} |\varphi_a'(z)|^2 \left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)\\
&\leq \|f\|^p_{\mathcal{B}^\alpha_\omega} \int\int_{\mathbf{D}_{\frac{1}{4}}} \left(\frac{\log\frac{1}{|z|}}{(1-|z|)}\right)^{p\alpha} |\varphi_a'(z)|^2\, dA(z)\\
&\leq \|f\|^p_{\mathcal{B}^\alpha_\omega} (\frac{4}{3})^{p\alpha+4} \int\int_{\mathbf{D}_{\frac{1}{4}}} \left(\log\frac{1}{|z|}\right)^{p\alpha} dA(z)\\
&= (\frac{4}{3})^{p\alpha+4} C(p,\alpha)\|f\|^p_{\mathcal{B}^\alpha_\omega},
\end{aligned}$$

where

$$C(p,\alpha) = \int\int_{\mathbf{D}_{\frac{1}{4}}} \left(\log\frac{1}{|z|}\right)^{p\alpha} dA(z) < \infty.$$

Now, for $z \in \mathbf{D}\backslash\mathbf{D}_{\frac{1}{4}}$, we know that $\log\frac{1}{|z|} \leq 4(1-|z|^2) \leq 8(1-|z|)$, then

$$\begin{aligned}
I_2(a) &\leq 8 \int\int_{\mathbf{D}\backslash\mathbf{D}_{\frac{1}{4}}} |f'(z)|^p \left(\log\frac{1}{|z|}\right)^{p\alpha} |\varphi_a'(z)|^2 \left(\frac{1}{\omega(1-|z|)}\right)^p dA(z)\\
&\leq 8^{p\alpha}\|f\|^p_{\mathcal{B}^\alpha_\omega} \int\int_{\mathbf{D}\backslash\mathbf{D}_{\frac{1}{4}}} |\varphi_a'(z)|^2 dA(z) \leq \lambda_1 \|f\|^p_{\mathcal{B}^\alpha_\omega}
\end{aligned}$$

where $\lambda_1$ is a positive constant. Hence, the quantity (E) is less than or equal to a constant times the quantity (A). The proof is complete. □

For $\mathcal{B}^\alpha_{\omega,0}$, we have the corresponding result with Theorem 2.1.

**Lemma 2.2.** *[13] Let $\omega : (0,1] \to (0,\infty)$ and let $1 \leq \alpha < \infty$. Then there are two functions $f_1$, $f_2 \in \mathcal{B}^\alpha_\omega$ such that*

$$|f_1'(z)| + |f_2'(z)| \approx \frac{\omega(1-|z|)}{(1-|z|)^\alpha}, \qquad z \in \mathbf{D}. \tag{2.1}$$

## 3 Boundedness of composition operators.

In this section we study the boundedness of composition operators on $Q_{K,\omega}(p,q)$ spaces and $\mathcal{B}_\omega^{\frac{q+2}{p}}$ spaces. We need the following notation.

$$\Phi_{\phi,K,\omega}(p,q;a) = \int_{\mathbf{D}} |\phi'(z)|^p \frac{(1-|z|^2)^q \omega^p(1-|\varphi_a(z)|)}{(1-|\phi(z)|^2)^{q+2}\omega^p(1-|z|)} K(g(z,a))dA(z),$$

for $0 < p < \infty, -2 < q < \infty, \omega : (0,1] \to (0,\infty)$, and $K : [0,\infty) \to [0,\infty)$.
Now, we give the following theorem:

**Theorem 3.1.** *Let $K(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$, $\omega : (0,1] \to (0,\infty)$, and let $\phi$ be an analytic self-map of $\mathbf{D}$. Then $C_\phi : \mathcal{B}_\omega^{\frac{q+2}{p}} \to Q_{K,\omega}(p,q)$ is bounded if and only if*

$$\sup_{a\in\mathbf{D}} \Phi_{\phi,K,\omega}(p,q;a) < \infty. \tag{3.1}$$

**Proof.** Let (3.1) hold and let $C_1^p (C_1 > 0)$ be the supremum in (3.1). If $f \in \mathcal{B}_\omega^{\frac{q+2}{p}}$, then for all $a \in \mathbf{D}$, we have

$$\begin{aligned}
\|C_\phi f\|^p_{Q_{K,\omega}(p,q)} &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |(f\circ\phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(\phi(z))\phi'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \\
&\le \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\phi'(z)|^p \frac{(1-|z|^2)^q \omega^p(1-|\varphi_a(z)|)}{(1-|\phi(z)|^2)^{q+2}\omega^p(1-|z|)} K(g(z,a)) dA(z) \\
&\le C_1^p \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} = \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \sup_{a\in\mathbf{D}} \Phi_{\phi,K,\omega}(p,q;a) < \infty.
\end{aligned}$$

