Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

Bull. Pure Appl. Sci. Sect. E Math. Stat.
38E(2), 540–549 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2019.00055.9

Production inventory system for deteriorating items with trapezoidal type demand *

Mihir S. Suthar[1], Kunal T. Shukla[2] and Nikhilkumar D. Abhanghi[3]

1. PDPIAS, Charotar University of Science and Technology,
Anand, Gujarat, India-388421.
2. Vishwakarma Government Engineering College, Sabarmati Koba Highway,
Chandkheda, Ahmedabad, Gujarat, India-382424.
3. Government Engineering College, Rajkot, Gujarat, India-360005.
1. E-mail: mihirsuthar86@gmail.com , 2. E-mail: drkunalshukla.maths@gmail.com
3. E-mail: nikhil.abhanghi@gmail.com

Abstract For the items like trendy goods, mobile phones and such others, it is examined that the demand rate is of trapezoidal type. The aim of this study is to present optimal production policy for deteriorating items, when the demand of an item is trapezoidal type. Rate of deterioration is assumed to be constant and rate of production depends upon demand rate. Shortages are not allowed. Mathematical formulation is derived in order to minimize the total cost of an inventory system. An easy to use algorithm is presented to decide an optimal production policy.

Key words Inventory system, EPQ model, Trapezoidal type demand, Deterioration, Minimizing Total Cost.

2010 Mathematics Subject Classification 90B05.

1 Introduction

In recent competitive market, it is observed that for items like fancy/seasonal/trendy goods, the demand of an item increases with respect to time over a period of time. Thereafter, the item consumed by level demand for a short period, followed by a decrease in demand with time again. Resh et al. [19] and Donaldson [9] made first attempts to integrate the linear trend in demand for an inventory system. For the first time, Hill [15] considered ramp type demand pattern to develop an optimal ordering policy. In case of ramp type demand rate, the rate increases linearly with time and thereafter it stabilizes. Such demand pattern is observed in newly introduced consumable items in the market. Many researchers have studied models with ramp type demand. The inventory models with the ramp type demand are studied by Wu et al. [28, 29], Wu and Ouyang [27], Wu [26], Giri et al. [11], Deng [7], Chen et al. [2], Deng et al. [8], Cheng and Wang [3], He et al. [14], Skouri et al. [23]. Moreover, the goods in the inventory system always retain their physical quality is not true in common. In current business environment, effect of deterioration is expected to be incorporated in an inventory system. The process

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received January 13, 2019 / Revised May 22, 2019 / Accepted June 13, 2019. Online First Published on December 24, 2019 at https://www.bpasjournals.com/.
Corresponding author Mihir S. Suthar, E-mail: mihirsuthar86@gmail.com

which depletes the present value or usefulness of an item and does not allow their original use, due to degradation, spoilage, evaporation etc. is known as deterioration. First of all Ghare and Schrader [10] incorporated deterioration in an inventory system. Covert and Philip [5] used Weibull distribution to extend the idea of Ghare and Schrader [10]. Dave and Patel [6], Goyal [13], Raafat [18], Shah and Shah [21], Goyal and Giri [12], Manna and Chaudhri [17], Skouri et al. [24], Ruxin et al. [20] and Bakker et al. [1] cite an up to date review on deteriorating inventory system. Cheng and Wang [3] extended this idea from ramp type demand to trapezoidal type demand. Cheng et al [4] integrated shortages with partial backlogging and deterioration in an inventory system to extend the idea of Cheng and Wang [3]. Recent articles on trapezoidal type demand by Shukla and Suthar [22], Wu et. al. [30, 31], Vandana and Shrivastava [25]. Manna et al. [16] present production inventory system for deteriorating items having ramp type demand. The demand rate increases with time up to a certain point of time and then stabilizes at constant level. Models were formulated without shortages and with two assumptions that: a) the demand rate is stabilized after the production stops and before the time when inventory level reaches to zero and b) the deterioration is constant.

Our present work is an extension of the work of Manna et al. [16] with the assumption that the production process stops before the inventory level reaches to zero and in one of the three situations: 1) before the demand rate becomes constant, 2) when the demand rate is constant and 3) after the demand rate is constant. In this paper, section 1 introduces the article in brief. Assumptions and notations are presented in section 2. Section 3 deals with the mathematical formulation for each of the three cases along with the computational algorithm and the conclusions are described in section 4.

2 Assumptions and notations

To formulate the proposed inventory system mathematically, the following assumptions are made and the notations to be used by us in this paper are also explained below:

1. The inventory system deals with a single item. Rate of replenishment rate is assumed to be finite and lead time is considered to be zero or negligible. The length of planning horizon is infinite. Inventory system does not possess shortages.
2. The function $Q(t)$ represents level of an stock at any instant of time t during $[0, T]$, where T is length of ordering cycle.
3. The demand is assumed to be trapezoidal type, say $R(t)$, were $R(t) = \begin{cases} a_1 + b_1 t & ; \quad 0 \le t \le \lambda_1 \\ a_1 + b_1 \lambda_1 & ; \quad \lambda_1 \le t \le \lambda_2 \\ a_2 - b_2 t & ; \quad \lambda_2 \le t \le T \end{cases}$, where a_1, a_2, b_1, b_2 are scaling parameters for rate of demand. During $[0, \lambda_1]$ demand increases with respect to time, then it stabilizes during $[\lambda_1, \lambda_2]$ and thereafter it decreases as t increases during $[\lambda_2, T]$.
4. The level of stock deteriorates with a constant rate say θ $(0 < \theta < 1)$ during the ordering cycle $[0, T]$. Again, deteriorated stock is neither repaired nor replaced during $[0, T]$.
5. Inventory system is assumed as production inventory system, where initial stock level is assumed to be zero at time $t = 0$. Production process starts at $t = 0$ and continues up to time $t = t_1$. At time $t = t_1$, stock level attains its maximum say S.
6. The unit production cost of an item is defined as $C_p = \alpha_1 (R(t))^{-\gamma}$, where $\gamma > 0$ and $\gamma \neq 2$(Manna et. al [16]). $\alpha_1 > 0$ as $C_p > 0$ and $R(t)$ is non – negative. As demand increases unit production cost will decrease, which validates that C_p and $R(t)$ are in inverse proportion.
7. We consider C_h is the holding cost / unit; C_d is the cost due to deterioration / unit; $C(t_1, T)$ is an average cost of an inventory system.

