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Solution of the Volterra integro-differential equations by the triple Elzaki transform *

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Abstract In this paper we discuss the application of the various properties of the triple Elzaki transform to solve the linear Volterra integro-differential equations in three-dimensions.

Key words Triple Elzaki transform, Inverse Triple Elzaki transform, partial integro-differential equations, Upadhyaya transform, triple Upadhyaya transform.

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1 Introduction

Linear integral equations are used to model many problems in engineering , chemistry, physics and many other disciplines of study. It is well known that most integro-differential equations give solutions in a closed form. It is therefore important to propose new methods of finding solutions to different integro-differential equations [1–3]. Of late, many methods are employed to find the solutions of the integro-differential equations in two and three dimensions like the multistep multiderivative methods, the homotopy perturbation method, the Adomian decomposition method, the Laplace transform method, the Sumudu transform method, the Elzaki transform method and many others. Some relevant references besides numerous others are [4–15]. One example of linear integro-differential equations is the three-dimension linear Volterra integro-differential equations (LVIDEs), which are obtained in the course of modeling engineering applications. The general three-dimensional LVIDEs are given in the following form (see, [14,15]):

$$\frac{\partial^3 u(x,y,t)}{\partial x\partial y\partial t}+u(x,y,t)=g(x,y,t)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{t_0}^{t}H(x,y,t,k,r,s,u(k,r,s))dkdrds \tag{1.1}$$

where $u(x,y,t)$ is the unknown function, and the functions H and g are analytic in the domain of interest.

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As mentioned above that the method of employing integral transforms is a very effective tool for finding the solutions of the integro-differential equations, the well known Laplace transform is the oldest one in this class. Many new integral transforms of this class are introduced by different researchers the world over during the recent times. A detailed description of most of these developments is very beautifully and very exhaustively narrated by Upadhyaya [16] in his classic and landmark work on the Upadhyaya transform, which is the most recent and the most prominent work in the direction of generalizing and unifying a large and wide variety of recently introduced integral transforms of the Laplace class by a large number of mathematicians and researchers across the globe. We underline here that in almost all the fields of applications of the classical Laplace transform, the Upadhyaya transform and its numerous other generalizations in the different directions, as pointed out by Upadhyaya [16] in his seminal work, promise a huge potential of applications to yield the more generalized and powerful solutions to most of the existing problems of engineering, applied mathematics, physics and other branches of study in which the classical Laplace transform and its other recently introduced variants have been so fruitfully employed during about the past twenty six years with the advent of the Sumudu transform of Watugala [17] in the year 1993! Motivated most strongly by the above stated stupendous work of Upadhyaya [16] we propose to exploit the full potential of the Upadhyaya transform [16] and its numerous generalizations as pointed out by Upadhyaya himself in his prodigious work [16] to the various domains of study in our future communications. For the interested reader we point out below how the Elzaki transform [12] introduced by the second author in the year 2011 and the triple Elzaki transform [4] introduced by both these authors in this very year in the month of March 2019 follow as a special case of the Upadhyaya transform [16] in the very notations of the paper of Upadhyaya [16]. With a view to make this paper self contained we give below the definition of the Elzaki transform [12] of the second author. The Elzaki transform is defined for a function of exponential order. Consider a set A defined as

$$A = \left\{ f(t) : \exists\, M, k_1, k_2 > 0, |f(t)| < M\, e^{\frac{|t|}{k_j}}, \text{if } t \in (-1)^j \times [0, \infty), j = 1, 2 \right\}.$$

For a given function $f(t)$ in the set A, the constant M must be finite, the numbers k_1, k_2 may be finite or infinite. The Elzaki Transform denoted by the operator E is defined as (see, [12])

$$E\left[f(t) : \rho\right] = \rho \int_0^\infty e^{-\frac{t}{\rho}} f(t)dt \tag{1.2}$$

in which the variable ρ is used to factorize the variable t.