For the other direction we use the fact that for each function $f \in \mathcal{B}_\omega^{\frac{q+2}{p}}$, the analytic function $C_\phi(f) \in Q_{K,\omega}(p,q)$. Then using the functions of Lemma 1.7 we get the following:

$$\begin{aligned}
& 2^p \left\{ \|C_\phi f_1\|^p_{Q_{K,\omega}(p,q)} + \|C_\phi f_2\|^p_{Q_{K,\omega}(p,q)} \right\} \\
&= 2^p \left\{ \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \Big[ |(f_1\circ\phi)'(z)|^p + |(f_2\circ\phi)'(z)|^p \Big] \times (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \right\} \\
&\ge \left\{ \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \Big[ |(f_1\circ\phi)'(z)| + |(f_2\circ\phi)'(z)| \Big]^p \times (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \right\} \\
&\ge \left\{ \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \Big[ |(f_1'(\phi(z))| + |(f_2(\phi(z))| \Big]^p \times |\phi'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \right\} \\
&\ge C \left\{ \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\phi'(z)|^p \frac{(1-|z|^2)^q \omega^p(1-|\varphi_a(z)|)}{(1-|\phi(z)|^2)^{q+2}\omega^p(1-|z|)} K(g(z,a)) \right\} \\
&\ge C \sup_{a\in\mathbf{D}} \Phi_{\phi,K,\omega}(p,q;a).
\end{aligned}$$

Hence $C_\phi$ is bounded, then (3.1) holds. The proof is completed.

The composition operator $C_\phi : \mathcal{B}_\omega^{\frac{q+2}{p}} \to Q_{K,\omega}(p,q)$ is compact if and only if for every sequence $\{f_n\}_{n\in\mathbb{N}} \subset Q_{K,\omega}(p,q)$ which is bounded in $Q_{K,\omega}(p,q)$ norm, $f_n \to 0, n \to \infty$, uniformly on the compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence

$$\|C_\phi(f_n)\|_{Q_{K,\omega}(p,q)} \to 0, \quad n \to \infty.$$

□

Now, we describe compactness in the following result.

**Theorem 3.2.** *Let $K(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \le r < \infty$, $\omega : (0,1] \to (0,\infty)$, and let $\phi$ be an analytic self-map of $\mathbf{D}$. Then $C_\phi : \mathcal{B}_\omega^{\frac{q+2}{p}} \to Q_{K,\omega}(p,q)$ is compact if and only if $\phi \in Q_{K,\omega}(p,q)$ and*

$$\lim_{r\to 1} \sup_{a\in\mathbf{D}} \Phi_{\phi,K,\omega}(p,q;a) = 0. \tag{3.2}$$

**Proof.** Let $C_\phi : \mathcal{B}_\omega^{\frac{q+2}{p}} \to Q_{K,\omega}(p,q)$ be compact. This means that $\phi \in Q_{K,\omega}(p,q)$.
Let

$$U_r^1 = \{z : |\phi(z)| > r, \;\; r \in (0,1)\}$$

and

$$U_r^2 = \{z : |\phi(z)| \le r, \;\; r \in (0,1)\}.$$

Let $f_n(z) = \frac{z^n}{n}$ if $\alpha = \frac{q+2}{p} \in [0,\infty)$ or $f_n(z) = \frac{z^n}{n^{1-\alpha}}$ if $\alpha \in (0,1)$. Without loss of generality, we only consider $\alpha \in (0,1)$. Since $\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \le M$ and $f_n(z) \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $\|C_\phi(f_n)\|_{Q_{K,\omega}(p,q)} \to 0,\ n\to\infty$. This means that for each $r \in (0,1)$ and for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\frac{N^{q+2}}{r^{p(1-N)}} \sup_{a\in\mathbf{D}} \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) < \varepsilon,$$

if we choose $r$ so that $(N^{q+2}\, r^{p(N-1)}) = 1$, then

$$\sup_{a\in\mathbf{D}} \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \ < \ \varepsilon. \tag{3.3}$$