3 Mathematical formulation and computational algorithm

As, $Q(t)$ represents level of stock at any instant of time t (say) during $[0, T]$. Initially level of stock is assumed to be zero at $t = 0$, at the same time production takes place and continues up to time $t = t_1$ with maximum stock level say S. Hence, during $[0, t_1]$

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\begin{cases}\left(\beta-1\right)R\left(t\right) & ; \quad 0\le t\le t_1\\ -R\left(t\right) & ; \quad t_1\le t\le T\end{cases}\tag{3.1}$$

Now depending upon the value of t_1 the following three different cases may arise:
Case 1: $0\le\lambda_1\le\lambda_2\le t_1\le T$, Case 2: $0\le\lambda_1\le t_1\le\lambda_2\le T$ and Case 3: $0\le t_1\le\lambda_1\le\lambda_2\le T$. Hence, we compute below separately for all the cases for the proposed production inventory system.

Case 1: $\boldsymbol{0\le\lambda_1\le\lambda_2\le t_1\le T}$

At time$t=0$, the production starts with zero stock level and stops at time $t=t_1$. Using (3.1)

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\begin{cases}\left(\beta-1\right)\left(a_1+b_1t\right) & ; \quad 0\le t\le\lambda_1\\ \left(\beta-1\right)\left(a_1+b_1\lambda_1\right) & ; \quad \lambda_1\le t\le\lambda_2\\ \left(\beta-1\right)\left(a_2-b_2t\right) & ; \quad \lambda_2\le t\le t_1\\ -\left(a_2-b_2t\right) & ; \quad t_1\le t\le T\end{cases}\tag{3.2}$$

Using initial condition $Q\left(0\right)=0$, we solve

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\left(\beta-1\right)\left(a_1+b_1t\right); \qquad 0\le t\le\lambda_1\tag{3.3}$$

Its solution is

$$Q\left(t\right)=\frac{\left(\beta-1\right)\left(e^{\theta t}b_1t\theta+e^{\theta t}a_1\theta-e^{\theta t}b_1-a_1\theta+b_1\right)e^{-\theta t}}{\theta^2}\tag{3.4}$$

Using continuity of $Q\left(t\right)$ at $t=\lambda_1$ we solve

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\left(\beta-1\right)\left(a_1+b_1\lambda_1\right); \qquad \lambda_1\le t\le\lambda_2\tag{3.5}$$

which results in

$$Q\left(t\right)=-\frac{\left(\begin{array}{l}e^{-\theta t}b_1\beta\lambda_1\theta-e^{\theta t}a_1\beta\theta+e^{\theta t}b_1\lambda_1\theta+e^{\theta\lambda_1}b_1\beta+e^{\theta t}a_1\theta\\ +a_1\beta\theta-e^{\theta\lambda_1}b_1-a_1\theta-b_1\beta+b_1\end{array}\right)e^{-\theta t}}{\theta^2}\tag{3.6}$$

Similarly, using continuity of $Q\left(t\right)$ at $t=\lambda_2$ we solve,

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\left(\beta-1\right)\left(a_2-b_2t\right); \qquad \lambda_2\le t\le t_1\tag{3.7}$$

which gives

$$Q\left(t\right)=\frac{1}{\theta^2}\left(e^{-\theta t}\left(\begin{array}{l}e^{\theta\lambda_2}b_1\beta\lambda_1\theta+e^{\theta\lambda_2}b_2\beta\lambda_2\theta-e^{\theta t}b_2\beta t\theta+e^{\theta\lambda_2}a_1\beta\theta-e^{\theta\lambda_2}a_2\beta\theta\\ -e^{\theta\lambda_2}b_1\lambda_1\theta-e^{\theta\lambda_2}b_2\lambda_2\theta+e^{\theta t}a_2\beta\theta+e^{\theta t}b_2t\theta-e^{\theta\lambda_2}a_1\theta+e^{\theta\lambda_2}a_2\theta\\ -e^{\theta\lambda_2}b_2\beta-e^{\theta t}a_2\theta+e^{\theta t}b_2\beta-e^{\theta\lambda_1}b_1\beta-a_1\beta\theta+e^{\theta\lambda_2}b_2-e^{\theta t}b_2\\ +e^{\theta\lambda_1}b_1+a_1\theta+b_1\beta-b_1\end{array}\right)\right)\tag{3.8}$$

Moreover, at time $t=t_1$, level of stock is maximum, using $Q\left(t_1\right)=S$ we solve,

$$\frac{dQ}{dt}+\theta Q\left(t\right)=-\left(a_2-b_2t\right); \qquad t_1\le t\le T\tag{3.9}$$

which yields

$$Q\left(t\right)=\frac{1}{\theta^2}\left(\begin{array}{l}e^{-\theta\left(t-t_1\right)}S\theta^2-e^{-\theta\left(t-t_1\right)}b_2t_1\theta+e^{-\theta\left(t-t_1\right)}a_2\theta\\ +b_2t\theta+e^{-\theta\left(t-t_1\right)}b_2-a_2\theta-b_2\end{array}\right)\tag{3.10}$$

Using the boundary condition $Q\left(T\right)=0$ in (3.10), we evaluate the maximum level of stock as under:

$$S=-\frac{1}{\theta^2}\left(b_2t_1\theta+a_2\theta+b_2T\theta e^{\theta\left(T-t_1\right)}+b_2-\left(a_2\theta+b_2\right)e^{\theta\left(T-t_1\right)}\right)\tag{3.11}$$

By substituting S in (3.10), we have $Q\left(t\right)$ as under, during $\left[t_1,T\right]$

$$Q\left(t\right)=-\frac{1}{\theta^2}\left(e^{\theta\left(T-t\right)}Tb_2\theta-e^{\theta\left(T-t\right)}a_2\theta-b_2t\theta-e^{\theta\left(T-t\right)}b_2+a_2\theta+b_2\right)\tag{3.12}$$

Using (3.4), (3.6), (3.8) and (3.12), the total inventory during $[0, T]$ is ,

$$TI_1 = \int_0^T Q(t)\ dt = \int_0^{\lambda_1} Q(t)\ dt + \int_{\lambda_1}^{\lambda_2} Q(t)\ dt + \int_{\lambda_2}^{t_1} Q(t)\ dt + \int_{t_1}^{T} Q(t)\ dt \tag{3.13}$$

where,

$$\int_0^{\lambda_1} Q(t)\ dt = \frac{1}{2\theta^3}\left(\begin{array}{l} \left(\begin{array}{l} b_1\beta\lambda_1^2\theta^2 + 2a_1\beta\lambda_1\theta^2 - b_1\lambda_1^2\theta^2 - 2a_1\theta^2\lambda_1 \\ -2b_1\beta\lambda_1\theta - 2a_1\beta\theta + 2b_1\lambda_1\theta + 2a_1\theta + 2b_1\beta - 2b_1 \end{array}\right) \\ +e^{-\theta\lambda_1}\left(2a_1\beta\theta - 2a_1\theta - 2\beta b_1 + 2b_1\right) \end{array}\right)$$