The triple Elzaki Transform [4] of a function $f(x, y, t)$ of three variables x, y and t that can be expressed as a convergent infinite series, and for $(x, y, t) \in \mathbb{R}_3^+$ is defined in the first octant of the xyt - plane by the triple integral:

$$E_3\left[f(x, y, t) : (\sigma, \rho, \delta)\right] = \sigma\rho\delta \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{x}{\sigma} - \frac{y}{\rho} - \frac{t}{\delta}} f(x, y, t)dxdydt \tag{1.3}$$

As pointed out above, the Elzaki transform of (1.2) is a special case of the Upadhyaya transform (see [16, (2.2), (2.3), p.473 and subsection 4.5, pp.476–477]) as

$$\mathcal{U}\left\{f\left(t\right); v, \frac{1}{v}\right\} = \mathfrak{u}\left(v, \frac{1}{v}, 1\right) = \Im\left[f\left(t\right), v\right] = T\left(\rho\right) \tag{1.4}$$

and the triple Elzaki transform of (1.3) is also a particular case of the Triple Upadhyaya Transform (TUT) (see, Upadhyaya [16, subsection 6.14, p.501])

$$\begin{aligned} \mathcal{U}_3\left\{f\left(x, y, t\right); \sigma, \tfrac{1}{\sigma}, 1, \rho, \tfrac{1}{\rho}, 1, \delta, \tfrac{1}{\delta}, 1\right\} &= \mathfrak{u}_3\left(\sigma, \tfrac{1}{\sigma}, 1, \rho, \tfrac{1}{\rho}, 1, \delta, \tfrac{1}{\delta}, 1\right) \\ &= \Im\left[f\left(x, y, t\right); \sigma, \rho, \delta\right] = T\left(\sigma, \rho, \delta\right) \end{aligned} \tag{1.5}$$

where, we refer the reader to Upadhyaya [16] for the notations used by us in the above two equations. The structure of the remaining part of the paper is organized as follows: In section 2, we discuss the existence and uniqueness conditions of of the solution of (1.1), and in section 3, we state the basic properties of the triple Elzaki transform [4] which are used by us in section 4 to solve the linear Volterra integro-differential equations (LVIDEs) in three-dimensions and finally the conclusions of this work are given in section 5 .

2 On the existence of the solution of the three-dimensional LVIDEs

In this section, we recall the conditions necessary for the existence and uniqueness of the solution of (1.1), on the complete metric space of complex valued continuous functions as follows (see [14]):

$$M = [C(S,d)]\ ,\ d\,(g,z) = \sup\{\,|g(x,y,t) - z(x,y,t)| : (x,y,t) \in S\,\}$$

where $S = [0,1] \times [0,1] \times [0,1]$.

Theorem 2.1. *Let g and H be continuous functions on $[0,1]^3$ and $[0,1] \times [0,1] \times \mathbb{C}$ respectively and there exists a nonnegative constant $L \leq 1$ such that*

$$\begin{aligned}&|H(x,y,t,k,r,s,u(k,r,s)) - H(x,y,t,k,r,s,v(k,r,s))| \\ &\quad \leq L\,|u(k,r,s) - v(k,r,s)|\end{aligned}$$

Then

$$u(x,y,t) = g(x,y,t) + \int_0^x \int_0^y \int_0^t H(x,y,t,k,r,s,u(k,r,s))dkdrds$$

has only one continuous solution u on S.

Proof. For the proof see [14, p.2944]. □

Corollary 2.2. *If the hypothesis of the Theorem 2.1 holds, then the equation (1.1)*

$$\frac{\partial^3 u(x,y,t)}{\partial x \partial y \partial t} + u(x,y,t) = g(x,y,t) + \int_{x_0}^x \int_{y_0}^y \int_{t_0}^t H(x,y,t,k,r,s,u(k,r,s))dkdrds$$

with initial condition

$$\begin{aligned}&u(0,0,0) = h_0\,, u(x,0,0) = h_1(x), u(0,y,0) = h_2(y), u(0,0,t) = h_3(t),\\ &u(x,y,0) = h_4(x,y), u(x,0,t) = h_5(x,t), u(0,y,t) = h_6(y,t)\end{aligned}$$

so by the Theorem 2.1 the equation has a unique continuous solution (see [15, p.5]).