Let now $f$ with $\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \leq 1$. We consider the functions $f_t(z) = f(tz), t \in (0,1)$. Then $f_t \to f$ uniformly on compact subset of the unit disk as $t \to 1$ and the family $(f_t)$ is bounded on $\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}}$, thus

$$\|(f_t\circ\phi) - (f\circ\phi)\| \to 0.$$

Due to compactness of $C_\phi$ we get that, for $\varepsilon > 0$ there is a $t \in (0,1)$ such that

$$\sup_{a\in\Delta} \int_{\mathbf{D}} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2} \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) < \varepsilon,$$

where

$$F_t(\phi(z)) = [(f\circ\phi)'(z) - (f_t\circ\phi)'(z)].$$

Thus, if we fix $t$, then

$$\begin{aligned}
&\sup_{a\in\mathbf{D}} \int_{U_r^1} |(f\circ\phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z)\\
\leq\ & 2^p \sup_{a\in\mathbf{D}} \int_{U_r^1} |F_t(\phi(z))|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z)\\
+\ & 2^p \sup_{a\in\mathbf{D}} \int_{U_r^1} |(f_t\circ\phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z)\\
\leq\ & 2^p\, \varepsilon\\
+\ & 2^p\, \|f_t'\|_{H^\infty}^p \sup_{a\in\mathbf{D}} \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z)\\
\leq\ & 2^p\, \varepsilon \ + \ 2^p\, \varepsilon\, ||f_t'||_{H^\infty}^p,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\sup_{a\in\mathbf{D}} \int_{U_r^1} |(f\circ\phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z)\\
&\leq 2^p\, \varepsilon\, (1 + \ ||f_t'||_{H^\infty}^p),
\end{aligned} \tag{3.4}$$

where we have used 3.3. On the other hand, for each $\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \leq 1$ and $\varepsilon > 0$, there exists a $\delta$ depending on $f, \varepsilon$, such that for $r \in [\delta, 1)$,

$$\sup_{a\in\mathbf{D}} \int_{U_r^1} |(f\circ\phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) < \varepsilon. \tag{3.5}$$

Since $C_\phi$ is compact, then it maps the unit ball of $\mathcal{B}_\omega^{\frac{q+2}{p}}$ to a relatively compact subset of $Q_{K,\omega}(p,q)$. Thus for each $\varepsilon > 0$ there exists a finite collection of functions $f_1, f_2, \ldots, f_n$ in the unit ball of $\mathcal{B}_\omega^{\frac{q+2}{p}}$ such that for each $\|f\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \leq 1$, there is $k \in \{1,2,3,\ldots,n\}$ such that

$$\sup_{a\in\mathbf{D}} \int_{\Delta} |F_k(\phi(z))|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) < \varepsilon,$$

where,

$$F_k(\phi(z)) = [(f \circ \phi)'(z) - (f_k \circ \phi)'(z)].$$

Using also (3.5), we get for $\delta = \max_{1\le k\le n} \delta(f_k, \varepsilon)$ and $r \in [\delta, 1)$, that

$$\sup_{a\in\mathbf{D}} \int_{U_r^1} |(f_k \circ \phi)'(z)|^p (1-|z|^2)^{\alpha p-2} \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) < \varepsilon.$$

Hence for any $f, \|f\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \le 1$, combining the two relations as above we get that

$$\sup_{a\in\mathbf{D}} \int_{U_r^1} |(f \circ \phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) < 2^p\varepsilon.$$

Therefore, we get that (3.2) holds.

