Bulletin of Pure and Applied Sciences. *Print version ISSN 0970 6569*
Vol.39 D (Physics), No.1, *Online version ISSN 2320 3218*
January-June 2020 P.124-129 *DOI: 10.5958/2320-3218.2020.00015.9*
Review Article *Available online at www.bpasjournals.com*

A simplistic approach to "The Uncertainty Principle"

Aviral Srivastava*

Author's Affiliations:

Aviral Srivastava Student, Integrated BSMS-2nd Semester, Indian Association for the Cultivation of Science (IACS**)**, Kolkata, West Bengal 700032, India

***Corresponding author:** **Aviral Srivastava,** Student, Integrated BSMS-2nd Semester, Indian Association for the Cultivation of Science (IACS**)**, Kolkata, West Bengal 700032, India
E-mail: aviralsrivastava12@gmail.com

Received on 22.02.2020
Accepted on 15.05.2020

ABSTRACT Throughout the world, almost all of the courses in Physics commence with an introduction to Newton's Laws! These laws elegantly describe "our" macroscopic world. These laws beautifully explained movement of planets, justified elliptical orbits and helped us in understanding this vast universe including our Solar System. These laws seemed flawless initially but further discoveries challenged Newtonian interpretation. Newton's laws couldn't explain the observations in the atomic arena and in the cosmic arena. Let us see what has transpired in atomic world since Newton through Uncertainty Principle deduced by Heisenberg. We must know following two variables about a system at any point of reference in time in order to study time and space evolution of the system: Position, Momentum. Newton stated that these physical quantities could be nimbly determined in an experiment to a great precision. However, things changed drastically with the advent of Quantum Mechanics. The birth of Quantum Mechanics was marked by the energy quantisation law given by Max Planck. This new law was the explanation for the Ultraviolet Catastrophe related to the black body radiation. Later, a lot of scientists built upon this subject in a collaborative effort. In this article, the main emphasis would be on Heisenberg approach which is based on Linear Algebra and matrix mechanics. Heisenberg stated that the position and momentum of a quantum particle can't be measured simultaneously with full accuracy. He challenged Newton's fundamentals with mathematical deductions. He named his new discovery very aptly as "The Uncertainty Principle". This article tries to deduce same mathematical results using a different approach.

KEYWORDS The Uncertainty Principle, Quantum Physics, Vector Space, Classical Mechanics

INTUITION

The Uncertainty Principle is derived purely on mathematical grounds; however, the physical intuition can be developed with a simple thought experiment. Consider an isolated electron in space.We have two objectives here:

i) to determine the electron' position and
ii) its momentum.

We need these two quantities to have the complete information about its state of motion. Let us employ EM radiation for this task. This radiation is completely determined by its wavelength say **λ**. Now, if one tries to find the whereabouts of the concerned electron by the means of the EMR then the argument is that this process of measuring the state will alter what we are trying to measure. Let us see how. Suppose such an EM wave is sent towards this electron. Before proceeding further, we must recollect these two assertions,

1) The electron is considered as a point particle in our thought experiment.
2) If an EM wave has a wavelength λ, then from wave optics we know that any entity having dimensions less than λ can't be resolved.

Coalescing these two, we can infer that an electron can't be resolved by an EM wave of any λ.

The second objective was to determine the electron's momentum. Now the same EM is treated as a particle. In this case there is a collision between the e^- and the photon. This collision in general may not be elastic and the coefficient of restitution is not known. Therefore, the fraction of energy transferred can't be determined. It means that there is always some discrepancy between the initial and final momentum because the momentum is not determined exactly. It can be seen how this simple thought provokes us to derive this fundamental principle of nature more pedantically.

DISCUSSION

Before proceeding with the derivation, some basic concepts need to be revised. Let's begin with Linear Algebra and Dirac's notation. These concepts will lead to better understanding of **Vector Space Approach to Quantum Mechanics**.