$$\int_{\lambda_1}^{\lambda_2} Q(t)\ dt =$$

$$-\frac{1}{\theta^3}\begin{pmatrix} b_1\beta{\lambda_1}^2\theta^2 - b_1\beta\lambda_1\lambda_2\theta^2 + a_1\beta\lambda_1\theta^2 - a_1\beta\lambda_2\theta^2 - b_1{\lambda_1}^2\theta^2 + b_1\lambda_1\lambda_2\theta^2 \\ -a_1\lambda_1\theta^2 + a_1\lambda_2\theta^2 + b_1\beta + a_1\beta\theta e^{-\theta\lambda_1} - b_1\beta e^{\theta(\lambda_1-\lambda_2)} - e^{-\theta\lambda_2}a_1\beta\theta - b_1 \end{pmatrix}$$
$$-\frac{1}{\theta^3}\left(-a_1\theta e^{-\theta\lambda_1} - b_1\beta e^{-\theta\lambda_1} + b_1 e^{\theta(\lambda_1-\lambda_2)} + e^{-\theta\lambda_2}a_1\theta + e^{-\theta\lambda_2}b_1\beta + e^{-\theta\lambda_1}b_1 - e^{-\theta\lambda_2 b_1}\right)$$

$$\int_{\lambda_2}^{t_1} Q(t)\ dt =$$

$$-\frac{1}{2\theta^3}\begin{pmatrix} -2b_2 + 2a_2\beta\lambda_2\theta^2 - 2a_2\beta t_1\theta^2 - 2b_2\beta t_1\theta - b_2\beta{\lambda_2}^2\theta^2 + b_2\beta{t_1}^2\theta^2 \\ -2b_1\beta\lambda_1\theta + 2b_2\beta + 2a_1\theta - 2a_2\theta - 2a_1\beta\theta + 2a_2\beta\theta + 2b_1\lambda_1\theta - 2a_2\lambda_2\theta^2 \\ +2a_2t_1\theta^2 + 2b_2t_1\theta + b_2{\lambda_2}^2\theta^2 - b_2{t_1}^2\theta^2 \end{pmatrix}$$
$$-\frac{1}{2\theta^3}e^{-\theta(\lambda_2+t_1)}\begin{pmatrix} 2e^{\theta t_1}b_1 - 2e^{\theta(\lambda_1+t_1)}b_1 + 2e^{\theta\lambda_2}a_1\theta - 2e^{\theta t_1}b_1\beta + 2e^{\theta(\lambda_1+\lambda_2)}b_1 - 2e^{\theta\lambda_2}a_1\beta\theta \\ -2e^{\theta t_1}a_1\theta + 2e^{\theta t_1}a_1\beta\theta - 2e^{2\theta\lambda_2}a_2\beta\theta + 2e^{2\theta\lambda_2}a_1\beta\theta - 2e^{2\theta\lambda_2}b_2\lambda_2\theta \\ -2e^{2\theta\lambda_2}b_1\lambda_1\theta + 2e^{2\theta\lambda_2}b_2 - 2e^{2\theta\lambda_2}a_1\theta + 2e^{2\theta\lambda_2}a_2\theta - 2e^{2\theta\lambda_2}b_2\beta \\ -2e^{\theta(\lambda_1+\lambda_2)}b_1\beta + 2e^{2\theta\lambda_2}b_1\beta\lambda_1\theta + 2e^{2\theta\lambda_2}b_2\beta\lambda_2\theta - 2e^{2\theta\lambda_2}b_1 + 2e^{2\theta\lambda_2}b_1\beta \end{pmatrix}$$

$$\int_{t_1}^{T} Q(t)\ dt = \frac{1}{2\theta^3}\left(\begin{array}{l} T^2b_2\theta^2 - b_2t_1^2\theta^2 - 2Ta_2\theta^2 + 2a_2\theta^2t_1 + 2b_2t_1\theta - 2e^{\theta(T-t_1)}Tb_2\theta \\ -2a_2\theta + 2e^{\theta(T-t_1)}a_2\theta - 2b_2 + 2e^{\theta(T-t_1)}b_2 \end{array}\right)$$

Now, total number of deteriorated units during $[0, T]$ is given by,
DI_1 =Production in $[0, \lambda_1]$+ Production in $[\lambda_1, \lambda_2]$ + Production in $[\lambda_2, t_1]$ – Demand in $[0, \lambda_1]$ – Demand in $[\lambda_1, \lambda_2]$ – Demand in $[\lambda_2, T]$

$$\begin{aligned} &= \beta\textstyle\int_0^{\lambda_1}(a_1+b_1t)\ dt + \beta\int_{\lambda_1}^{\lambda_2}(a_1+b_1\lambda_1)\ dt + \beta\int_{\lambda_2}^{t_1}(a_2-b_2t)\ dt \\ &\quad -\textstyle\int_0^{\lambda_1}(a_1+b_1t)\ dt - \int_{\lambda_1}^{\lambda_2}(a_1+b_1\lambda_1)\ dt - \int_{\lambda_2}^{T}(a_2-b_2t)\ dt \end{aligned}$$

$$\begin{aligned} &= -\tfrac{1}{2}\beta b_1\lambda_1^2 + \beta b_1\lambda_1\lambda_2 + \beta a_1\lambda_2 + \tfrac{1}{2}\beta b_2\lambda_2^2 - \tfrac{1}{2}\beta b_2t_1^2 - \beta a_2\lambda_2 \\ &+\beta a_2t_1 + \tfrac{1}{2}b_1\lambda_1^2 - b_1\lambda_1\lambda_2 - a_1\lambda_2 + \tfrac{1}{2}T^2b_2 - \tfrac{1}{2}b_2\lambda_2^2 - a_2T + a_2\lambda_2 \end{aligned} \tag{3.14}$$

Cost of production during $[0, t_1]$ is given by,

$$PC_1 = \int_0^{t_1} \beta\alpha_1\left(R(t)\right)^{(1-\gamma)}du$$

$$= \int_0^{\lambda_1}\beta\alpha_1\left(a_1+b_1u\right)^{(1-\gamma)}du + \int_{\lambda_1}^{\lambda_2}\beta\alpha_1\left(a_1+b_1\lambda_1\right)^{(1-\gamma)}du + \int_{\lambda_2}^{t_1}\beta\alpha_1\left(a_2-b_2u\right)^{(1-\gamma)}du$$

$$= \beta\alpha_1\left(\frac{-a_1^\gamma + (b_1\lambda_1+a_1)^\gamma}{b_1\gamma} + (b_1\lambda_1+a_1)^{\gamma-1}(\lambda_2-\lambda_1) + \frac{(-b_2\lambda_2+a_2)^\gamma - (a_2-Tb_2)^\gamma}{b_2\gamma}\right) \tag{3.15}$$