For the sufficiency we use that $\phi \in Q_{K,\omega}(p,\, q)$ and (3.2) holds. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the unit ball of $\mathcal{B}_\omega^{\frac{q+2}{p}}$, such that $f_n \to 0$ as $n \to \infty$, uniformly on the compact subsets of the unit disk. Let also $r \in (0,1)$. Then

$$\begin{aligned}
&\|f_n \circ \phi\|^p_{Q_{K,\omega}(p,q)} \le 2^p\ |f_n(\phi(0))| \\
&+2^p \sup_{a\in\mathbf{D}} \int_{U_r^2} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \\
&+2^p \sup_{a\in\mathbf{D}} \int_{U_r^1} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^{\alpha p-2} \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \\
&= 2^p(I_1 + I_2 + I_3).
\end{aligned}$$

Since $f_n \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $I_1 = |f_n(\phi(0))|$ goes to zero as $n \to \infty$ and for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for each $n > N$,

$$\begin{aligned}
I_2 &= \sup_{a\in\mathbf{D}} \int_{U_r^2} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) \\
&\le \varepsilon\ \|\phi\|^p_{Q_{K,\omega}(p,\, q)}.
\end{aligned}$$

We also observe that

$$\begin{aligned}
I_3 &= \sup_{a\in\mathbf{D}} \int_{U_r^1} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^{\alpha p-2} \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)}\, dA(z) \\
&\le \|f_n\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \\
&\quad \times \sup_{a\in\mathbf{D}} \int_{U_r^1} |\phi'(z)|^p \frac{(1-|z|^2)^q \omega^p(1-|\varphi_a(z)|)}{(1-|\phi(z)|^2)^{q+2}\omega^p(1-|z|)} K(g(z,a)) dA(z).
\end{aligned}$$

Under the assumption that (3.2) holds, then for every $n > N$ and for every $\varepsilon > 0$ there exists $r_1$ such that for every $r > r_1$, $I_3 < \varepsilon$. Thus if $\phi \in Q_{K,\omega}(p,\, q)$ we get

$$\begin{aligned}
\|f_n \circ \phi\|^p_{Q_{K,\omega}(p,\, q)} &\le 2^p\ \{0 + \varepsilon\ \|\phi\|^p_{Q_{K,\omega}(p,\, q)} + \varepsilon\} \\
&\le C\ \varepsilon
\end{aligned}$$

Combining the above, we get that $\|C_\phi(f_n)\|^p_{Q_{K,\omega}(p,\, q)} \to 0$ as $n \to \infty$, which proves compactness. The proof of our theorem is therefore established. □

Now we consider the composition operators from the Dirichlet space $\mathcal{D}$ into $Q_{K,\omega}(p,q)$ spaces. Our result is stated as follows.

**Theorem 3.3.** *Let $2 \le p < \infty$, $-2 < q < \infty$ and $K : (0,\infty) \to (0,\infty)$. If $\phi$ is an analytic self-map of the unit disk, then the composition operator $C_\phi : \mathcal{D} \to Q_{K,\omega}(p,q)$ is compact if and only if*

$$\lim_{|a|\to 1} \|C_\phi \varphi_a\|_{Q_{K,\omega}(p,q)} = 0. \tag{3.6}$$

**Proof.** Assume that $C_\phi : \mathcal{D} \to Q_{K,\omega}(p,q)$ is compact. If $\lim_{|a|\to 1} \|C_\phi \varphi_a\|_{Q_{K,\omega}(p,q)} \neq 0$, then there exist an $\epsilon_0 > 0$ and a subsequence $\varphi_{a_n}$ such that

$$\|C_\phi(\varphi_{a_n} - a_n)\|_{Q_{K,\omega}(p,q)} = \|C_\phi(\varphi_{a_n})\|_{Q_{K,\omega}(p,q)} \geq \epsilon_0$$

for $n = 1, 2, \cdots$. Since $C_\phi$ is compact, there is a subsequence $\{\varphi_{a_{n_k}} - a_{n_k}\}$ of $\{\varphi_{a_n} - a_n\}$ and a function $g$ in $Q_{K,\omega}(p,q)$ such that

$$\lim_{k\to\infty} \|C_\phi(\varphi_{a_{n_k}} - a_{n_k})\|_{Q_{K,\omega}(p,q)} = 0.$$

