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Creeping flow of a viscous fluid past a pair of porous separated spheres *

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Abstract In this paper we consider the problem of creeping or the Stokes' flow of a viscous fluid past a pair of porous separated spheres with the problem formulation done in the bipolar coordinate system. Stokesian approximation of the Navier-Stokes equations for the Newtonian fluid model is taken to describe the fluid flow in the region exterior to the porous spheres, while the classical Darcy's law is for the flow inside the porous spheres. An analytical solution to this problem is found wherein the expressions for stream function, pressure and velocity are derived in terms of the Legendre functions, the hyperbolic trigonometric functions and the Gegenbauer functions. Also, the expression for the drag experienced by each of the spheres is found and we carry out numerical evaluations to compute the values of drag in the cases where the two spheres are of equal radii and the case where they are of unequal radii. The plots of streamlines and pressure contours are presented and discussed.

Key words Porous separated spheres, Bipolar, Gegenbauer functions, Legendre function, Stream function, Drag.

2020 Mathematics Subject Classification 76S05, 76S99.

1 Introduction

The problem of fluid flow past separated spheres or flow past two spheres is a classical problem that had its origin at the beginning of the 19$^{\text{th}}$ century. Jeffery [1] found the solution to the Laplace equation in the bipolar coordinate system while considering the flow of viscous fluid past separated spheres. Later, Stimson and Jeffery [2] formulated an equivalent problem to the problem of viscous fluid flow past separated spheres in the bipolar coordinate system, where the motion set up in the viscous fluid at rest at infinity by two solid spheres moving with small constant and equal velocities was studied. They determined the stream function as an infinite series in terms of the Legendre polynomials and hyperbolic trigonometric functions and calculated the forces necessary to maintain the motion of the two spheres. Dyuro [3] considered the potential flow of fluid past two solid spheres in the bipolar system and derived

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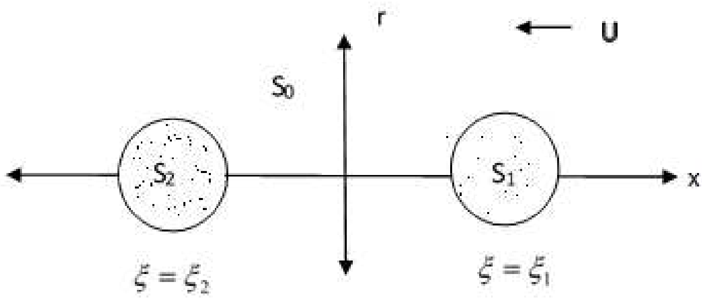

Fig. 1: Schematic diagram of flow past separated porous spheres in bipolar coordinates.

an expression for the velocity potential as an infinite series in associated Legendre polynomials (see, for instance, [16]). Sneddon and Fulton [4] solved the problem of irrotational flow of a perfect fluid past two spheres using two sets of spherical polar coordinates (one for each sphere) to solve the Laplace equation governing the flow. Payne and Pell [5] studied Stokes flow for a class of axially symmetric bodies with the problem of flow past solid separated spheres formulated in the peripolar coordinate system. They derived an expression for the stream function in terms of the associated Legendre polynomials. Later, many investigations were carried out on the problem of fluid flow past two or cluster of solid spheres and analytical or numerical solutions are presented as in the references [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Though the problem of flow past porous spheres has been studied in several works, to our knowledge,the problem of flow past porous separated spheres was given sparse attention.

Thus, in the present work, we aim to solve the problem of viscous flow past porous separated spheres using Stokesian approximation of the Navier-Stokes equation to describe the flow of fluid in the region around (exterior to) the porous spheres while the classical Darcy's law holds for the flow within the spheres. We present analytical solutions to this problem wherein, we derive expressions for the stream function and the pressure function in terms of the Legendre functions, the hyperbolic trigonometric functions and the Gegenbauer functions. An analytical expression for the drag experienced by each of the porous spheres. Further we carry out numerical computations to find the values of the drag experienced by each of the spheres in two cases: where the spheres are of equal radii and when the spheres are of unequal radii. We also plot and discuss the streamlines and pressure contours.

The paper is organized as follows: section 2 presents the mathematical formulation of the problem in the bipolar coordinate system. The subsections 2.1 and 2.2 detail the solution in the region S_0, the region outside the porous spheres. The subsection 2.3 presents the solution in the porous regions $S_i, i = 1, 2$.. In section 3 we derive the equations to determine the arbitrary constants in the solution using appropriate boundary conditions. Section 4 consists of the detailed working on the formula for computing drag on each sphere. In subsection 4.1 we derive the expression for drag on each sphere and tabulate their numerical values. Section 5 consists of the plots and discussions on streamlines and the pressure function and the conclusions are summarized in the section 6.

2 Mathematical formulation of the problem

Consider the flow of a viscous fluid (with a uniform velocity U at infinity) past a pair of separated porous spheres fixed in the flow domain, as shown in Fig. 1.

The bipolar system represented by the coordinates (ξ, η, ϕ), with $(\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\varphi)$ as base vectors and(h_ξ, h_η, h_ϕ) as the corresponding scale factors are taken to describe the flow domain, where,

$$x = \frac{a \sinh \xi}{\cosh \xi - \cos \eta} \quad \text{and} \quad r = \frac{a \sin \eta}{\cosh \xi - \cos \eta} \tag{2.1}$$

$$h_\xi = \frac{a}{\cosh\xi - \cos\eta}; h_\eta = \frac{a}{\cosh\xi - \cos\eta}; h_\phi = \frac{a\sin\eta}{\cosh\xi - \cos\eta} \tag{2.2}$$

for $-\infty < \xi < \infty, 0 \leq \eta < \pi$.
Further, $\xi = c > 0$ where, c is a constant, represents the spheres on the positive x-axis with centre at a distance of $a\coth c$ from the origin (along the x-axis) and with radius equal to $a\,\mathrm{cosech}\, c$ and $\xi = c < 0$ describes the spheres on the negative x-axis with their centres at a distance of $-a\coth c$ from the origin (along the x-axis) and with radius equal to $a\,\mathrm{cosech}\, c$.
Let $\left(\bar{q}^{(i)}, p^{(i)}\right)$ denote the velocity and pressure in the region $S_i, i = 1,2$ whereS_1 represents the region inside the porous sphere $\xi = \xi_1$ and S_2, the region inside the porous sphere $\xi = \xi_2$.
Equations governing the fluid flow in the region S_0:
Let $\left(\bar{q}^{(0)}, p^{(0)}\right)$ be the velocity vector and pressure in the regionS_0. Assuming that the flow (in the region S_0) is axi-symmetric, we have $\bar{q}^{(0)} = u^{(0)}(\xi,\eta)\,\hat{e}_\xi + v^{(0)}(\xi,\eta)\,\hat{e}_\eta$ and the pressure as $p^{(0)}(\xi,\eta)$. Further, considering the fluid to be incompressible and the flow as steady, the momentum equations under Stokesian approximation take the form:

$$\mathrm{grad}\, p^{(0)} + \mu\,\mathrm{curl}\left(\mathrm{curl}\,\bar{q}^{(0)}\right) = 0 \tag{2.3}$$

Now, introducing the stream function through,

$$h_\eta h_\phi u^{(0)} = -\frac{\partial\psi^{(0)}}{\partial\eta}; h_\xi h_\phi v^{(0)} = \frac{\partial\psi^{(0)}}{\partial\xi} \tag{2.4}$$

we see that

$$\mathrm{curl}\,\vec{q}^{(0)} = \left\{\frac{1}{h_\phi}E^2\psi^{(0)}\right\}\vec{e}_\varphi \tag{2.5}$$

$$\mathrm{curl}\,\mathrm{curl}\,\vec{q}^{(0)} = \frac{1}{h_\xi h_\eta h_\phi}\left\{h_\xi\frac{\partial}{\partial\eta}\left(E^2\psi^{(0)}\right)\vec{e}_\xi - h_\eta\frac{\partial}{\partial\xi}\left(E^2\psi^{(0)}\right)\vec{e}_\eta\right\} \tag{2.6}$$

in which the Stokes stream function operator E^2 is given by

$$E^2 = \frac{h_\phi}{h_\xi h_\eta}\left\{\frac{\partial}{\partial\xi}\left(\frac{h_\eta}{h_\xi h_\phi}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\xi}{h_\eta h_\phi}\frac{\partial}{\partial\eta}\right)\right\} \tag{2.7}$$