3 Theorems and properties of the triple Elzaki transform

In this section we state some properties of the triple Elzaki transform which will be used in the next section for solving the LVIDEs.

Theorem 3.1. Linearity of the triple Elzaki transform: *Let $f(x,y,t)$ and $g(x,y,t)$ be the functions whose triple Elzaki transforms exist then*

$$E_3\left[\alpha f(x,y,t) + \beta g(x,y,t)\right] = \alpha E_3\left[f(x,y,t)\right] + \beta E_3\left[g(x,y,t)\right]$$

where α and β are constants.

Theorem 3.2. *If $E_3\left[f(x,y,t)\right] = T(\sigma,\rho,\delta)$ then,*

$$E_3\left[f(x-\alpha, y-\beta, t-\kappa)H(x-\alpha, y-\beta, t-\kappa)\right] = e^{\frac{-\alpha}{\sigma} - \frac{\beta}{\rho} - \frac{\kappa}{\delta}}\, T(\sigma,\rho,\delta)$$

where $H(x,y,t)$ is the Heaviside unit step function defined by

$$H(x-\theta, y-\beta, t-\kappa) = \begin{cases} 1, & \text{when, } x > \theta\,, y > \beta\,, t > \kappa \\ 0, & \text{when, } x < \theta\,, y < \beta\,, t < \kappa. \end{cases}$$

Theorem 3.3. Convolution theorem: *If $E_3\left[F(x,y,t)\right] = f(\sigma,\rho,\delta)$, $E_3\left[G(x,y,t)\right] = g(\sigma,\rho,\delta)$, then the convolution of $F(x,y,t)$ and $G(x,y,t)$ is defined by $E_3\left[(F***G)\,(x,y,t)\right] = \sigma\rho\delta \int_0^x \int_0^y \int_0^t F(x-\alpha, y-\beta, t-\kappa)\,G(\alpha,\beta,\kappa)\,dxdydt$ and we have $E_3\left[(F***G)\,(x,y,t)\right] = E_3\left[F(x,y,t)\right].E_3\left[G(x,y,t)\right] = f(\sigma,\rho,\delta).g(\sigma,\rho,\delta)$.*

Theorem 3.4. Operational Formula: *If $f(x,y,t) = \frac{\partial^3 f(x,y,t)}{\partial x \partial y \partial t}$, then,*

$$E_3\left[\frac{\partial^3 f(x,y,t)}{\partial x \partial y \partial t} : (\sigma,\rho,\delta)\right] = -\sigma\rho\delta T(0,0,0) + \frac{\rho\delta}{\sigma} T(\sigma,0,0) + \frac{\sigma\delta}{\rho} T(0,\rho,0) - \frac{\delta}{\sigma\rho} T(\sigma,\rho,0) + \frac{\sigma\rho}{\delta} T(0,0,\delta) - \frac{\rho}{\sigma\delta} T(\sigma,0,\delta) - \frac{\sigma}{\rho\delta} T(0,\rho,\delta) + \frac{1}{\sigma\rho\delta} T(\sigma,\rho,\delta).$$

4 Application of triple Elzaki transform to find the solution of linear Volterra integro-differential equations

We now apply the various properties of the triple Elzaki transform to determine the solution of linear Volterra integro-differential equations in this section. We consider the following two illustrations and work out their respective solutions.