Case 2: $\boldsymbol{0 \le \lambda_1 \le t_1 \le \lambda_2 \le T}$

In this case, from (3.1),

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\begin{cases}\left(\beta-1\right)\left(a_1+b_1t\right) & ; \quad 0\le t\le\lambda_1\\ \left(\beta-1\right)\left(a_1+b_1\lambda_1\right) & ; \quad \lambda_1\le t\le t_1\\ -\left(a_1+b_1\lambda_1\right) & ; \quad t_1\le t\le\lambda_2\\ -\left(a_2-b_2t\right) & ; \quad \lambda_2\le t\le T\end{cases} \tag{3.16}$$

We solve differential equation given below, using initial condition $Q\left(0\right)=0$,

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\left(\beta-1\right)\left(a_1+b_1t\right); \qquad 0\le t\le\lambda_1 \tag{3.17}$$

whose solution is

$$Q\left(t\right)=\frac{\left(\beta-1\right)}{\theta^2}\left(b_1t\theta+a_1\theta-b_1-\left(a_1\theta-b_1\right)e^{-\theta t}\right) \tag{3.18}$$

We use continuity of $Q\left(t\right)$ at $t=\lambda_1$ to solve the differential equation,

$$\frac{dQ}{dt}+\theta Q\left(t\right)=\left(\beta-1\right)\left(a_1+b_1\lambda_1\right); \qquad \lambda_1\le t\le t_1 \tag{3.19}$$

which gives

$$Q\left(t\right)=-\frac{1}{\theta^2}\left(\begin{array}{l}b_1\beta\lambda_1\theta-a_1\beta\theta+b_1\lambda_1\theta+b_1\beta e^{\theta(\lambda_1-t)}+a_1\theta+a_1\beta\theta e^{-\theta t}\\ -b_1e^{\theta(\lambda_1-t)}-\left(a_1\theta+b_1\beta-b_1\right)e^{-\theta t}\end{array}\right) \tag{3.20}$$

Now at $t=t_1$ the production process stops and the maximum level of stock is S, i.e. $Q\left(t_1\right)=S$. We solve,

$$\frac{dQ}{dt}+\theta Q\left(t\right)=-\left(a_1+b_1\lambda_1\right); \qquad t_1\le t\le\lambda_2 \tag{3.21}$$

for which we have

$$Q\left(t\right)=\frac{1}{\theta}\left(e^{-\theta(t-t_1)}S\theta+e^{-\theta(t-t_1)}b_1\lambda_1+e^{-\theta(t-t_1)}a_1-b_1\lambda_1-a_1\right) \tag{3.22}$$

Using continuity of $Q\left(t\right)$ at $t=\lambda_2$, we solve the differential equation,

$$\frac{dQ}{dt}+\theta Q\left(t\right)=-\left(a_2-b_2t\right); \qquad \lambda_2\le t\le T \tag{3.23}$$

which yields

$$Q\left(t\right)=-\frac{1}{\theta^2}\left(\begin{array}{l}e^{\theta(\lambda_2-t)}b_1\lambda_1\theta+e^{\theta(\lambda_2-t)}b_2\lambda_2\theta-e^{\theta(t_1-t)}S\theta^2-e^{\theta(t_1-t)}b_1\lambda_1\theta\\ +e^{\theta(\lambda_2-t)}a_1\theta-e^{\theta(\lambda_2-t)}a_2\theta-e^{\theta(t_1-t)}a_1\theta-b_2t\theta-e^{\theta(\lambda_2-t)}b_2\\ +a_2\theta+b_2\end{array}\right) \tag{3.24}$$

Using the boundary condition $Q\left(T\right)=0$ in (3.24), we evaluate the maximum level of stock as under:

$$S=\frac{1}{\theta^2e^{-\theta(T-t_1)}}\left(\begin{array}{l}b_1\lambda_1\theta e^{-\theta(T-\lambda_2)}+b_2\lambda_2\theta e^{-\theta(T-\lambda_2)}-b_1\lambda_1\theta e^{-\theta(T-t_1)}+a_1\theta e^{-\theta(T-\lambda_2)}\\ -a_2\theta e^{-\theta(T-\lambda_2)}-a_1\theta e^{-\theta(T-t_1)}-b_2T\theta-b_2e^{-\theta(T-\lambda_2)}+a_2\theta+b_2\end{array}\right) \tag{3.25}$$

By substituting S in (3.22) and (3.24), we have $Q\left(t\right)$ as under,

$$Q\left(t\right)=-\frac{1}{\theta^2}\left(\begin{array}{l}e^{\theta(T-t)}Tb_2\theta-e^{\theta(\lambda_2-t)}b_1\lambda_1\theta-e^{\theta(\lambda_2-t)}b_2\lambda_2\theta-e^{\theta(T-t)}a_2\theta\\ -e^{\theta(\lambda_2-t)}a_1\theta+e^{\theta(\lambda_2-t)}a_2\theta+b_1\lambda_1\theta-e^{\theta(T-t)}b_2+e^{\theta(\lambda_2-t)}b_2+a_1\theta\end{array}\right) \tag{3.26}$$

$$Q\left(t\right)=-\frac{1}{\theta^2}\left(e^{\theta(T-t)}Tb_2\theta-e^{\theta(T-t)}a_2\theta-b_2t\theta-e^{\theta(T-t)}b_2+a_2\theta+b_2\right) \tag{3.27}$$

Using (3.18), (3.20), (3.26) and (3.27),
The total inventory during $[0,T]$ is ,

$$TI_2=\int_0^T Q\left(t\right)\ dt=\int_0^{\lambda_1} Q\left(t\right)\ dt+\int_{\lambda_1}^{t_1} Q\left(t\right)\ dt+\int_{t_1}^{\lambda_2} Q\left(t\right)\ dt+\int_{\lambda_2}^{T} Q\left(t\right)\ dt \tag{3.28}$$

where,

$$\int_0^{\lambda_1} Q(t)\ dt = \frac{1}{2\theta^3}\left(\begin{array}{l} b_1\beta\lambda_1^2\theta^2 + 2a_1\beta\lambda_1\theta^2 - b_1\lambda_1^2\theta^2 - 2a_1\lambda_1\theta^2 - 2b_1\beta\lambda_1\theta \\ -2a_1\beta\theta + 2b_1\lambda_1\theta + 2a_1\theta + 2b_1\beta + 2a_1\beta\theta e^{-\theta\lambda_1} - 2b_1 - \\ 2a_1\theta e^{-\theta\lambda_1} - 2b_1\beta e^{-\theta\lambda_1} + 2b_1 e^{-\theta\lambda_1} \end{array}\right)$$