Using the expressions (2.5) and (2.6), (2.3) takes the form

$$\frac{1}{h_\xi}\frac{\partial p^{(0)}}{\partial\xi} + \frac{\mu}{h_\eta h_\phi}\frac{\partial}{\partial\eta}\left(E^2\psi^{(0)}\right) = 0 \tag{2.8}$$

$$\frac{1}{h_\eta}\frac{\partial p^{(0)}}{\partial\eta} - \frac{\mu}{h_\xi h_\varphi}\frac{\partial}{\partial\xi}\left(E^2\psi^{(0)}\right) = 0 \tag{2.9}$$

Eliminating $p^{(0)}$ from (2.8) and (2.9) gives

$$E^4\psi^{(0)} = 0 \tag{2.10}$$

which is the equation governing the fluid flow in the region S_0.
Equations governing the fluid flow in regions $S_i, i = 1, 2$:
In the regions $S_i, i = 1, 2$, we consider the classical Darcy's law given by

$$\mathrm{div}\,\bar{q}^{(i)} = 0 \tag{2.11}$$

$$\bar{q}^{(i)} = -k^{(i)}\,\mathrm{grad}\;p^{(i)} \tag{2.12}$$

where $k^{(i)}$,$i = 1, 2$ is the permeability of the porous medium S_i.
Eliminating $\bar{q}^{(i)}$from equations (2.11) and (2.12), we get

$$\nabla^2 p^{(i)} = 0, i = 1, 2 \tag{2.13}$$

which is the governing equation in regions $S_i, i = 1, 2$.

Here ∇^2 is the Laplacian operator given by

$$\nabla^2 = \frac{1}{h_\xi h_\eta h_\phi}\left\{\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\phi}{h_\xi}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\xi h_\phi}{h_\eta}\frac{\partial}{\partial\eta}\right)\right\} \tag{2.14}$$

Boundary Conditions:
The determination of the relevant flow field variables $\psi^{(0)}$ and $p^i, i = 0, 1, 2$ is subject to the following boundary and regularity conditions:

(i) Continuity of the normal velocity component at interfaces:

$$u^{(i)} = u^{(0)} \text{ on } S_i, i = 1, 2 \tag{2.15}$$

(ii) Tangential velocity component vanish at interfaces:

$$v^{(0)} = 0 \text{ on } S_i, i = 1, 2 \tag{2.16}$$

(iii) Continuity of pressure at interfaces:

$$p^{(i)} = p^{(0)} \text{ on } S_i, i = 1, 2 \tag{2.17}$$

(iv) The velocities are regular on the axis, and far away from S_0, the flow is a uniform stream which means, at infinity

$$\psi^{(0)} = -\frac{1}{2}Ur^2, \text{ i.e., } \lim_{\xi\to 0} u^{(0)} = -U, \lim_{\xi\to 0} v^{(0)} = 0. \tag{2.18}$$

2.1 Solution to the equations governing the flow in S_0

To find the solution to the problem in the region S_0 given by (2.10) we make use of its linear nature and, thus, we have to solve

$$E^2\psi^{(0)} = f, \tag{2.19}$$

where f is the solution to

$$E^2 f = 0. \tag{2.20}$$

Solution to the equation (2.20):
(2.20) in bi-polar coordinate system is

$$\frac{\cosh\xi-\cos\eta}{a^2}\left(a\sin\eta\frac{\partial}{\partial\xi}\left(\frac{\cosh\xi-\cos\eta}{a\sin\eta}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left((\cosh\xi-\cos\eta)\frac{\partial}{\partial\eta}\right)\right)f = 0 \tag{2.21}$$

Following [2], let $\cos\eta = \tau$. Then equation (2.21) takes the form

$$\frac{\cosh\xi-\tau}{a^2}\left(\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)\frac{\partial}{\partial\xi}\right)+\left(1-\tau^2\right)\frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)\frac{\partial}{\partial\tau}\right)\right)f = 0 \tag{2.22}$$

Now, following the method of separation of variables, let us assume the solution to (2.22) as

$$f(\xi,\tau) = (\cosh\xi-\tau)^n g(\xi,\tau) \tag{2.23}$$

Substituting expression shown in (2.23) in equation (2.22) and after a straight forward calculation, we get

$$f(\xi,\tau) = (\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau) \tag{2.24}$$

where A_n, B_n are arbitrary constants.
Using the expression(2.24) in (2.19), we get

$$\begin{aligned}&\frac{\cosh\xi-\tau}{a^2}\left(\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)\frac{\partial}{\partial\xi}\right)+\left(1-\tau^2\right)\frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)\frac{\partial}{\partial\tau}\right)\right)\psi^{(0)}\\ &= (\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau).\end{aligned} \tag{2.25}$$

Again, using the method of separation of variables, we assume the solution of (2.25) in the form

$$\psi^{(0)}(\xi,\tau) = a^2(\cosh\xi-\tau)^{-3/2}\sum_{n=1}^{\infty} h_n(\xi)\,\vartheta_{n+1}(\tau). \tag{2.26}$$

Now, substitute the above expression in (2.25) and using the following relations in Gegenbauer functions:

$$\left(1-x^2\right)\vartheta'_{n+1}(x) = -(n+1)\,x\vartheta_{n+1}(x) + (n-1)\,\vartheta_n(x), \tag{2.27}$$

$$(n+1)\,\vartheta_{n+1}(x) = (2n-1)\,x\vartheta_n(x) - (n-2)\,\vartheta_{n-1}(x), \tag{2.28}$$

and the orthogonality property of the Legendre polynomials, we get

$$\psi^{(0)}(\xi,\tau) = a^2(\cosh\xi-\tau)^{-3/2}\times$$
$$\sum_{n=1}^{\infty}\left(C_n\cosh\left(n-\frac{1}{2}\right)\xi + D_n\sinh\left(n-\frac{1}{2}\right)\xi + E_n\cosh\left(n+\frac{3}{2}\right)\xi + F_n\sinh\left(n+\frac{3}{2}\right)\xi\right)\vartheta_{n+1}(\tau), \tag{2.29}$$

where,

$$A_n = -(2n-1)\,C_n + (2n+3)\,E_n \text{ and } B_n = -(2n-1)\,D_n + (2n+3)\,F_n. \tag{2.30}$$

2.2 Expression for pressure in the region S_0

We now derive the expression for the pressure function from (2.8) and (2.9). For this, we substitute the expressions for the scale factors from (2.2) in (2.8) and (2.9) to get

$$\frac{\partial p^{(0)}}{\partial\xi} = -\frac{\mu(\cosh\xi-\tau)}{a}\frac{\partial}{\partial\tau}\left(E^2\psi^{(0)}\right) \tag{2.31}$$

$$\frac{\partial p^{(0)}}{\partial\tau} = -\frac{\mu(\cosh\xi-\tau)}{(1-\tau^2)}\frac{\partial}{\partial\xi}\left(E^2\psi^{(0)}\right). \tag{2.32}$$

Eliminating $\psi^{(0)}$ from these equations, we get

$$\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)^{-1}\frac{\partial p^{(0)}}{\partial\xi}\right) + \frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)^{-1}\left(1-\tau^2\right)\frac{\partial p^{(0)}}{\partial\tau}\right) = 0. \tag{2.33}$$

Using the method of separation of variables, we get

$$p^{(0)}(\xi,\tau) = (\cosh\xi-\tau)^{1/2}\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau) \tag{2.34}$$

where $P_n(\tau)$'s are the Legendre's polynomials.
Substituting the expression for pressure from (2.34) in equation (2.31), we have

$$\begin{gathered}
\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sinh\xi\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau) + \\
(\cosh\xi-\tau)^{1/2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(H_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi + G_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau) \\
= \frac{\mu}{a}\left(\begin{array}{l}\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi + B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau) + \\ (\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi + B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta'_{n+1}(\tau)\end{array}\right)
\end{gathered} \tag{2.35}$$

Multiplying on both sides of (2.35) by $(\cosh\xi-\tau)^{-1}$ and integrating the resulting equation with respect to τ between the limits -1 and 1 gives,

$$\begin{aligned}&\frac{1}{2}\sinh\xi\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{P_n\left(\tau\right)}{\left(\cosh\xi-\tau\right)^{3/2}}d\tau+\\&\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(H_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{P_n\left(\tau\right)}{\left(\cosh\xi-\tau\right)^{1/2}}d\tau\\&=-\frac{\mu}{2a}\left(\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}\left(\tau\right)}{\left(\cosh\xi-\tau\right)^{3/2}}d\tau\right)\end{aligned}\tag{2.36}$$

Using the formulae given in expressions (2.37) and (2.38), the integrals in the above expression can be evaluated to find the equation involving the constants G_n, H_n, A_n, B_n.