Example 4.1. Consider the linear volterraintegro-differential equation

$$\frac{\partial^3 u(x,y,t)}{\partial x \partial y \partial t} + u(x,y,t) = x\cos t - \frac{x^2 y \sin t}{2} + \int_0^x \int_0^y \int_0^t u(k,r,s)dkdrds, \tag{4.1}$$

with

$$u(x,y,0) = x, u(x,0,t) = x\cos t, u(0,y,t) = 0, u(0,0,t) = 0, u(x,0,0) = x, u(0,y,0) = 0, u(0,0,0) = 0. \tag{4.2}$$

Solution. By taking the triple Elzaki transform of (4.1), and the double Elzaki transform of initial conditions (4.2) we obtain

$$\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta) + \frac{\sigma\rho}{\delta}\bar{u}(0,0,\delta) + \frac{\rho\delta}{\sigma}\bar{u}(\sigma,0,0) + \frac{\sigma\delta}{\rho}\bar{u}(0,\rho,0) - \frac{\sigma}{\rho\delta}\bar{u}(0,\rho,\delta) - \frac{\rho}{\sigma\delta}\bar{u}(\sigma,0,\delta) -$$

$$\frac{\delta}{\sigma\rho}\bar{u}(\sigma,\rho,0) - \sigma\rho\delta\bar{u}(0,0,0) + \bar{u}(\sigma,\rho,\delta) = \sigma^3\rho^2\frac{\delta^2}{\delta^2+1} - \sigma^4\rho^3\frac{\delta^3}{\delta^2+1} + \sigma\rho\delta\bar{u}(\sigma,\rho,\delta) \tag{4.3}$$

and

$$\bar{u}(0,\rho,\delta) = 0\ ,\ \bar{u}(\sigma,0,\delta) = \sigma^3\frac{\delta^2}{\delta^2+1}\ ,\ \bar{u}(\sigma,\rho,0) = \sigma^3\rho^2\ ,$$

$$\bar{u}(0,0,\delta) = 0\,, \bar{u}(0,\rho,0) = 0\,, \bar{u}(\sigma,0,0) = \sigma^3\,,\ \bar{u}(0,0,0) = 0. \tag{4.4}$$

Substituting (4.3) in (4.2), and on simplifying it we obtain,

$$\begin{aligned}
&\bar{u}(\sigma,\rho,\delta)\left[\tfrac{1}{\sigma\rho\delta} + 1 - \sigma\rho\delta\right] + \tfrac{\sigma\rho}{\delta}[0] + \tfrac{\rho\delta}{\sigma}\left[\sigma^3\right] + \tfrac{\sigma\delta}{\rho}[0] - \tfrac{\sigma}{\rho\delta}[0] - \tfrac{\rho}{\sigma\delta}\left[\sigma^3\tfrac{\delta^2}{\delta^2+1}\right] - \\
&\quad \tfrac{\delta}{\sigma\rho}\left[\sigma^3\rho^2\right] - \sigma\rho\delta\,[0] = \sigma^3\rho^2\tfrac{\delta^2}{\delta^2+1} - \sigma^4\rho^3\tfrac{\delta^3}{\delta^2+1} \\
&\bar{u}(\sigma,\rho,\delta)\left[\tfrac{1}{\sigma\rho\delta} + 1 - \sigma\rho\delta\right] = \sigma^3\rho^2\tfrac{\delta^2}{\delta^2+1} - \sigma^4\rho^3\tfrac{\delta^3}{\delta^2+1} + \tfrac{\sigma^2\rho\delta}{\delta^2+1} \\
&\bar{u}(\sigma,\rho,\delta)\left[\tfrac{1}{\sigma\rho\delta} + 1 - \sigma\rho\delta\right] = \sigma^3\rho^2\tfrac{\delta^2}{\delta^2+1}\left[1 - \sigma\rho\delta + \tfrac{1}{\sigma\rho\delta}\right] \\
&\bar{u}(\sigma,\rho,\delta) = \sigma^3\rho^2\tfrac{\delta^2}{\delta^2+1}.
\end{aligned}$$

Taking the inverse triple Elzaki transform of both the sides we have

$$u(x,y,t) = x\cos t$$

which is the desired solution to (4.1).