First and the foremost we define the following:

Vector Space: A vector space (to be denoted as VS) is a set of elements obeying a mathematical construct. The mathematical construct differentiates a vector space from a set which doesn't obey this construct. This construct defines a vector space **S**, which has to have the following properties satisfied, for all a, b, c in **S,** in order to qualify as a VS: -

a) **Closure**: **S** should be closed under Addition and Scaler multiplication. i.e. if a,b ε **S** then λa+b must be in **S.**
b) **Commutativity:** a+b = b+a for all a,b ε **S.**
c) **Associativity:** (a+b)+c = a+(b+c) for all a, b, c ε **S.**
d) **Existence of Additive Identity:** There exists an element 0 in **S** such that a+0=a for all a in **S**.
e) **Existence of Additive Inverse:** For every a in **S**, there exists a b such that a+b=0, b is then called the additive inverse of a and b = -a.
f) **Existence of Multiplicative Identity:** 1a=a for every a in **S.**
g) **Distributive Properties:** a(b+c) = ab+ac

$$\lambda(a+b) = \lambda a+\lambda b,\ \lambda \in \mathbf{F}.$$

Each element of a vector space is called as a Vector and in Dirac's notation a vector is represented as $|a\rangle$, this notation would be frequently used in our discussion.

Let us revise a few postulates of Quantum Mechanics:

- Wave Function: Each system in QM is identified by a wave function associated with it.
- Operators: In Quantum Mechanics an 'operator' represents the interaction between the observer and the system. These operators are defined on an infinite vector space called Hilbert Space (HS).
- The outcome of the interaction is reflected by the eigenvalue of the corresponding eigenfunction and operator.
- The operators are derived from their corresponding classical counterparts, just like the energy operator is constructed from its classical expression.

Now, let us define,
$\Delta x = \hat{x} - < \hat{x} >$ $\Delta p = \hat{p} - < \hat{p} >$

In position basis, the operators $\hat{x}$ and $\hat{p}$ are nothing but the position x and the momentum p, which are of course real and complex respectively in the Hilbert Space.

Henceforth $\hat{x}$ and $\hat{p}$ will be denoted simply x and p, and
$<x_i>$ is the expectation of the i^{th} operator.

Squaring both the equations,
=>$\Delta x^2 = (x - \langle x \rangle)^2$ $\Delta p^2 = (p - \langle p \rangle)^2$
=>$\Delta x^2 = x^2 - 2x\langle x \rangle + \langle x \rangle^2$ $\Delta p^2 = p^2 - 2p\langle p \rangle + \langle p \rangle^2$
=>$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$

We know that
$\sigma_i^2 = \langle \Delta x_i^2 \rangle$

We now define an operation called inner product as follows,
$z = \langle f | g \rangle$

Where f, g are vectors in HS and z ε **C**. The inner product is an operation on operators which map vectors (operators in HS) to a scaler in the Complex Field.

Another quantity of interest is the expectation of an operator which is given by

$\langle f \rangle = \langle \mathbf{f} | \mathbf{f} \rangle$

Let $|u\rangle = \Delta x = x - \langle x \rangle$ and $|v\rangle = \Delta p = p - \langle p \rangle$

It can be easily shown that (will be done later in the discussion),
$\sigma_x^2 = \langle \mathbf{u} | \mathbf{u} \rangle$ (1)

$\sigma_p^2 = \langle \mathbf{v} | \mathbf{v} \rangle$ (2)

Taking product of equations (1) and (2) we obtain,

$\sigma_x^2 * \sigma_p^2 = \langle \mathbf{v} | \mathbf{v} \rangle \langle \mathbf{u} | \mathbf{u} \rangle$ (3)

Cauchy-Schwarz inequality states,

$\langle \mathbf{u} | \mathbf{u} \rangle \langle \mathbf{v} | \mathbf{v} \rangle \geq | \langle \mathbf{u} | \mathbf{v} \rangle |^2 = | \langle \mathbf{u} | \mathbf{v} \rangle \langle \mathbf{v} | \mathbf{u} \rangle |$

⇨ $\sigma_x^2 * \sigma_p^2 \geq \langle \mathbf{u} | \mathbf{v} \rangle^2$ (4)

Where $\langle \mathbf{u} | \mathbf{v} \rangle$ and $\langle \mathbf{v} | \mathbf{u} \rangle$ are complex conjugates of each other in the Complex Field.