$$\int_{-1}^{1}\frac{P_n\left(x\right)}{\left(\cosh\xi-x\right)^{1/2}}dx=\frac{2\sqrt{2}}{2n+1}e^{-\left(n+\frac{1}{2}\right)|\xi|}\tag{2.37}$$

$$\int_{-1}^{1}\frac{P_n\left(x\right)}{\left(\cosh\xi-x\right)^{3/2}}dx=\frac{2\sqrt{2}}{sinh\,|\xi|}e^{-\left(n+\frac{1}{2}\right)|\xi|}.\tag{2.38}$$

Substituting the expression for pressure from (2.34) in (2.32), we have

$$\begin{aligned}&-\frac{1}{2}\left(\cosh\xi-\tau\right)^{-1/2}\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n\left(\tau\right)+\\&\left(\cosh\xi-\tau\right)^{1/2}\sum_{n=1}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)P'_n\left(\tau\right)\\&=-\frac{\mu}{a\left(1-\tau^2\right)}\left(\begin{array}{l}-\frac{1}{2}\left(\cosh\xi-\tau\right)^{-1/2}\sinh\xi\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}\left(\tau\right)+\\\left(\cosh\xi-\tau\right)^{1/2}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\left(A_n\sinh\left(n+\frac{1}{2}\right)\xi+B_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}\left(\tau\right)\end{array}\right)\end{aligned}\tag{2.39}$$

Multiplying on both sides of (2.39) by $(\cosh\xi-\tau)^{-1}$ and integrating the resulting equation with respect to τ between the limits -1 and 1 gives,

$$\begin{aligned}&-\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\frac{d}{d\tau}\left(\left(1-\tau^2\right)P_n\left(\tau\right)\right)}{\left(\cosh\xi-\tau\right)^{1/2}}d\tau+\\&\sum_{n=1}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi+G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\left(1-\tau^2\right)P'_n\left(\tau\right)}{\left(\cosh\xi-\tau\right)^{1/2}}d\tau\\&=-\frac{\mu}{a}\left(\begin{array}{l}-\frac{1}{2}\sinh\xi\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}\left(\tau\right)}{\left(\cosh\xi-\tau\right)^{3/2}}d\tau+\\\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\left(A_n\sinh\left(n+\frac{1}{2}\right)\xi+B_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}\left(\tau\right)}{\left(\cosh\xi-\tau\right)^{1/2}}d\tau\end{array}\right)\end{aligned}\tag{2.40}$$

Using the relations in (2.37), (2.38) and the recurrence relations in Legendre polynomials [16] (shown in detail in section 4 of this paper) we can derive the expressions for G_n's and H_n 's in terms of A_n's and B_n's. Again using the relations given in (2.30), these, in turn, can be written in terms of C_n, D_n, E_n and F_n, thus the pressure function is completely determined.

2.3 Solution to the equations governing the flow in regions $S_i, i = 1, 2$

From equations (2.13) and (2.14), we have

$$\frac{\partial}{\partial \xi}\left((\cosh \xi-\tau)^{-1} \frac{\partial p^{(i)}}{\partial \xi}\right)+\frac{\partial}{\partial \tau}\left((\cosh \xi-\tau)^{-1}\left(1-\tau^{2}\right) \frac{\partial p^{(i)}}{\partial \tau}\right)=0, i=1,2. \quad (2.41)$$

Its solution is

$$p^{(1)}(\xi, \tau)=(\cosh \xi-\tau)^{1/2} \sum_{n=0}^{\infty}\left(L_{n+1} \cosh \left(n+\frac{1}{2}\right) \xi+M_{n+1} \sinh \left(n+\frac{1}{2}\right) \xi\right) P_{n}(\tau), \quad (2.42)$$

and

$$p^{(2)}(\xi, \tau)=(\cosh \xi-\tau)^{1/2} \sum_{n=0}^{\infty}\left(R_{n+1} \cosh \left(n+\frac{1}{2}\right) \xi+S_{n+1} \sinh \left(n+\frac{1}{2}\right) \xi\right) P_{n}(\tau) \quad (2.43)$$

where L_n, M_n, R_n, S_n are arbitrary constants.
Let us introduce stream function $\psi^{(i)}, i = 1, 2$ by defining relations similar to those given in (2.4). Now, substituting the expression for the stream function and eliminating $p^{(i)}$ from (2.11) and (2.12), we get,

$$\frac{\partial}{\partial \xi}\left((\cosh \xi-\mu) \frac{\partial \psi^{(i)}}{\partial \xi}\right)+\left(1-\mu^{2}\right) \frac{\partial}{\partial \mu}\left((\cosh \xi-\mu) \frac{\partial \psi^{(i)}}{\partial \mu}\right)=0. \quad (2.44)$$

Using the method of separation of variables, we find its solution to be

$$\psi^{(i)}=(\cosh \xi-\mu)^{-1/2} \sum_{n=1}^{\infty}\left(U_{n} \cosh (n+1/2) \xi+V_{n} \sinh (n+1/2) \xi\right) \vartheta_{n+1}(\mu). \quad (2.45)$$

Substituting the expressions for $\psi^{(i)}$and $p^{(i)}$, $i = 1$ in equation (2.11), we can express U_n, V_n in terms of L_n, M_n and thus the stream function in the region $S_1$is determined. Similarly, the stream function in the region S_2 can also be determined.

3 Determination of Arbitrary constants

The eight sets of arbitrary constants L_n, M_n, R_n, S_n,C_n, D_n, E_n and F_n in the expressions (2.29), (2.36), (2.37) are to be determined using the boundary conditions given in (2.15)–(2.18).

(i) Continuity of the normal velocity component at interfaces:
$u^{(0)} = u^{(1)}$ on $\xi = \xi_1$ gives,

$$\frac{\partial \psi^{(0)}}{\partial \tau}=K^{(1)} a(\cosh \xi-\mu)^{-1} \frac{\partial p^{(1)}}{\partial \xi} \text{ on } \xi=\xi_{1}, \quad (3.1)$$

Integrating the above expression with respect to τ between the limits -1 and 1 gives,

$$\begin{aligned} &\frac{\sinh \xi_{1}}{2} \sum_{n=0}^{\infty}\left(L_{n+1} \cosh \left(n+\frac{1}{2}\right) \xi_{1}+M_{n+1} \sinh \left(n+\frac{1}{2}\right) \xi_{1}\right) \frac{e^{-(n+1/2)\left|\xi_{1}\right|}}{\sinh \left|\xi_{1}\right|}+ \\ &\sum_{n=0}^{\infty}(n+1/2)\left(L_{n+1} \sinh \left(n+\frac{1}{2}\right) \xi_{1}+M_{n+1} \cosh \left(n+\frac{1}{2}\right) \xi_{1}\right) \frac{e^{-(n+1/2)\left|\xi_{1}\right|}}{(2 n+1)}=0, \end{aligned} \quad (3.2)$$

and $u^{(0)} = u^{(2)}$ on $\xi = \xi_2$ gives,

$$\frac{\partial \psi^{(0)}}{\partial \tau}=K^{(2)} a(\cosh \xi-\mu)^{-1} \frac{\partial p^{(2)}}{\partial \xi} \text{ on } \xi=\xi_{2} \quad (3.3)$$

$$\begin{aligned} \Rightarrow &\frac{\sinh \xi_{2}}{2} \sum_{n=0}^{\infty}\left(R_{n+1} \cosh \left(n+\frac{1}{2}\right) \xi_{2}+S_{n+1} \sinh \left(n+\frac{1}{2}\right) \xi_{2}\right) \frac{e^{-(n+1/2)\left|\xi_{2}\right|}}{\sinh \left|\xi_{2}\right|}+ \\ &\sum_{n=0}^{\infty}(n+1/2)\left(R_{n+1} \sinh \left(n+\frac{1}{2}\right) \xi_{2}+S_{n+1} \cosh \left(n+\frac{1}{2}\right) \xi_{2}\right) \frac{e^{-(n+1/2)\left|\xi_{2}\right|}}{(2 n+1)}=0. \end{aligned} \quad (3.4)$$