$$\int_{\lambda_1}^{t_1} Q(t)\ dt = -\frac{1}{\theta^3}\left(\begin{array}{l} b_1\beta\lambda_1^2\theta^2 - b_1\beta\lambda_1 t_1\theta^2 + a_1\beta\lambda_1\theta^2 - a_1\beta t_1\theta^2 - b_1\lambda_1^2\theta^2 \\ +b_1\lambda_1 t_1\theta^2 - a_1\lambda_1\theta^2 + a_1 t_1\theta^2 + b_1\beta + a_1\beta\theta e^{-\theta\lambda_1} - b_1\beta e^{\theta(\lambda_1-t_1)} \\ -a_1\beta\theta e^{-\theta t_1} - b_1 - a_1\theta e^{-\theta\lambda_1} - b_1\beta e^{-\theta\lambda_1} + b_1 e^{\theta(\lambda_1-t_1)} + a_1\theta e^{-\theta t_1} \\ +b_1\beta e^{-\theta t_1} + b_1 e^{-\theta\lambda_1} - b_1 e^{-\theta t_1} \end{array}\right)$$

$$\int_{t_1}^{\lambda_2} Q(t)\ dt = -\frac{1}{\theta^3}\left(\begin{array}{l} b_1\lambda_1\lambda_2\theta^2 - b_1\lambda_1 t_1\theta^2 + a_1\lambda_2\theta^2 - a_1 t_1\theta^2 + b_1\lambda_1\theta + b_2\lambda_2\theta - b_1\lambda_1\theta e^{\theta(\lambda_2-t_1)} \\ -b_2\lambda_2\theta e^{\theta(\lambda_2-t_1)} + Tb_2\theta e^{\theta(T-t_1)} - Tb_2\theta e^{\theta(T-\lambda_2)} + a_1\theta - a_2\theta - a_1\theta e^{\theta(\lambda_2-t_1)} \\ +a_2\theta e^{\theta(\lambda_2-t_1)} - a_2\theta e^{\theta(T-t_1)} + a_2\theta e^{\theta(T-\lambda_2)} - b_2 + b_2 e^{\theta(\lambda_2-t_1)} \\ -b_2 e^{\theta(T-t_1)} + b_2 e^{\theta(T-\lambda_2)} \end{array}\right)$$

$$\int_{\lambda_2}^{T} Q(t)\ dt = -\frac{1}{2\theta^3}\left(\begin{array}{l} -T^2 b_2\theta^2 + \theta^2 b_2\lambda_2^2 + 2Ta_2\theta^2 - 2\theta^2 a_2\lambda_2 + 2Tb_2\theta e^{\theta(T-\lambda_2)} \\ -2b_2\lambda_2\theta - 2a_2\theta e^{\theta(T-\lambda_2)} + 2a_2\theta - 2b_2 e^{\theta(T-\lambda_2)} + 2b_2 \end{array}\right)$$

Now, total number of deteriorated units during $[0,T]$ is given by,
DI_2 =Production in $[0,\lambda_1]$+ Production in $[\lambda_1,t_1]$- Demand in $[0,\lambda_1]$ – Demand in $[\lambda_1,\lambda_2]$ – Demand in $[\lambda_2,T]$

$$\begin{array}{l} = \beta\int_0^{\lambda_1}(a_1+b_1t)\ dt + \beta\int_{\lambda_1}^{t_1}(a_1+b_1\lambda_1)\ dt \\ -\int_0^{\lambda_1}(a_1+b_1t)\ dt - \int_{\lambda_1}^{\lambda_2}(a_1+b_1\lambda_1)\ dt - \int_{\lambda_2}^{T}(a_2-b_2t)\ dt \end{array}$$

$$= -\frac{1}{2}b_1\beta\lambda_1^2 + b_1\beta\lambda_1 t_1 + \beta t_1 a_1 + \frac{1}{2}b_1\lambda_1^2 - b_1\lambda_1\lambda_2 - \lambda_2 a_1 + \frac{1}{2}T^2 b_2 - \frac{1}{2}b_2\lambda_2^2 - a_2T + a_2\lambda_2 \tag{3.29}$$

Cost of production during $[0,t_1]$ is given by,

$$PC_2 = \int_0^{t_1}\beta\alpha_1(R(t))^{(1-\gamma)}du$$

$$= \int_0^{\lambda_1}\beta\alpha_1(a_1+b_1u)^{(1-\gamma)}du + \int_{\lambda_1}^{t_1}\beta\alpha_1(a_1+b_1\lambda_1)^{(1-\gamma)}du$$

$$= \beta\alpha_1\left(\frac{-a_1^{\gamma}+(b_1\lambda_1+a_1)^{\gamma}}{b_1\gamma} + (b_1\lambda_1+a_1)^{\gamma-1}(t_1-\lambda_1)\right) \tag{3.30}$$

Case 3: $\boldsymbol{0 \le t_1 \le \lambda_1 \le \lambda_2 \le T}$
For this case, using (3.1),

$$\frac{dQ}{dt} + \theta Q(t) = \left\{\begin{array}{ccc} (\beta-1)(a_1+b_1t) & ; & 0 \le t \le t_1 \\ -(a_1+b_1t) & ; & t_1 \le t \le \lambda_1 \\ -(a_1+b_1\lambda_1) & ; & \lambda_1 \le t \le \lambda_2 \\ -(a_2-b_2t) & ; & \lambda_2 \le t \le T \end{array}\right. \tag{3.31}$$

To solve (3.31), we use the initial condition $Q(0)=0$,

$$\frac{dQ}{dt} + \theta Q(t) = (\beta-1)(a_1+b_1t); \qquad 0 \le t \le t_1 \tag{3.32}$$

The solution is

$$Q(t) = \frac{1}{\theta^2}(\beta-1)\left(b_1t\theta + a_1\theta - b_1 - (a_1\theta-b_1)e^{-\theta t}\right) \tag{3.33}$$

At time $t=t_1$ the production stops and the level of stock is at its maximum value S, i.e. $Q(t_1)=S$ and we solve,

$$\frac{dQ}{dt} + \theta Q(t) = -(a_1+b_1t); \qquad t_1 \le t \le \lambda_1 \tag{3.34}$$

which results in

$$Q(t) = \frac{1}{\theta^2}\left(\begin{array}{l} e^{-\theta(t-t_1)}S\theta^2 + e^{-\theta(t-t_1)}b_1t_1\theta + a_1\theta e^{-\theta(t-t_1)} - b_1t\theta \\ -b_1e^{-\theta(t-t_1)} - a_1\theta - b_1 \end{array}\right) \tag{3.35}$$