Consider,
$\langle \mathbf{u} | \mathbf{v} \rangle = \langle \Psi^* | u | v | \Psi \rangle$ (Here a relation between inner product and expectation is set up)
$= \langle \Psi^* | \ x - \langle x \rangle \ | \ p - \langle p \rangle | \Psi \rangle$

$\langle \mathbf{u} | \mathbf{v} \rangle = <xp> - <x><p>$ (5)

Similarly,
$\langle \mathbf{v} | \mathbf{u} \rangle = \langle \Psi^* | v | u | \Psi \rangle$.
$= \langle \Psi^* | p - \langle p \rangle | \ x - \langle x \rangle \ | \Psi \rangle$
$\langle \mathbf{v} | \mathbf{u} \rangle = \langle px \rangle - \langle x \rangle \langle p \rangle$ (6)

Proof of the above Equation:
Consider a normalised wavefunction Ψ, in vector notation it is written as: -
$| \Psi \rangle = \int \langle x | \Psi \rangle * | x \rangle dx$
$= \int \Psi(x) | x \rangle dx$ (i)
Taking complex conjugates,
$\langle \Psi | = \int \Psi^*(x) \langle x | dx$ (ii)

Taking product of the two equations,
$\langle \Psi | \Psi^* \rangle = [\int \Psi(x) | x \rangle dx]*[\int \Psi^*(x) \langle x | dx]$
$= [\int \Psi^*(x)\Psi(x)dx]*[\int | x \rangle\langle x | dx]$
$= \int \Psi^*(x)\Psi(x)dx$ (by definition of delta function)
$\langle \Psi | \Psi^* \rangle = 1$ (Ψ was normalised)– (iii)

Now let $\langle u | v \rangle$ be x
$\Rightarrow \langle u | v \rangle = x$
$\Rightarrow$ $1 * \langle u | v \rangle = x * 1$ (multiplying both sides by 1)
$\Rightarrow \langle \Psi | \Psi^* \rangle * \langle u | v \rangle = x$ {from equation (iii)}
$\Rightarrow \langle \Psi | u | v | \Psi \rangle = x$
$\Rightarrow \langle \Psi | u | v | \Psi \rangle = \langle u | v \rangle$
Hence proved.

Now it will be shown that,
$\sigma_x^2 = \langle \mathbf{u} | \mathbf{u} \rangle$ - (1)
$\sigma_p^2 = \langle \mathbf{v} | \mathbf{v} \rangle$ - (2)

Proof: -
σ_x^2 was defined above as,
$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$
Now consider,
$\langle \mathbf{u} | \mathbf{u} \rangle = \langle \Psi | u | v | \Psi \rangle \langle x - \langle x \rangle | x - \langle x \rangle \rangle$
$= \langle \Psi | x - \langle x \rangle | x - \langle x \rangle | \Psi \rangle$
$= \langle \Psi | x^2 - 2x\langle x \rangle + \langle x \rangle^2 | \Psi \rangle$
$= \langle x^2 \rangle - \langle x \rangle^2$

Subtracting (6) from (5),
$\langle \mathbf{u} | \mathbf{v} \rangle - \langle \mathbf{v} | \mathbf{u} \rangle = \langle xp \rangle - \langle px \rangle$
$= \langle xp - px \rangle$
$= \langle [x,p] \rangle$
$= i\hbar$ (7)

Since, $\langle \mathbf{v} | \mathbf{u} \rangle = \langle \mathbf{u} | \mathbf{v} \rangle^*$ so let's call $\langle \mathbf{u} | \mathbf{v} \rangle$ z and $\langle \mathbf{v} | \mathbf{u} \rangle$ z^*.