(ii) Tangential velocity component vanishes at interfaces:

$$v^{(0)} = 0 \text{ on } \xi = \xi_1 \text{ gives, } \frac{\partial \psi^{(0)}}{\partial \xi} = 0 \text{ on } \xi = \xi_1. \tag{3.5}$$

i.e.,

$$\begin{aligned}
&-\frac{3}{2}\frac{\sinh \xi_1}{\sinh |\xi_1|}\sum_{n=1}^{\infty}\left(C_n \cosh\left(n-\frac{1}{2}\right)\xi_1 + D_n \sinh\left(n-\frac{1}{2}\right)\xi_1\right.\\
&\left.+E_n \cosh\left(n+\frac{3}{2}\right)\xi_1 + F_n \sinh\left(n+\frac{3}{2}\right)\xi_1\right)\left(\frac{e^{-(n-1/2)|\xi_1|} - e^{-(n+3/2)|\xi_1|}}{2n+1}\right)\\
&+\sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\left(n-\frac{1}{2}\right)C_n \sinh\left(n-\frac{1}{2}\right)\xi_1 + \left(n-\frac{1}{2}\right)D_n \cosh\left(n-\frac{1}{2}\right)\xi_1\right.\\
&\left.+\left(n+\frac{3}{2}\right)E_n \sinh\left(n+\frac{3}{2}\right)\xi_1 + \left(n+\frac{3}{2}\right)F_n \cosh\left(n+\frac{3}{2}\right)\xi_1\right)\times\\
&\left(\frac{e^{-(n-1/2)|\xi_1|}}{2n-1} - \frac{e^{-(n+3/2)|\xi_1|}}{2n+3}\right) = 0,
\end{aligned} \tag{3.6}$$

Furthermore,

$$v^{(0)} = 0 \text{ on } \xi = \xi_2 \text{ gives } \frac{\partial \psi^{(0)}}{\partial \xi} = 0 \text{ on } \xi = \xi_2. \tag{3.7}$$

$$\begin{aligned}
&-\frac{3}{2}\frac{\sinh \xi_2}{\sinh |\xi_2|}\sum_{n=1}^{\infty}\left(C_n \cosh\left(n-\frac{1}{2}\right)\xi_2 + D_n \sinh\left(n-\frac{1}{2}\right)\xi_2\right.\\
&\left.+E_n \cosh\left(n+\frac{3}{2}\right)\xi_2 + F_n \sinh\left(n+\frac{3}{2}\right)\xi_2\right)\left(\frac{e^{-(n-1/2)|\xi_2|} - e^{-(n+3/2)|\xi_2|}}{2n+1}\right)\\
&+\sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\left(n-\frac{1}{2}\right)C_n \sinh\left(n-\frac{1}{2}\right)\xi_2 + \left(n-\frac{1}{2}\right)D_n \cosh\left(n-\frac{1}{2}\right)\xi_2\right.\\
&\left.+\left(n+\frac{3}{2}\right)E_n \sinh\left(n+\frac{3}{2}\right)\xi_2 + \left(n+\frac{3}{2}\right)F_n \cosh\left(n+\frac{3}{2}\right)\xi_2\right)\times\\
&\left(\frac{e^{-(n-1/2)|\xi_2|}}{2n-1} - \frac{e^{-(n+3/2)|\xi_2|}}{2n+3}\right) = 0.
\end{aligned} \tag{3.8}$$

(iii) Continuity of pressure at interfaces: $p^{(1)} = p^{(0)}$ on $\xi = \xi_1$ gives

$$\begin{aligned}
&(\cosh \xi_1 - \tau)^{1/2}\sum_{n=0}^{\infty}\left(L_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi_1 + M_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi_1\right)P_n(\tau)\\
&= (\cosh \xi_1 - \tau)^{1/2}\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi_1 + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi_1\right)P_n(\tau),
\end{aligned} \tag{3.9}$$

and $p^{(2)} = p^{(0)}$ on $\xi = \xi_2$ gives

$$\begin{aligned}
&(\cosh \xi_2 - \tau)^{1/2}\sum_{n=0}^{\infty}\left(R_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi_2 + S_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi_2\right)P_n(\tau)\\
&= (\cosh \xi_2 - \tau)^{1/2}\sum_{n=0}^{\infty}\left(H_{n+1}\cosh\left(n+\frac{1}{2}\right)\xi_2 + G_{n+1}\sinh\left(n+\frac{1}{2}\right)\xi_2\right)P_n(\tau).
\end{aligned} \tag{3.10}$$

(iv) Regularity condition at Infinity:

$$\lim_{\xi \to 0} u^{(0)} = -U$$

$$\Rightarrow \frac{3}{2}(1-\tau)^{-1/2}\sum_{n=1}^{\infty}(C_n + E_n)\vartheta_{n+1}(\tau) - (1-\tau)^{1/2}\sum_{n=1}^{\infty}(C_n + E_n)\vartheta'_{n+1}(\tau) = -U, \tag{3.11}$$

$$\lim_{\xi\to 0} v^{(0)} = 0 \Rightarrow \sum_{n=1}^{\infty}\left(\left(n-\frac{1}{2}\right)D_n + \left(n+\frac{3}{2}\right)F_n\right)\vartheta_{n+1}(\tau) = 0. \tag{3.12}$$

Let us introduce the non-dimensional quantities as

$$\psi* = \psi/(Ua^2); p^{(0)} = p^{(0)*}\mu U/a; p^{(i)} = p^{(i)*}\mu U/a, i = 0,1,2; \text{ and } k^{(i)*} = \frac{k^{(i)}a}{\mu}. \tag{3.13}$$

We see that the above sets of equations to determine the arbitrary constants are infinite series in terms of infinite sets of constants. Solving these equations for the constants is the most crucial and a complex task. We handle the complexity in the following way - for this, let us recall the definition of equality of two infinite series: we say that two infinite series are equal if and only if their corresponding partial sums are equal.

Thus, equating the sum to first one term of the two series (here, we take the right-hand side of the above equations as the zero series) we get eight equations in eight unknowns L_1, M_1, R_1, S_1,$C_1, D_1, E_1$and F_1 that can be readily solved for these unknowns. Now, equating the sum to first two terms of the two series, we get eight equations in eight unknowns, namely, L_1, M_1, R_1, S_1,C_1, D_1, E_1,F_1 and L_2, M_2, R_2, S_2,C_2, D_2, E_2,F_2. Since the values L_1, M_1, R_1, S_1,C_1, D_1, E_1,F_1 are known, we substitute these values in these equation and get eight equations in eight unknowns that can be solved for L_n, M_n, R_n, S_n,C_n, D_n, E_n,F_n for $n = 2$. We then equate the sum to first three terms of the two series to get the values of L_n, M_n, R_n, S_n,C_n, D_n, E_n,F_n for $n = 3$. This process continues until the difference in the values of the expressions on both the sides of the equations (3.1) – (3.12) is up to a desired degree of accuracy. Thus, knowing the values of the infinite sets of arbitrary constants, we use them to derive analytical expressions for the non-dimensional stream and the pressure functions in all the three regions. In our study, we computed the values of L_n, M_n, R_n, S_n,C_n, D_n, E_n,F_n for $n = 1, 2, 3$ and the difference is in the order 10^{-9} to 10^{-31}.

The reader may note that, in the latter part of the text, '*' is dropped from the notations of stream function and pressure function, but still they continue to represent the non-dimensional quantities.

4 Determination of drag

In view of the assumption that the flow is axisymmetric, the stress vector $\vec{t}$ on the boundary of a sphere is given by

$$\bar{t} = t_{\xi\xi}\ \hat{e}_\xi + t_{\xi\eta}\ \hat{e}_\eta. \tag{4.1}$$

We have,

$$t_{\xi\xi} = -p + 2\mu e_{\xi\xi}, \tag{4.2}$$

where,

$$\begin{aligned} e_{\xi\xi} &= \tfrac{\partial}{\partial\xi}(u/h_\xi) + \tfrac{1}{h_\xi}\bar{q}.\nabla h_\xi, \\ &= \tfrac{1}{a}\left(-v\sqrt{1-\tau^2} + (cosh\xi - \tau)\,\tfrac{\partial u}{\partial\xi}\right), \end{aligned} \tag{4.3}$$

and

$$t_{\xi\eta} = 2\mu e_{\xi\eta}, \tag{4.4}$$

where,

$$\begin{aligned} e_{\xi\eta} &= \tfrac{1}{2}\left(\tfrac{1}{h_\xi}\left(\tfrac{\partial v}{\partial\xi} - \tfrac{u}{h_\eta}\tfrac{\partial h_\xi}{\partial\eta}\right) + \tfrac{1}{h_\eta}\left(\tfrac{\partial u}{\partial\mu} - \tfrac{v}{h_\xi}\tfrac{\partial h_\eta}{\partial\xi}\right)\right) \\ &= \tfrac{1}{2a}\left(u\sqrt{1-\tau^2} + v\sinh\xi + (\cosh\xi - \tau)\left(\tfrac{\partial u}{\partial\eta} + \tfrac{\partial v}{\partial\xi}\right)\right), \end{aligned} \tag{4.5}$$