Example 4.2. Consider the linear Volterra integro-differential equation

$$\frac{\partial^3 u(x,y,t)}{\partial x \partial y \partial t} + u(x,y,t) = xyt + 1 - \frac{x^2 y^2 t^2}{8} + \int_0^x \int_0^y \int_0^t u(k,r,s)dkdrds, \tag{4.5}$$

with

$$u(x,y,0) = u(x,0,t) = u(0,y,t) = 0. \tag{4.6}$$

Solution. By taking the triple Elzaki transform of (4.5), and the double Elzaki transform of the initial conditions (4.6) we get

$$\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta) + \frac{\sigma\rho}{\delta}\bar{u}(0,0,\delta) + \frac{\rho\delta}{\sigma}\bar{u}(\sigma,0,0) + \frac{\sigma\delta}{\rho}\bar{u}(0,\rho,0) - \frac{\sigma}{\rho\delta}\bar{u}(0,\rho,\delta) - \frac{\rho}{\sigma\delta}\bar{u}(\sigma,0,\delta) -$$

$$\frac{\delta}{\sigma\rho}\bar{u}(\sigma,\rho,0) - \sigma\rho\delta\bar{u}(0,0,0) + \bar{u}(\sigma,\rho,\delta) = \sigma^3\rho^3\delta^3 + \sigma^2\rho^2\delta^2 - \sigma^4\rho^4\delta^4 + \sigma\rho\delta\bar{u}(\sigma,\rho,\delta) \tag{4.7}$$

and

$$\bar{u}(\sigma,\rho,0)=0 \quad , \bar{u}(\sigma,0,\delta)=0 \quad , \bar{u}(0,\rho,\delta)=0. \tag{4.8}$$

On substituting (4.8) in (4.7) and simplifying and we obtain,

$$\begin{aligned}
\bar{u}(\sigma,\rho,\delta)\left[\tfrac{1}{\sigma\rho\delta}+1-\sigma\rho\delta\right] &= \sigma^3\rho^3\delta^3-\sigma^4\rho^4\delta^4+\sigma^2\rho^2\delta^2\\
\bar{u}(\sigma,\rho,\delta)\left[\tfrac{1}{\sigma\rho\delta}+1-\sigma\rho\delta\right] &= \sigma^3\rho^3\delta^3\left[1-\sigma\rho\delta+\tfrac{1}{\sigma\rho\delta}\right]\\
\bar{u}(\sigma,\rho,\delta) &= \sigma^3\rho^3\delta^3
\end{aligned}$$

Finally taking the inverse triple Elzaki transform of both sides of the above relation gives the solution of (4.5) in the form

$$u(x,y,t)=xyt.$$

Example 4.3. Consider the linear Volterra integro-differential equation

$$\frac{\partial^3 u(x,y,t)}{\partial x\partial y\partial t}+u(x,y,t)=\frac{x^2yt+xy^2t+xyt^2}{2}+x+y+t-\int_0^x\int_0^y\int_0^t u(k,r,s)dkdrds \tag{4.9}$$

with

$$\begin{gathered}
u(x,y,0)=x+y, u(x,0,t)=x+t, u(0,y,t)=y+t, u(0,0,t)=t,\\
u(x,0,0)=x, u(0,y,0)=y, u(0,0,0)=0.
\end{gathered} \tag{4.10}$$