Since $g \in Q_{K,\omega}(p,q) \subset \mathcal{B}_\omega^{\frac{q+2}{p}}$, we have for some $r_0 \in (0,1)$,

$$\begin{aligned} &|C_\phi(\varphi_{a_{n_k}} - a_{n_k})(z) - g(z)| \\ \leq\ & \|C_\phi(\varphi_{a_{n_k}} - a_{n_k}) - g\|_{Q_{K,\omega}(p,q)} \left(1 + \frac{1}{2r_0(\pi K(\log \frac{1}{r_0}))^{\frac{1}{2}}}\right) \log \frac{1+|z|}{1-|z|}. \end{aligned}$$

Thus, $C_\phi(\varphi_{a_n} - a_n) \to g$ uniformly on compact subsets of $\mathbf{D}$ as $k \to \infty$. It means that we must have $g = 0$ which contradicts that $\lim_{|a|\to 1} \|C_\phi \varphi_a\|_{Q_{K,\omega}(p,q)} \neq 0$.
Conversely, let $\{f_n\} \in \mathcal{D}$ be a bounded sequence. Since $f_n \in \mathcal{D} \subset \mathcal{B}$, for $z \in \mathbf{D}$

$$|f_n(z)| \leq \sup_n \|f_n\|_{\mathcal{D}} \left(1 + \frac{1}{2} \log \frac{1+|z|}{1-|z|}\right).$$

Hence, $\{f_n\}$ is a normal family. Thus, there is a subsequence $\{f_{n_k}\}$, which converges to $f$ analytic on $\mathbf{D}$ and both $f_{n_k} \to f$ and $f'_{n_k} \to f'$ uniformly on compact subsets of $\mathbf{D}$. It is easy to show that $f \in \mathcal{D}$. We replace $f$ by $C_\phi f$, we remark that $C_\phi$ is compact by showing

$$\|C_\phi f_{n_k} - C_\phi f\|_{Q_{K,\omega}(p,q)} \longrightarrow 0 \quad \text{as } |k| \to \infty.$$

We write

$$\begin{aligned} \|C_\phi \varphi_a\|^p_{Q_{K,\omega}(p,q)} &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |(\varphi_a \circ \phi)'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\ &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|a|^2)^p}{|1-\overline{a}\phi(z)|^{2p}} |\phi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\ &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|a|^2)^p}{|1-\overline{a}w|^{2p}} \big(N^{p,q}_{K,\omega}(\phi,w,a)\big) dA(w). \end{aligned}$$

Here,

$$N^{p,q}_{K,\omega}(\phi,w,a) = \sum_{z\in\phi^{-1}(w)} |\phi'(z)|^{p-2} (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}$$

is the counting function. Thus (3.6) is equivalent to

$$\lim_{|a|\to 1} \sup_{a\in\Delta} \int_\Delta \frac{(1-|a|^2)^p}{|1-\overline{a}w|^{2p}} \big(N^{p,q}_{K,\omega}(\phi,w,a)\big) dA(w) = 0. \tag{3.7}$$

Hence, for any $\varepsilon > 0$ there exists a $\delta, 0 < \delta < 1$, such that for $|a| > \delta$ and all $a \in \mathbf{D}$,

$$\sup_{a\in\Delta} \int_{S(2(1-|z|),\theta)} N^{p,q}_{K,\omega}(\phi,w,a) dA(w) \ < \ \varepsilon\,(1-|a|)^p. \tag{3.8}$$

The mean value property for analytic functions $f'_{n_k}$ and $f'$ yields that,

$$f'_{n_k}(w) - f'(w) = \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<\frac{1-|w|}{2}} (f'_{n_k}(z) - f'(z)) dA(z).$$

Then by Jensen's inequality (see [15, Theorem 3.3]), we have

$$\begin{aligned} |f'_{n_k}(w) - f'(w)|^p &= \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<\frac{1-|w|}{2}} |f'_{n_k}(z) - f'(z)|^p dA(z) \\ &\leq \frac{4}{\pi(1-|w|)^2} \int_{\mathbf{D}} |f'_{n_k}(z) - f'(z)|^p dA(z). \end{aligned}$$