Now, the component of the stress vector in the direction of the axis of symmetry is

$$\begin{aligned} (\text{stress})_{\text{axial}} &= \bar{t}.\nabla x \\ &= \frac{1}{\cosh - \tau}\left((1 - \tau\cosh\xi)\,t_{\xi\xi} - \left(\sqrt{1-\tau^2}\ \text{sinh}\xi\right)t_{\xi\eta}\right). \end{aligned} \tag{4.6}$$

Thus, the drag D experienced by each sphere can be written in the form

$$D = 2\pi a^2 \int_{-1}^{1} \frac{(\text{stress})_{\text{axial}}}{(\cosh\xi - \tau)^2} d\tau. \tag{4.7}$$

4.1 Drag on the sphere $\xi = \xi_1$

Using the expressions for pressure given in (2.36), we find the velocity components using the relation given in (2.12). We, then compute the strain components using the following formulae

$$e_{\xi\xi}|_{\xi=\xi_1} = \frac{(\cosh\xi_1 - \tau)}{a}\frac{\partial u^{(1)}}{\partial\xi}\bigg|_{\xi=\xi_1}, \tag{4.8}$$

and

$$e_{\xi\eta}|_{\xi=\xi_1} = \frac{1}{2}\left(u^{(1)}\sqrt{1-\tau^2} + +(\cosh\xi_1 - \tau)\left(\sqrt{1-\tau^2}\frac{\partial u^{(1)}}{\partial\tau} + \frac{\partial v^{(1)}}{\partial\xi}\right)\right)\bigg|_{\xi=\xi_1}. \tag{4.9}$$

Now, the stress components are to be evaluated by using the expressions given in (4.2) and (4.4). For this, let us denote

$$I_1(n) = \int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi - x)^{1/2}}dx = \frac{2\sqrt{2}}{2n+1}e^{-(n+1/2)|\xi|}, \tag{4.10}$$

$$I_2(n) = \int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi - x)^{3/2}}dx = \frac{2\sqrt{2}}{\sinh|\xi|}e^{-(n+1/2)|\xi|}, \tag{4.11}$$

$$I_3(n) = \int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi - x)^{5/2}}dx = \frac{4\sqrt{2}}{3(\sinh|\xi|)^2}\left(\frac{2n+1}{2} + \coth|\xi|\right)e^{-(n+1/2)|\xi|}. \tag{4.12}$$

Then, we have

$$I_4(n) = \int_{-1}^{1}\frac{\left(1-x^2\right)P_n(x)}{(\cosh\xi - x)^{3/2}}dx = \frac{2}{2n+1}\left(n(n-1)I_1(n-1) - (n+1)(n+2)I_1(n+1)\right), \tag{4.13}$$

$$I_5(n) = \int_{-1}^{1}\frac{\left(1-x^2\right)P_n(x)}{(\cosh\xi - x)^{5/2}}dx = \frac{2}{3(2n+1)}\left(n(n-1)I_2(n-1) - (n+1)(n+2)I_2(n+1)\right), \tag{4.14}$$

$$I_6(n) = \int_{-1}^{1}\frac{xP_n(x)}{(\cosh\xi - x)^{1/2}}dx = \frac{(n+1)I_1(n+1) + nI_1(n-1)}{2n+1}, \tag{4.15}$$

$$I_7(n) = \int_{-1}^{1}\frac{xP_n(x)}{(\cosh\xi - x)^{3/2}}dx = \frac{(n+1)I_2(n+1) + nI_2(n-1)}{2n+1}, \tag{4.16}$$

$$I_8(n) = \int_{-1}^{1}\frac{xP_n(x)}{(\cosh\xi - x)^{5/2}}dx = \frac{(n+1)I_3(n+1) + nI_3(n-1)}{2n+1}, \tag{4.17}$$

$$I_9(n) = \int_{-1}^{1}\frac{\left(1-x^2\right)P'_n(x)}{(\cosh\xi - x)^{1/2}}dx = \frac{n(n+1)}{2n+1}\left(I_1(n-1) - I_1(n+1)\right), \tag{4.18}$$

$$I_{10}(n) = \int_{-1}^{1}\frac{\left(1-x^2\right)P'_n(x)}{(\cosh\xi - x)^{3/2}}dx = \frac{n(n+1)}{2n+1}\left(I_2(n-1) - I_2(n+1)\right), \tag{4.19}$$

$$I_{11}(n) = \int_{-1}^{1}\frac{\left(1-x^2\right)P'_n(x)}{(\cosh\xi - x)^{5/2}}dx = \frac{n(n+1)}{2n+1}\left(I_3(n-1) - I_3(n+1)\right). \tag{4.20}$$

Table 1: Values of the non-dimensional drag in different cases.

		Non-dimensional Drag	
		$k^{(1)} = k^{(2)}$ =0.01, 0.005	
Case-1	$\xi_1 = 1$	-108853	-112085
Spheres are of equal radius	$\xi_2 = -1$	-6956.82	-6355.21
Case-2	$\xi_1 = 2$	739.12	207.9
Spheres are of equal radius	$\xi_2 = -2$	-166.361	-105.075
Case-3	$\xi_1 = 1$	-1923.78	-2702.24
Spheres are of unequal radius	$\xi_2 = -2$	-213.48	-134.836
Case-4	$\xi_1 = 2$	-1.535	-0.4319
Spheres are of unequal radius	$\xi_2 = -1$	3.011	2.561

Using the above expressions, the non-dimensional drag (shown in expression (4.7)) on the sphere $\xi = \xi_1$ given by $D* = D/\mu a U$ simplifies to

$$D = 2\pi \sum_{n=0}^{\infty} \left\{ \begin{array}{l} -\left(L_{n+1} \cosh\left(n+1/2\right)\xi_1 + M_{n+1} \sinh\left(n+1/2\right)\xi_1\right)\left(I_3 - I_8 \cosh\xi\right) \\ -2k^{(1)} \left(\begin{array}{l} \left\{1/4(\sinh\xi_1)^2 \left(I_2 - I_8 \cosh\xi_1\right) + 1/2 \cosh\xi_1 \left(I_2 - I_7 \cosh\xi_1\right)\right. \\ \left. +(n+1/2)^2 \left(I_1 - I_6 \cosh\xi_1\right)\right\} (L_{n+1} \cosh\left(n+1/2\right)\xi_1 \\ +M_{n+1} \sinh\left(n+1/2\right)\xi_1) + 2 \sinh\xi_1 \left(I_2 - I_8 \cosh\xi_1\right)\left(n+1/2\right) \times \\ \left(L_{n+1} \sinh\left(n+1/2\right)\xi_1 + M_{n+1} \cosh\left(n+1/2\right)\xi_1\right) \end{array} \right) \\ +k^{(1)} \left\{ \begin{array}{l} \left(\dfrac{I_5}{4} - \dfrac{I_{10}}{2}\right) \sinh\xi_1 \left(L_{n+1} \cosh\left(n+1/2\right)\xi_1 \; +M_{n+1} \sinh\left(n+1/2\right)\xi_1\right) \\ + \left(\dfrac{3I_4}{2} - I_9\right)\left(n+1/2\right) (L_{n+1} \sinh\left(n+1/2\right)\xi_1 \\ +M_{n+1} \cosh\left(n+1/2\right)\xi_1) \end{array} \right\} \\ +k^{(1)} \left\{ \begin{array}{l} \left(-\dfrac{I_5}{4}\dfrac{3I_{10}}{2}\right) \sinh\xi_1 (L_{n+1} \cosh\left(n+1/2\right)\xi_1 \\ +M_{n+1} \sinh\left(n+1/2\right)\xi_1) + \left(-\dfrac{I_4}{2} + I_9\right)\left(n+1/2\right) \times \\ \left(L_{n+1} \sinh\left(n+1/2\right)\xi_1 + M_{n+1} \cosh\left(n+1/2\right)\xi_1\right) \end{array} \right\} \\ +k^{(1)} \left\{ \begin{array}{l} \dfrac{\sinh\xi_1}{2} I_5 \left(L_{n+1} \cosh\left(n+1/2\right)\xi_1 + M_{n+1} \sinh\left(n+1/2\right)\xi_1\right) \\ +I_4 \left(n+1/2\right) (L_{n+1} \sinh\left(n+1/2\right)\xi_1 \\ +M_{n+1} \cosh\left(n+1/2\right)\xi_1) \end{array} \right\} \end{array} \right\}. \tag{4.21}$$

It may be noted here that in the above expression all the integrals all evaluated for $\xi = \xi_1$. Similarly, the expression for the drag on the sphere $\xi = \xi_2$ can be derived. In the Table 1 we present the values of the non-dimensional drag computed in different cases.