Solution. By taking the triple Elzaki transform of (4.9) and the double Elzaki transform of initial conditions (4.10) we get

$$E_3\left[\frac{\partial^3 u}{\partial x\partial y\partial t}\right]+E_3\left[u(x,y,t)\right]=E_3\left[\frac{x^2yt+xy^2t+xyt^2}{2}+x+y+t-\int\limits_0^x\int\limits_0^y\int\limits_0^t u(k,r,s)dkdrds\right] \tag{4.11}$$

or,

$$\begin{aligned}
&\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta)+\frac{\sigma\rho}{\delta}\bar{u}(0,0,\delta)+\frac{\rho\delta}{\sigma}\bar{u}(\sigma,0,0)+\frac{\sigma\delta}{\rho}\bar{u}(0,\rho,0)-\frac{\sigma}{\rho\delta}\bar{u}(0,\rho,\delta)-\\
&\frac{\rho}{\sigma\delta}\bar{u}(\sigma,0,\delta)-\frac{\delta}{\sigma\rho}\bar{u}(\sigma,\rho,0)-\sigma\rho\delta\bar{u}(0,0,0)+\bar{u}(\sigma,\rho,\delta)=\sigma^4\rho^3\delta^3+\\
&\sigma^3\rho^4\delta^3+\sigma^3\rho^3\delta^4+\sigma^3\rho^2\delta^2+\sigma^2\rho^3\delta^2+\sigma^2\rho^2\delta^3-\sigma\rho\delta\bar{u}(\sigma,\rho,\delta)
\end{aligned}$$

and

$$\begin{aligned}
&\bar{u}(\sigma,\rho,0)=\left[\sigma^3\rho^2+\sigma^2\rho^3\right], \ \bar{u}(\sigma,0,\delta)=\left[\sigma^3\delta^2+\sigma^2\delta^3\right] \ , \bar{u}(0,\rho,\delta)=\left[\rho^3\delta^2+\rho^2\delta^3\right]\\
&\bar{u}(\sigma,0,0)=\sigma^3 \ , \ \bar{u}(0,\rho,0)=\rho^3 \ , \ \bar{u}(0,0,\delta)=\delta^3 \ , \ \bar{u}(0,0,0)=0.
\end{aligned} \tag{4.12}$$

Substittution from (4.12) into (4.11) followed by simplification gives

$$\begin{aligned}
&\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta)+\frac{\sigma\rho}{\delta}\left[\delta^3\right]+\frac{\rho\delta}{\sigma}\left[\sigma^3\right]+\frac{\sigma\delta}{\rho}\left[\rho^3\right]-\frac{\sigma}{\rho\delta}\left[\rho^3\delta^2+\rho^2\delta^3\right]-\\
&\frac{\rho}{\sigma\delta}\left[\sigma^3\delta^2+\sigma^2\delta^3\right]-\frac{\delta}{\sigma\rho}\left[\sigma^3\rho^2+\sigma^2\rho^3\right]-\sigma\rho\delta\left[0\right]+T(\sigma,\rho,\delta)=\sigma^4\rho^3\delta^3+\\
&\sigma^3\rho^4\delta^3+\sigma^3\rho^3\delta^4+\sigma^3\rho^2\delta^2+\sigma^2\rho^3\delta^2+\sigma^2\rho^2\delta^3-\sigma\rho\delta\bar{u}(\sigma,\rho,\delta)
\end{aligned}$$

or,

$$\begin{aligned}
&\bar{u}(\sigma,\rho,\delta)\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]=\sigma^3\rho^2\delta^2\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]+\sigma^2\rho^3\delta^2\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]+\\
&\sigma^2\rho^2\delta^3\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]
\end{aligned}$$

i.e.,

$$\bar{u}(\sigma,\rho,\delta)=\sigma^3\rho^2\delta^2+\sigma^2\rho^3\delta^2+\sigma^2\rho^2\delta^3.$$

which on taking the inverse Elzaki transform yields the solution of (4.9) as

$$u(x,y,t)=x+y+t.$$

5 Conclusion

In this paper we successfully applied the various properties of the triple Elzaki transform to find the exact solutions of certain linear Volterra integro-differential equations in three dimensions subject to some initial conditions. The method exhibited by us in this paper is also applicable to the solution of problems in engineering, applied mathematics, physics and other fields where the researchers often search for the solutions of the LVIDEs arising in connection with their research problems.

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