Now, using continuity of the function $Q(t)$ at $t = \lambda_1$ we solve,

$$\frac{dQ}{dt} + \theta Q(t) = -(a_1 + b_1\lambda_1); \qquad \lambda_1 \le t \le \lambda_2 \tag{3.36}$$

whose solution is

$$Q(t) = \frac{1}{\theta^2}\begin{pmatrix} e^{-\theta(t-t_1)}S\theta^2 + b_1t_1\theta e^{-\theta(t-t_1)} + a_1\theta e^{-\theta(t-t_1)} - b_1e^{-\theta(t-t_1)} \\ -b_1\lambda_1\theta + b_1e^{-\theta(t-\lambda_1)} - a_1\theta \end{pmatrix} \tag{3.37}$$

Again, using the continuity of $Q(t)$ at $t = \lambda_2$, we solve the differential equation,

$$\frac{dQ}{dt} + \theta Q(t) = -(a_2 - b_2t); \qquad \lambda_2 \le t \le T \tag{3.38}$$

which gives

$$Q(t) = -\frac{1}{\theta^2}\begin{pmatrix} e^{\theta(\lambda_2-t)}b_1\lambda_1\theta + e^{\theta(\lambda_2-t)}b_2\lambda_2\theta - e^{\theta(t_1-t)}S\theta^2 - e^{\theta(t_1-t)}b_1t_1\theta \\ +e^{\theta(\lambda_2-t)}a_1\theta - e^{\theta(\lambda_2-t)}a_2\theta - e^{\theta(t_1-t)}a_1\theta - b_2t\theta - e^{\theta(\lambda_2-t)}b_2 \\ +e^{\theta(t_1-t)}b_1 - e^{\theta(\lambda_1-t)}b_1 + a_2\theta + b_2 \end{pmatrix} \tag{3.39}$$

As similar to other cases, we use the boundary condition $Q(T) = 0$ in (3.39), the maximum level of stock is as below:

$$S = -\frac{1}{\theta^2e^{-\theta(T-t_1)}}\begin{pmatrix} -b_1\lambda_1\theta e^{-\theta(T-\lambda_2)} - b_2\lambda_2\theta e^{-\theta(T-\lambda_2)} + b_1t_1\theta e^{-\theta(T-t_1)} - a_1\theta e^{-\theta(T-\lambda_2)} \\ +a_2\theta e^{-\theta(T-\lambda_2)} + a_1\theta e^{-\theta(T-t_1)} + b_2T\theta + b_2e^{-\theta(T-\lambda_2)} - b_1e^{-\theta(T-t_1)} \\ +b_1e^{-\theta(T-\lambda_1)} - a_2\theta - b_2 \end{pmatrix} \tag{3.40}$$

By substituting S in (3.35), (3.37) and (3.39), we have $Q(t)$ as under,

$$Q(t) = \frac{1}{\theta^2}\begin{pmatrix} b_1\lambda_1\theta e^{\theta(\lambda_2-t)} + b_2\lambda_2\theta e^{\theta(\lambda_2-t)} + a_1\theta e^{\theta(\lambda_2-t)} - a_2\theta e^{\theta(\lambda_2-t)} \\ -Tb_2\theta e^{\theta(T-t)} - b_2e^{\theta(\lambda_2-t)} - b_1e^{\theta(\lambda_1-t)} + a_2\theta e^{\theta(T-t)} + b_2e^{\theta(T-t)} \\ -b_1t\theta - a_1\theta + b_1 \end{pmatrix} \tag{3.41}$$

$$Q(t) = \frac{1}{\theta^2}\begin{pmatrix} b_1\lambda_1\theta e^{\theta(\lambda_2-t)} + b_2\lambda_2\theta e^{\theta(\lambda_2-t)} + a_1\theta e^{\theta(\lambda_2-t)} - a_2\theta e^{\theta(\lambda_2-t)} \\ -Tb_2\theta e^{\theta(T-t)} - b_2e^{\theta(\lambda_2-t)} + a_2\theta e^{\theta(T-t)} + b_2e^{\theta(T-t)} - b_1\lambda_1\theta - a_1\theta \end{pmatrix} \tag{3.42}$$

$$Q(t) = -\frac{1}{\theta^2}\left(Tb_2\theta e^{\theta(T-t)} - a_2\theta e^{\theta(T-t)} - b_2t\theta - b_2e^{\theta(T-t)} + a_2\theta + b_2\right) \tag{3.43}$$

Using (3.33), (3.41), (3.42) and (3.43),
Total inventory during $[0, T]$ is ,

$$TI_3 = \int_0^T Q(t)\ dt = \int_0^{t_1} Q(t)\ dt + \int_{t_1}^{\lambda_1} Q(t)\ dt + \int_{\lambda_1}^{\lambda_2} Q(t)\ dt + \int_{\lambda_2}^{T} Q(t)\ dt \tag{3.44}$$

where,

$$\int_0^{t_1} Q(t)\ dt = \frac{1}{2\theta^3}\begin{pmatrix} b_1\beta t_1^2\theta^2 + 2a_1\beta t_1\theta^2 - b_1t_1^2\theta^2 - 2a_1t_1\theta^2 - 2b_1\beta t_1\theta \\ -2a_1\beta\theta + 2b_1t_1\theta + 2a_1\theta + 2b_1\beta - 2b_1 + (2a_1\beta\theta - 2a_1\theta - 2b_1\beta + 2b_1)e^{-\theta t_1} \end{pmatrix}$$

$$\int_{t_1}^{\lambda_1} Q(t)\ dt = -\frac{1}{2\theta^3}\begin{pmatrix} b_1\lambda_1^2\theta^2 - b_1t_1^2\theta^2 + 2a_1\lambda_1\theta^2 - 2a_1t_1\theta^2 - 2b_1\lambda_1\theta e^{\theta(\lambda_2-t_1)} \\ -2b_2\lambda_2\theta e^{\theta(\lambda_2-t_1)} + 2b_1\lambda_1\theta e^{\theta(\lambda_2-\lambda_1)} + 2b_2\lambda_2\theta e^{\theta(\lambda_2-\lambda_1)} \\ +2Tb_2\theta e^{\theta(T-t_1)} - 2Tb_2\theta e^{\theta(T-\lambda_1)} - 2b_1\lambda_1\theta + 2b_1t_1\theta \\ -2a_1\theta e^{\theta(\lambda_2-t_1)} + 2a_2\theta e^{\theta(\lambda_2-t_1)} + 2a_1\theta e^{\theta(\lambda_2-\lambda_1)} - 2a_2\theta e^{\theta(\lambda_2-\lambda_1)} \\ -2a_2\theta e^{\theta(T-t_1)} + 2a_2\theta e^{\theta(T-\lambda_1)} + 2b_2e^{\theta(\lambda_2-t_1)} - 2b_2e^{\theta(\lambda_2-\lambda_1)} - 2b_2e^{\theta(T-t_1)} \\ +2b_2e^{\theta(T-\lambda_1)} + 2b_1e^{\theta(\lambda_1-t_1)} - 2b_1 \end{pmatrix}$$