Observations:

1. In case the two spheres are of equal radius, (Case 1 and Case 2 mentioned above in the Table 1),

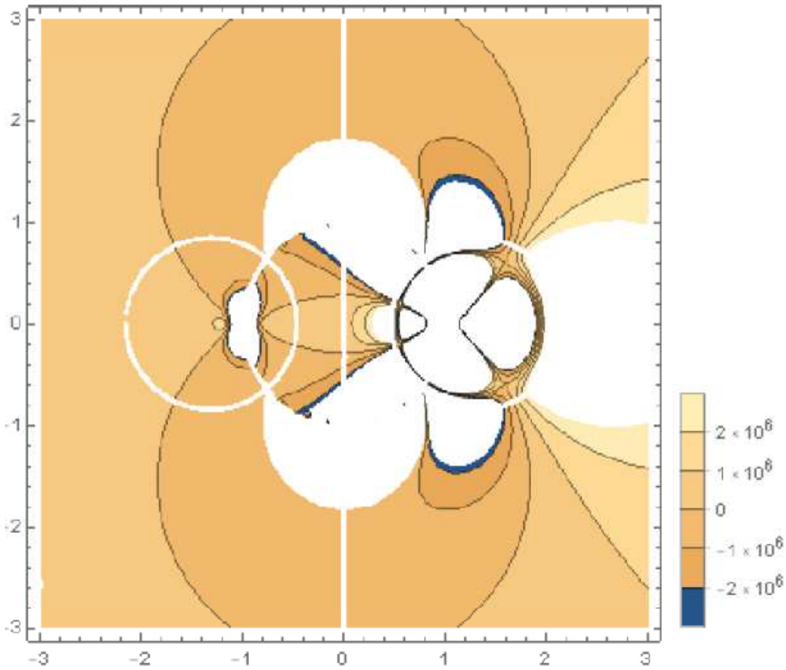

Fig. 2: Pressure contours in the xy plane when $\xi_1 = 1, \xi_2 = -1$.

the sphere placed to the negative x-axis ($\xi = \xi_2$) experiences lesser drag (in magnitude) than the one placed ($\xi = \xi_1$) on the positive x-axis.

2. When the radius of the sphere placed on the positive x-axis is small (Case 3 and Case 4 mentioned above in the Table 1), the drag experienced by the two spheres are of opposite signs indicating an attractive force between them. We find this result to be consistent with the observations made by Kotsev in his work [17] on the viscous flow around spherical particles in different arrangements.

5 Plots of the streamlines and the pressure function

Case(I): The two spheres are of equal radii:
The figures Fig. 2 and Fig. 3 – Fig. 6 respectively present the pressure contours and streamlines in the case of equal spheres with $\xi_1 = 1$,$\xi_2 = -1$. The figures Fig. 7 – Fig. 11 depict the same when $\xi_1 = 2$,$\xi_2 = -2$.
From Fig. 2 and Fig. 7 we observe that the pressure in the region in between the larger spheres is greater than that of smaller spheres. It could be because the drag experienced by the smaller spheres is lesser than the drag experienced by the larger spheres, as seen from Table 1 presented in section 4 above. Further, we see from (2.13) that the pressure function does not depend on the permeability of the porous region and hence the pressure contours are presented as functions of the radii of the spheres.
The figures Fig. 3 – Fig. 6 and Fig. 8 – Fig. 11 depict the streamline for different values of the permeability parameter in case of equal spheres. The figures Fig. 3, Fig. 4, Fig. 8 and Fig. 9 show the streamlines in the case where the two spheres are of different permeability. We see that the streamlines are dense near the sphere with greater permeability.
The figures Fig. 5, Fig. 6, Fig. 10 and Fig. 11 show the streamline pattern in the case when two spheres have equal permeability. Here again, the streamlines are dense in the case when both the spheres have greater permeability.

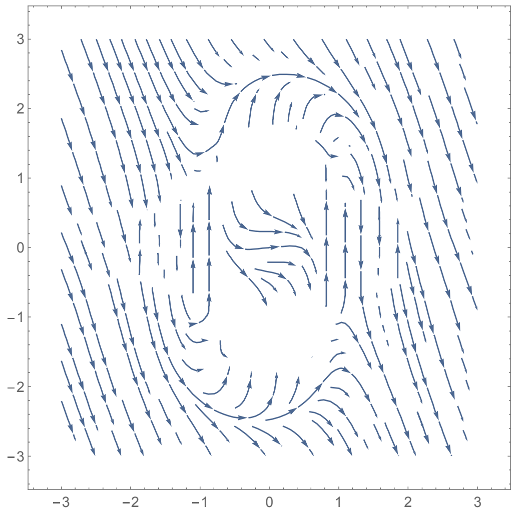

Fig. 3: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.005$.

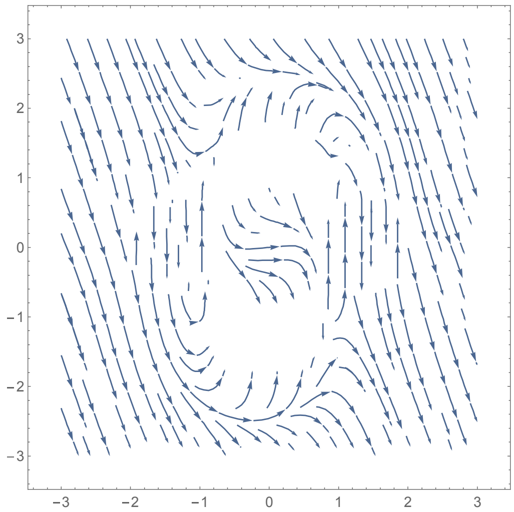

Fig. 4: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.01$.

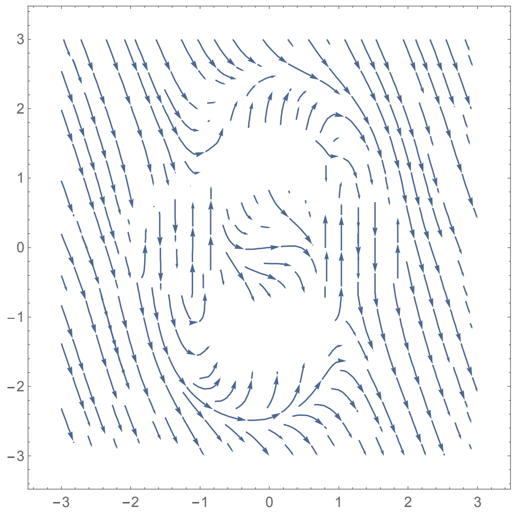

Fig. 5: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.005$.

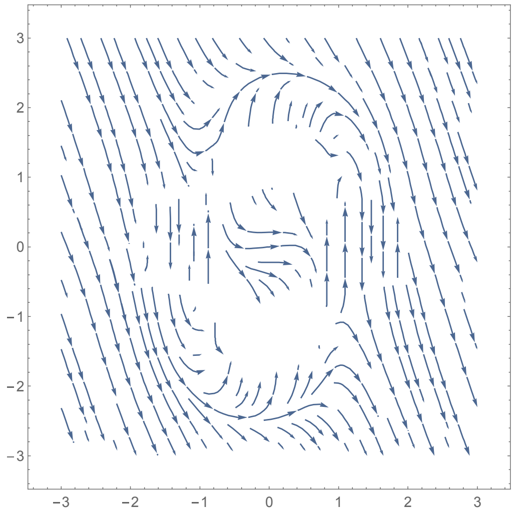

Fig. 6: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.01$.