For any complex number the following hold true,
$z * z^* = |z|^2$

$z - z^* = 2Im(z)$ {Im(z) denotes Imaginary part of z}

Let Im(z)=y

⇨ $z-z^{*}= 2y$, on squaring both sided we obtain

⇨ $(z-z^{*})^2 = 4y^2$

⇨ $2|z|^2 - z^2 - (z^{*})^2 = -4y^2$

⇨ $2|z|^2 - \{z^2 +(z^{*})^2\}= -4y^2$ (8)

Claim: $z^2 +(z^{*})^2> 0$

Suppose z = a + ib => z^{*}= a – ib where a and b are Reals.

Upon squaring the two equations we obtain,
$z^2 = a^2 + i2ab -b^2$ and $z^{*2} = a^2 -i2ab - b^2$,

Adding z^2 and z^{*2}
$z^2 + z^{*2} = 2(a^2 - b^2)$

z was defined earlier to be <u|v>.

$z = < x- <x> |\ p -<p>>$
$= \langle\Psi^{*}|u|v|\Psi\rangle$
$= \langle\Psi^{*}|\ x - \langle x\rangle\ |\ p - \langle p\rangle|\Psi\rangle$

z = <xp> - <x><p>, it can be seen that z is a real quantity as it is sum of expectations therefore b=0.

Now the sum becomes 2a² as shown,

$z^2 + z^{*2} = 2a^2> 0$ for all a in **R**

Proceeding further, we can argue that the following inequality holds

⇨ $|z|^2 \geq -y^2$

It can be justified because, in equation 8, R.H.S is negative and equality holds when a positive quantity is subtracted from $2|z^2|$, therefore $2|z^2|$ is greater than R.H.S.
Substituting y.

⇨ $|z|^2 \geq -(z-z^{*})^2/4$

⇨ $|\langle \mathbf{u}|\mathbf{v}\rangle|^2 \geq -\frac{(\langle\mathbf{u}|\mathbf{v}\rangle-\langle\mathbf{v}|\mathbf{u}\rangle)^2}{4}$

⇨ $|\langle \mathbf{u}|\mathbf{v}\rangle|^2 \geq -(i\hbar)^2/4$ {using equation (7)}

⇨ $|\langle \mathbf{u}|\mathbf{v}\rangle|^2 \geq \hbar^2/4$, substituting this in equation (4) we obtain,

$\boldsymbol{\sigma_x^{2} * \sigma_p^2} \geq \hbar^2/4$

⇨ $\boldsymbol{\sigma_x * \sigma_p} \geq \hbar/2$

Generally, the inequality is represented by the uncertainty notation rather than the standard deviation so that it looks more convincing.

$\boldsymbol{\Delta x * \Delta p} \geq \hbar/2$

It can be inferred from above that any change in position automatically changes the momentum such that the product of the two remains greater than or equal to half h cross.

CONCLUSION

We see how High School Complex Algebra coupled with some Linear Algebra led us to one of the most powerful and celebrated principles of Science. The Uncertainty Principle has been of immense importance and has been successful in explaining:

- The Interference pattern of the double slit experiment with light as well as electrons, which contributed to the idea of dual nature of light and matter.
- In all the successful atomic models like the Bohr or Schrodinger model, the electrons were in motion around the nucleus however they were never found within the nucleus. When this principle was coupled with relativity one could thoroughly arrive at an explanation for this observation.
- One of Bohr's postulates which stated "the electrons move in fixed orbits around the nucleus" was rendered obsolete by 'Uncertainty Principle'. As it would have meant that one could determine the position with full accuracy which is prohibited by this principle. This result helped in formulating the probability wave model of the Quantum Mechanics.

So, the next time if someone asks "Why can't an electron reside within the nucleus" we can simply refer to Heisenberg' principle.

Abbreviations: None

REFERENCES

[1]. Linear Algebra Done Right-Sheldon Axler
[2]. Wolfram Alpha-https://mathworld.wolfram.com/InnerProduct.html
[3]. Introduction to Quantum Mechanics, David J. Griffiths