$$\int_{\lambda_1}^{\lambda_2} Q(t)\ dt = \frac{1}{\theta^3}\begin{pmatrix} b_1\lambda_1^2\theta^2 - b_1\lambda_1\lambda_2\theta^2 + a_1\lambda_1\theta^2 - a_1\lambda_2\theta^2 - b_1\lambda_1\theta - b_2\lambda_2\theta \\ +Tb_2\theta e^{\theta(T-\lambda_2)} + b_1\lambda_1\theta e^{\theta(\lambda_2-\lambda_1)} + b_2\lambda_2\theta e^{\theta(\lambda_2-\lambda_1)} - Tb_2\theta e^{\theta(T-\lambda_1)} \\ -a_1\theta + a_2\theta - a_2\theta e^{\theta(T-\lambda_2)} + a_1\theta e^{\theta(\lambda_2-\lambda_1)} - a_2\theta e^{\theta(\lambda_2-\lambda_1)} + a_2\theta e^{\theta(T-\lambda_1)} \\ +b_2 - b_2e^{\theta(T-\lambda_2)} - b_2e^{\theta(\lambda_2-\lambda_1)} + b_2e^{\theta(T-\lambda_1)} \end{pmatrix}$$

$$\int_{\lambda_2}^{T} Q(t)\ dt = \frac{1}{2\theta^3}\begin{pmatrix} T^2 b_2\theta^2 - \lambda_2^2 b_2\theta^2 - 2Ta_2\theta^2 + 2a_2\theta^2\lambda_2 + 2b_2\lambda_2\theta \\ -2Tb_2\theta e^{\theta(T-\lambda_2)} - 2a_2\theta + 2a_2\theta e^{\theta(T-\lambda_2)} - 2b_2 + 2b_2 e^{\theta(T-\lambda_2)} \end{pmatrix}$$

Now, the total number of deteriorated units during $[0, T]$ is given by,
DI_3 =Production in $[0, t_1]$ – Demand in $[0, \lambda_1]$ – Demand in $[\lambda_1, \lambda_2]$ – Demand in $[\lambda_2, T]$

$$= \beta\int_0^{t_1} (a_1 + b_1 t)\ dt - \int_0^{\lambda_1} (a_1 + b_1 t)\ dt - \int_{\lambda_1}^{\lambda_2} (a_1 + b_1\lambda_1)\ dt - \int_{\lambda_2}^{T} (a_2 - b_2 t)\ dt$$

$$= \frac{1}{2}b_1\beta t_1^2 + a_1\beta t_1 + \frac{1}{2}b_1\lambda_1^2 - b_1\lambda_1\lambda_2 - a_1\lambda_2 + \frac{1}{2}T^2 b_2 - \frac{1}{2}b_2\lambda_2^2 - a_2 T + a_2\lambda_2 \tag{3.45}$$

Cost of production during $[0, t_1]$ is given by,

$$PC_3 = \int_0^{t_1} \beta\alpha_1 (R(t))^{(1-\gamma)}\, du = \int_0^{t_1} \beta\alpha_1 (a_1 + b_1 u)^{(1-\gamma)}\, du$$

$$= \beta\alpha_1 \left(\frac{-a_1^\gamma + (b_1 t_1 + a_1)^\gamma}{b_1\gamma}\right) \tag{3.46}$$

The Total cost of an inventory system is defined as the sum of the Inventory holding cost, the Deterioration cost and the Purchase cost. For all the three cases discussed above, we denote and define this as under:

$$C(t_1, T) = \begin{cases} C_1(t_1, T) = \frac{1}{T}[C_h TI_1 + C_d DI_1 + PC_1] & ; \quad 0 \le \lambda_1 \le \lambda_2 \le t_1 \le T \\ C_2(t_1, T) = \frac{1}{T}[C_h TI_2 + C_d DI_2 + PC_2] & ; \quad 0 \le \lambda_1 \le t_1 \le \lambda_2 \le T \\ C_3(t_1, T) = \frac{1}{T}[C_h TI_3 + C_d DI_3 + PC_3] & ; \quad 0 \le t_1 \le \lambda_1 \le \lambda_2 \le T \end{cases} \tag{3.47}$$

To find the optimal values of t_1 and T, which minimizes total cost of an inventory system $C(t_1, T)$, we use fundamental calculus. Solve $\frac{\partial C_i}{\partial t_1} = 0$ and $\frac{\partial C_i}{\partial T} = 0$ simultaneously to find t_1 and T, provided that they satisfy the sufficient conditions $\frac{\partial^2 C_i}{\partial t_1^2} > 0$, $\frac{\partial^2 C_i}{\partial T^2} > 0$ and $\frac{\partial^2 C_i}{\partial t_1^2}\frac{\partial^2 C_i}{\partial T^2} - \left(\frac{\partial^2 C_i}{\partial t_1\,\partial T}\right)^2 > 0$, where $i = 1, 2, 3$.

Computational Algorithm

Step 1: Assign values to the parameters.

Step 2: Compute t_1 and Tusing $\frac{\partial C_i}{\partial t_1} = 0$, $\frac{\partial C_i}{\partial T} = 0$ for $i = 1, 2, 3$.

Step 3: If $0 \le \lambda_1 \le \lambda_2 \le t_1 \le T$ then $C_1(t_1, T)$ is optimal, compute $C_1(t_1, T)$ and S using (3.11), else go to Step 4.

Step 4: If $0 \le \lambda_1 \le t_1 \le \lambda_2 \le T$ then $C_2(t_1, T)$ is optimal, compute $C_2(t_1, T)$ and S using (3.25), else $C_3(t_1, T)$ is optimal, compute $C_3(t_1, T)$ and S using (3.40).