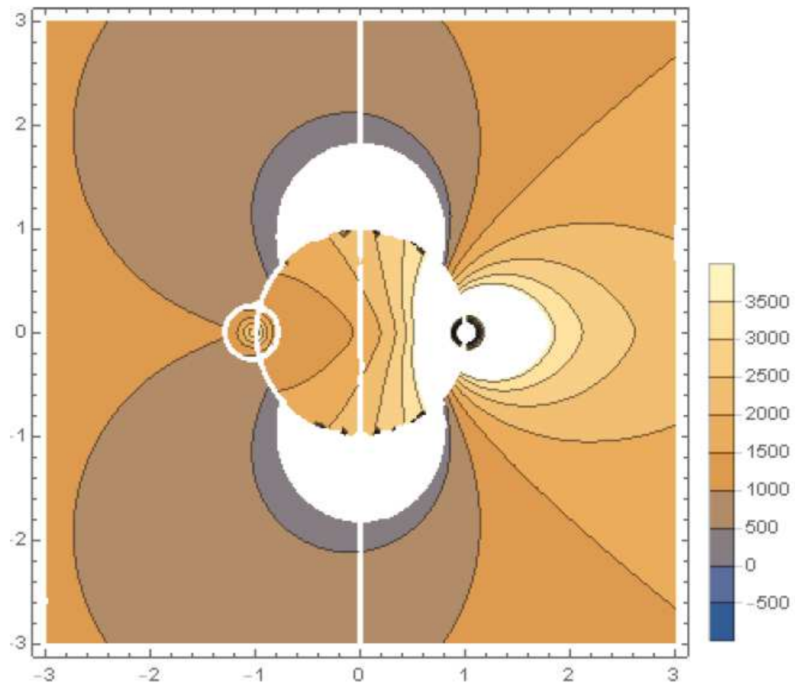

Fig. 7: Pressure contours in the xy plane when $\xi_1 = 2, \xi_2 = -2$.

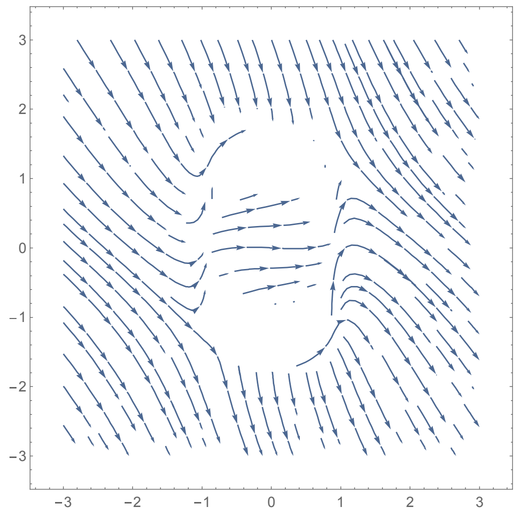

Fig. 8: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.005$.

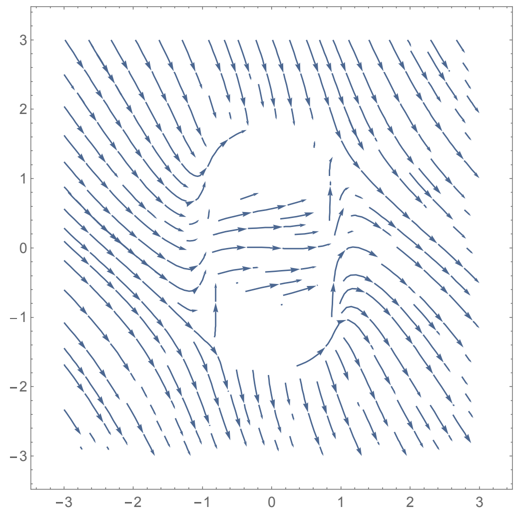

Fig. 9: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.01$.

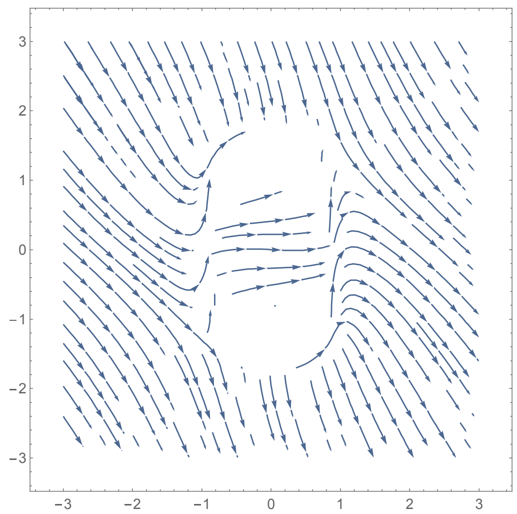

Fig. 10: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.005$.

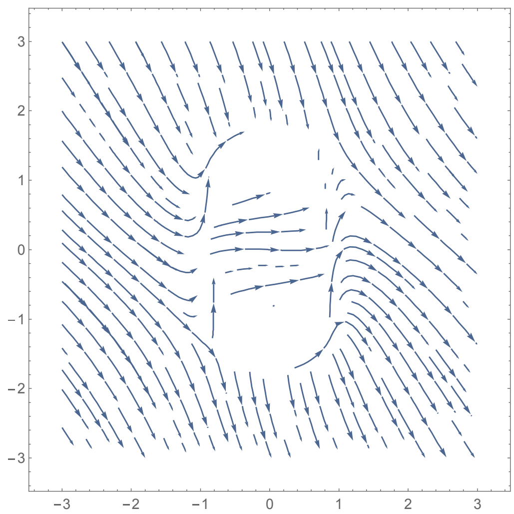

Fig. 11: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.01$.

Case(II): The spheres are of unequal radii:

The figures Fig. 12 and Fig. 13 – Fig. 16 depict respectively, the pressure contours and the streamlines in the case where the sphere to the right of the origin is larger than the one towards the other side of the origin.

Further, the figures Fig. 17 and Fig. 18 – Fig. 21 depict respectively, the pressure contours and the streamlines in the case where the sphere to the left of the origin is larger than the one towards the other side of the origin.

We see from the plots of the pressure contours in the figures Fig. 12 and Fig. 17 that the pressure in the region in between the two spheres is more when the smaller sphere is towards the right of the larger sphere. It is because of the lesser drag experienced by the smaller sphere.

6 Conclusion

In this work, we presented the analytical solution to the problem of Stokes flow of a viscous fluid past a pair of separated porous spheres. The flow within the porous spheres is taken to obey Darcy's law, and we derived analytical expressions for the drag experienced by each of the spheres and computed its values for some assumed values of the parameters. We also presented and discussed the pressure contours and streamlines in the cases when the two spheres are of equal radii and when they have unequal radii.

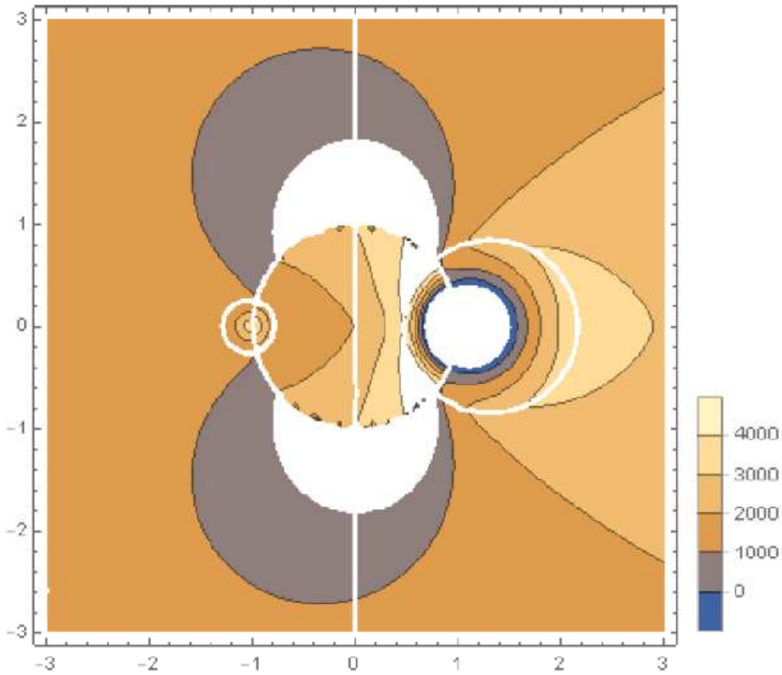

Fig. 12: Pressure contours in the xy plane when $\xi_1 = 1, \xi_2 = -2$.

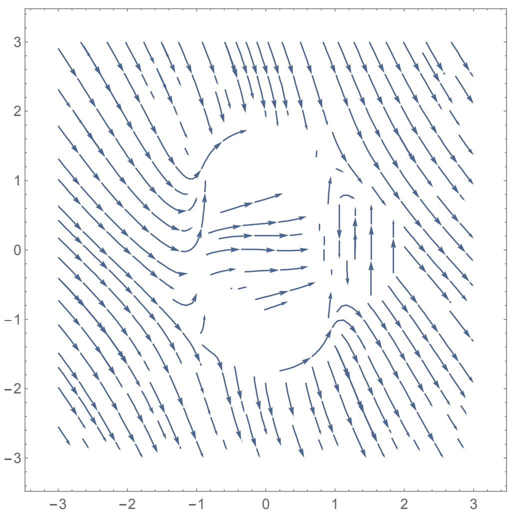

Fig. 13: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.005$.