Note that if $|w-z|<\frac{1-|w|}{2}$, then $w\in S(2(1-|z|),\theta)$ and $\frac{1}{(1-|w|)^2}\leq\frac{C}{(1-|z|)^2}$ (see [18]).
Then, for $G_1=\{z\in\mathbf{D}:|z|>1-\frac{\delta}{2}\}$ and $G_2=\{z\in\mathbf{D}:|z|\leq 1-\frac{\delta}{2}\}$ by Fubini's theorem (see [15, Theorem 8.8]), we deduce that

$$\begin{aligned}
&\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|f'_{n_k}(w)-f'(w)|^p\big(N^{p,q}_{K,\omega}(\phi,w,a)\ \big)dA(w)\\
\leq\ & C\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\left\{\frac{4}{\pi(1-|w|)^2}\int_{|w-z|<\frac{1-|w|}{2}}|f'_{n_k}(z)-f'(z)|^p dA(z)\right\}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)dA(z)\\
\leq\ & C\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\frac{|f'_{n_k}(z)-f'(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)dA(z)\\
=\ & C\sup_{a\in\mathbf{D}}\int_{G_1}\frac{|f'_{n_k}(z)-f'(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)dA(z)\\
+\ & C\sup_{a\in\mathbf{D}}\int_{G_2}\frac{|f'_{n_k}(z)-f'(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)dA(z)\\
=\ & C\{I_1+I_2\},
\end{aligned}$$

For one hand, since $f_{n_k},f\in\mathcal{D}\subset\mathcal{B}$ and $2\leq p<\infty$, we have

$$\begin{aligned}
I_1\ =\ & \sup_{a\in\mathbf{D}}\int_{G_1}\frac{|f'_{n_k}(z)-f'(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)dA(z)\\
\leq\ & 2^p\varepsilon\sup_{a\in\Delta}\int_{G_1}|f'_{n_k}(z)-f'(z)|^p(1-|z|)^{p-2}\ dA(z)\\
\leq\ & C\varepsilon\ \|f_{n_k}-f\|^{p-2}_{\mathcal{B}}\sup_{a\in\mathbf{D}}\int_{G_1}|f'_{n_k}(z)-f'(z)|^2\ dA(z)\\
\leq\ & C\varepsilon\ \|f_{n_k}-f\|^{p-2}_{\mathcal{B}}\ \|f_{n_k}-f\|^2_{\mathcal{D}}\\
\leq\ & C_1\varepsilon\ \|f_{n_k}-f\|^2_{\mathcal{D}},
\end{aligned}$$

where $C$ and $C_1$ are constants. On the other hand,

$$\begin{aligned}
I_2\ =\ & \sup_{a\in\mathbf{D}}\int_{G_2}\frac{|f'_{n_k}(z)-f'(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)dA(z)\\
\leq\ & C\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)\int_{G_2}|f'_{n_k}(z)-f'(z)|^p dA(z)\\
\leq\ & C\ \varepsilon,
\end{aligned}$$

for $n$ large enough since $f'_{n_k}(z)-f'(z)\longrightarrow 0$ uniformly on $G_2$. Therefore, for sufficiently large $k$, the above discussion gives

$$\begin{aligned}
&\|C_\phi f_{n_k}-C_\phi f\|^p_{Q_{K,\omega}(p,q)}\\
=\ & \sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|(f_{n_k}\circ\phi)'(z)-(f\circ\phi)'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)\\
=\ & \sup_{a\in\Delta}\int_{\Delta}|f'_{n_k}(z)-f'(z)|^p(1-|z|^2)^q N^{p,q}_{K,\omega}(\phi,w,a)\ dA(w)<\ C\ \varepsilon.
\end{aligned}$$

It follows that $C_\phi$ is a compact operator. Therefore, the proof is finished. □

**Remark 3.4.** It should be remarked that our $Q_{K,\omega}(p,q)$ classes are more general than many classes of analytic functions. If $\omega\equiv 1$, we obtain $Q_K(p,q)$ type spaces (cf. [20]) . If $q=p=2$, and $\omega(t)=t$, we obtain $Q_K$ spaces as studied recently in [6,7,9,19,20,21] and others. If $q=p=2$, $\omega(t)=t$ and $K(t)=t^p$, we obtain $Q_p$ spaces as studied in [2,3,22] and others. If $\omega\equiv 1$ and $K(t)=t^s$, then $Q_{K,\omega}=F(p,q,s)$ classes (cf. [1,23]).

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