4 Concluding remarks

To implement the computational algorithm, one needs to evaluate optimal values of t_1 and T by solving the equations $\frac{\partial C_i}{\partial t_1} = 0$, $\frac{\partial C_i}{\partial T} = 0$ case wise for $i = 1, 2, 3$. As equations $\frac{\partial C_i}{\partial t_1} = 0$ and $\frac{\partial C_i}{\partial T} = 0$ are highly nonlinear functions of t_1 and T, one may use Newton Raphson method for suitable values of $a_1, b_1, a_2, b_2, \lambda_1, \lambda_2, \theta, \gamma$ and α_1, to find optimal values of $t_1^*$and T^*. So, the corresponding minimum value of the average cost of an inventory system $C(t_1^*, T^*)$ is minimum. Convexity of cost function is difficult to discuss analytically. This says $C(t_1^*, T^*)$ may not be global minimum but may be local minimum. One may extend this model, using advance optimization techniques like, PSO, GA etc. to find the minimum cost of an inventory system.

References

[1] Bakker, M., Riezebos, J. and Teunter, R.H. (2012). Review of inventory systems with deterioration since 2001, *European Journal of Operation Research*, 221, 275–284.

[2] Chen, H.L., Ouyang, L.Y. and Teng, J.T. (2006). On an EOQ model with ramp type demand rate and time dependent deterioration rate, *International Journal of Information and Management Sciences*, 17(4) , 51–66.

[3] Cheng, M. and Wang, G. (2009). A note on the inventory model for deteriorating items with trapezoidal type demand rate, *Computers and Industrial Engineering*, 56, 1296–1300.

[4] Cheng, M., Zhang, B. and Wang, G. (2011). Optimal policy for deteriorating items with trapezoidal type demand and partial backlogging, *Applied Mathematical Modeling*, 35, 3552–3560.

[5] Covert, R.P. and Philip, G.C. (1973). An EOQ model for items with Weibull distribution deterioration. *AIIE Transactions*, 5, 323–326.

[6] Dave, U. and Patel, L.K. (1981). (T, Si) policy inventory model for deteriorating items with time proportional demand, *The Journal of the Operational Research Society*, 32, 137–142.

[7] Deng, P.S. (2005). Improved inventory models with ramp type demand and Weibull deterioration, *International Journal of Information and Management Sciences*, 16(4), 79–86.

[8] Deng, P.S., Lin, R.H.J. and Chu, P. (2007). A note on the inventory models for deteriorating items with ramp type demand rate, *European Journal of Operational Research*, 178(1), 112–120.

[9] Donaldson, W.A. (1977). Inventory replenishment policy for a linear trend in demand: an analytic solution, Operational Research Quarterly, 28, 663–670.

[10] Ghare, P.M. and Schrader, G.F. (1963). A model for exponentially decaying inventories, *Journal of Industrial Engineering*, 14, 238–243.

[11] Giri, B.C., Jalan, A.K. and Chaudhuri, K.S. (2003). Economic order quantity model with Weibull deterioration distribution, shortage and ramp-type demand, *International Journal of Systems Science*, 34(4), 237–243.

[12] Goyal, S.K. and Giri, B.C. (2001). Recent trends in modeling of deteriorating inventory, *J. European Journal of Operation Research*, 134 (1), 1–16.

[13] Goyal, S.K. (1986). On improving replenishment policies for linear trend in demand, *Engineering Costs and Production Economics*, 10, 73–76.

[14] He, Y., Wang, S.Y. and Lai, K.K. (2010). An optimal production-inventory model for deteriorating items with multiple-market demand, *European Journal of Operational Research*, 203(3), 593–600.

[15] Hill, R.M. (1995). Inventory model for increasing demand followed by level demand, *The Journal of the Operational Research Society*, 46, 1250–1259.

[16] Manna, P., Manna, S. and Giri, B.C. (2016). An economic order quantity model with ramp type demand rate, constant deterioration rate and unit production cost, *Yugoslav Journal of Operation Research*, 26 (3), 305–316.

[17] Manna, S.K. and Chaudhuri, K.S. (2006). An EOQ model with ramp type demand rate, time dependent deterioration rate, unit production cost and shortages, *European Journal of Operational Research*, 171(2), 557–566.

[18] Raafat, F. (1991). Survey of literature on continuously deteriorating inventory model, *The Journal of the Operational Research Society*, 42, 27–37.

[19] Resh, M., Friedman, M. and Barbosa, L.C. (1976). On a general solution of the deterministic lot size problem with time-proportional demand, *Operations Research*, 24, 718–725.

[20] Ruxian, L., Hongjie, L. and Mawhinney, J.R. (2010). A review on deteriorating inventory study. *Journal of Service Science and Management*, 3, 117–129.

[21] Shah, N.H. and Shah, Y.K. (2000). Literature survey on inventory models for deteriorating items. *Ekonomskianali*, 44 (145), 221–237.

[22] Shukla, K.T. and Suthar, M.S. (2016). A framework for an inventory model for deteriorating items with expiration under trapezoidal type demand and partial backlogging, *Logforum*, 12(2), 269–281.

[23] Skouri, K., Konstantaras, I., Manna, S.K. and Chaudhuri, K.S. (2011). Inventory models with ramp type demand rate, time dependent deterioration rate, unit production cost and shortages, *Annals of Operations Research*, 191(1), 73–95.

[24] Skouri, K., Konstantaras, I., Papachristos, S. and Ganas, I. (2009). Inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate, *European Journal of Operational Research*, 192(1), 79–92.

[25] Vandana and Shrivastava, H.M. (2017). An inventory model for ameliorating / deteriorating items with trapezoidal type demand and complete backlogging under inflation and time discounting, *Mathematical Methods in the Applied Sciences*, 40(8), 2980–2993.

[26] Wu, K.S. (2001). An EOQ inventory model for items with Weibull distribution deterioration, ramp type demand rate and partial backlogging, *Production Planning and Control*, 12, 787–793.

[27] Wu, K.S. and Ouyang, L.Y. (2000). A replenishment policy for deteriorating items with ramp type demand rate, *Proceedings of the National Science Council, Republic of China (A)*, 24, 279-286.

[28] Wu, J.W., Lin, C., Tan, B. and Lee, W.C. (1999). An EOQ model with ramp type demand rate for items with Weibull deterioration, *International Journal of Information and Management Sciences*, 10, 41–51.

[29] Wu, K.S., Ouyang, L.Y. and Yang, C.T. (2008). Retailers optimal ordering policy for deteriorating items with ramp-type demand under stock-dependent consumption rate, *International Journal of Information and Management Sciences*, 19(2), 245–262.

[30] Wu, J., Skouri, K., Teng, J.T. and Hu, Y. (2016). Two inventory systems with trapezoidal type demand rate and time dependent deterioration and backlogging, *Expert Systems with Applications*, 46, 367–379.

[31] Wu. J., Teng, J.T. and Skouri, K. (2018). Optimal inventory policies for deteriorating items with trapezoidal type demand patterns and maximum life times under upstream and downstream trade credits, *Annals of Operation Research*, 264(1), 459–476.