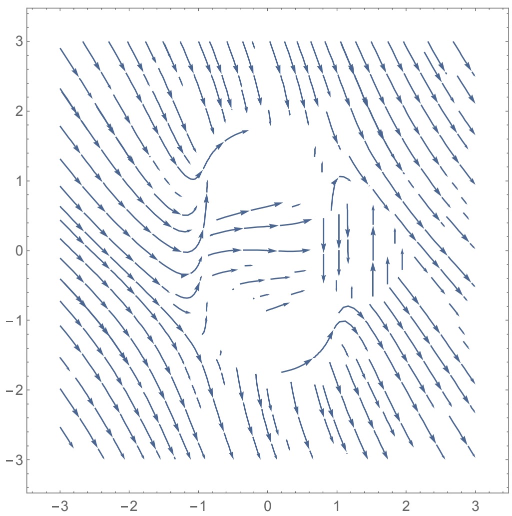

Fig. 14: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.01$.

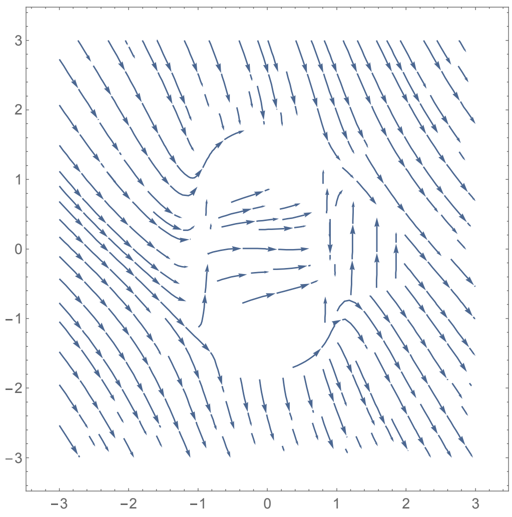

Fig. 15: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.005$.

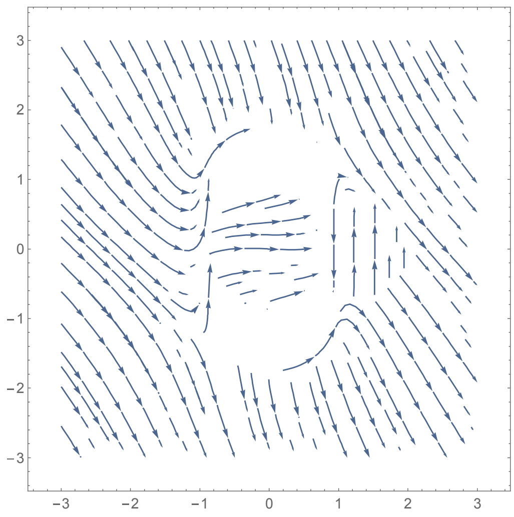

Fig. 16: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.01$.

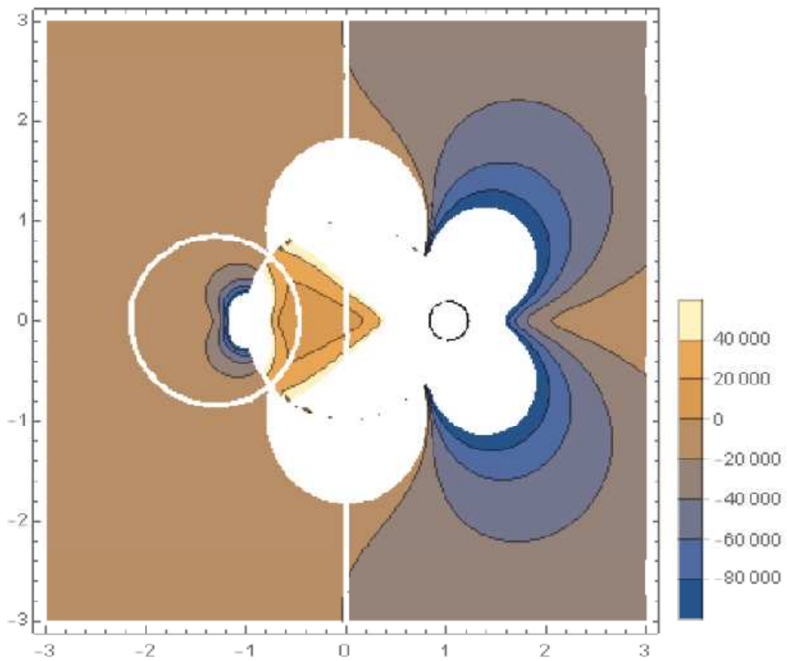

Fig. 17: Pressure contours in the xy plane when $\xi_1 = 2, \xi_2 = -1$.

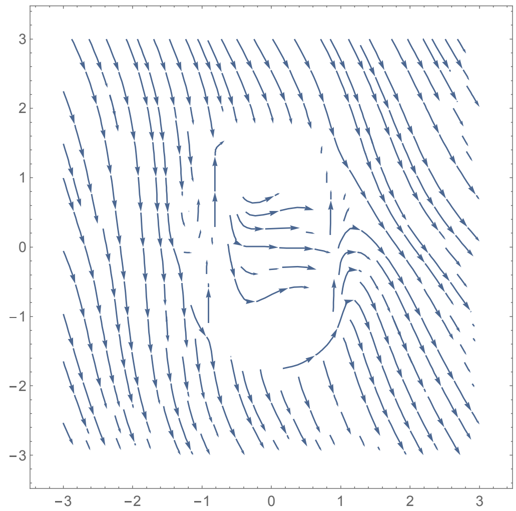

Fig. 18: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.005$.

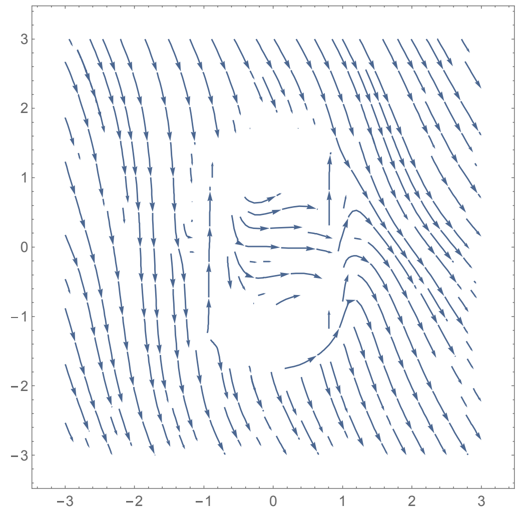

Fig. 19: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.01$.

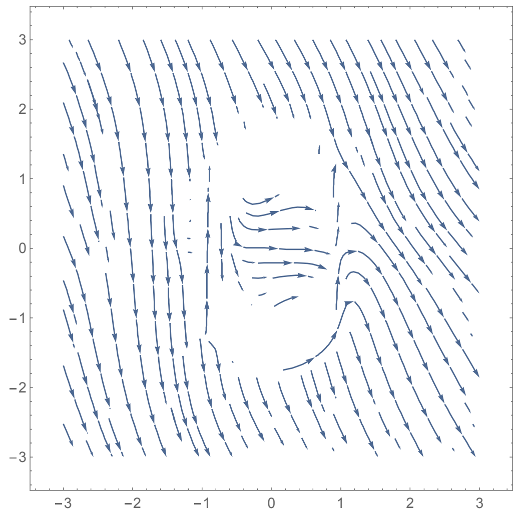

Fig. 20: Plot of streamlines for $k^{(1)} = 0.005, k^{(2)} = 0.005$.

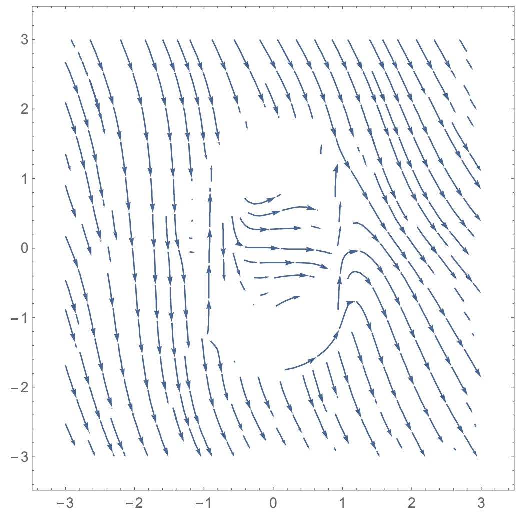

Fig. 21: Plot of streamlines for $k^{(1)} = 0.01, k^{(2)} = 0.01$.

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