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A Plausible Formal Correspondence between Tetrahedral Condensates/TSC and PT-Symmetric Crystals model of CMNS (aka. Low-Energy Nuclear Reactions)

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ABSTRACT

Akito Takahashi's Tetrahedral Symmetric Condensate (TSC) model, detailed in several of his earlier works[1] proposes a mechanism for condensed matter nuclear science (CMNS) aka. low-energy nuclear reactions (LENR) within palladium lattices. The model centres on the formation of a tetrahedral cluster of deuterons, enhancing the probability of nuclear fusion. Here, we explore the possibility of extending this framework by considering the TSC within a more general crystalline solid with tetrahedral symmetry, and by approximating the screening potential experienced by the deuterons using PT-symmetric potentials.

KEYWORDS

Low energy nuclear reaction (LENR), Tetrahedral Symmetric Condensate, condensed matter cluster nuclear science, fusion energy, PT-symmetric potential, crystal solid, palladium lattices.

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INTRODUCTION: TAKAHASHI'S (TSC) MODEL

Akito Takahashi's Tetrahedral Symmetric Condensate (TSC) model, detailed in several of his earlier works proposes a mechanism for low-energy nuclear reactions (LENR) within palladium lattices (ICCF-19, 2015). The model centres on the formation of a tetrahedral cluster of deuterons, enhancing the probability of nuclear fusion.

Prof. Takahashi's TSC model postulates that under specific conditions within a palladium lattice, four deuterons can arrange themselves in

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[1] cf. for instance, *Proc. ICCF-19*, Prof. A. Takahashi, vixra.org/1504.0191v1 (2015).

a tetrahedral configuration. This arrangement enhances the overlap of their wave functions, significantly increasing the probability of fusion. The model emphasizes the importance of the lattice structure in facilitating this condensation.

In the present article, we explore the possibility of extending this framework by considering the TSC within a more general crystalline solid with tetrahedral symmetry, and by approximating the screening potential experienced by the deuterons using PT-symmetric potential. The present article can be considered as a follow up to our previous article at *Infinite Energy Magazine*, 2008 [2].

GENERALIZING TO TETRAHEDRAL SYMMETRIC CRYSTALS

We can envision the TSC model as a specific manifestation of a more general phenomenon within crystalline solids that exhibit tetrahedral symmetry. Such solids, beyond palladium, may offer environments conducive to the formation of similar tetrahedral clusters.

Starting with hypothesizing that Akito Takahashi's Tetrahedral Symmetric Condensate (TSC) model might be a specific instance of a broader physical phenomenon, we can consider crystalline solids as a general framework. In essence, the TSC model suggests that specific arrangements of deuterons, particularly the tetrahedral configuration, are favoured within the lattice structure of certain materials. Expanding on this hypothesis, we propose that crystalline solids exhibiting tetrahedral symmetry, extending beyond just palladium, may provide an environment where similar tetrahedral clusters can form and stabilize. The critical factor can be found in the lattice's inherent ability to generate potential wells, which act as traps, facilitating the assembly and maintenance of these clusters.

UNDERSTANDING CRYSTALLINE SOLIDS

Crystalline solids are characterized by their highly ordered, repeating atomic or molecular structure. Unlike amorphous solids, where atoms are arranged randomly, crystalline materials possess a long-range order, forming a regular, three-dimensional lattice. This lattice structure is defined by a unit cell, the smallest repeating unit that, when translated in three dimensions, generates the entire crystal.

Key characteristics of crystalline solids include:

- Long-Range Order: Atoms or molecules are arranged in a periodic pattern that extends throughout the material.
- Lattice Structure: The regular arrangement of atoms forms a lattice, which can be described by a unit cell.
- Defined Melting Point: Crystalline solids have a sharp melting point, as the ordered structure breaks down at a specific temperature.
- Anisotropy: Many physical properties of crystalline solids, such as conductivity and mechanical strength, can vary depending on the direction within the crystal.

TETRAHEDRAL SYMMETRY IN CRYSTALLINE LATTICES

Tetrahedral symmetry, a specific type of spatial arrangement, is characterized by four equivalent vertices forming a tetrahedron. This symmetry is observed in various crystalline structures, including diamond cubic, zinc blende, and certain metallic alloys. When a crystal lattice exhibits this symmetry, it can create potential wells that favour the formation of tetrahedral clusters.

These potential wells arise from the electrostatic interactions between the atoms or ions in the lattice. The arrangement of atoms creates regions of lower potential energy, where other atoms or ions can be trapped. In the context of Takahashi's TSC model, these potential wells can trap deuterons, allowing them to form the tetrahedral clusters necessary for enhanced fusion.

POTENTIAL WELLS AND CLUSTER FORMATION

The depth and shape of the potential wells are crucial for the formation and stability of tetrahedral clusters. Deeper wells can confine the deuterons more tightly, increasing the probability of their interaction. The shape of the well can also influence the arrangement of the deuterons, favoring the tetrahedral configuration.

The lattice's ability to create these potential wells is influenced by:

- **Atomic Spacing:** The distance between atoms in the lattice determines the size and shape of the potential wells.
- **Electronic Configuration:** The electronic configuration of the atoms influences the electrostatic interactions and, consequently, the potential landscape.
- **Lattice Vibrations (Phonons):** Phonons can modulate the potential wells, affecting the stability of the tetrahedral clusters.
- **Defects and Impurities:** Lattice defects and impurities can create additional potential wells or modify existing ones, influencing the formation of clusters.

IMPLICATIONS FOR LOW-ENERGY NUCLEAR REACTIONS

By recognizing that the TSC model might be a specific occurrence within a broader class of tetrahedral symmetric crystalline solids, we open up new avenues for exploring LENR. This approach allows us to consider a wider range of materials and lattice structures that might support the formation of tetrahedral clusters.

Furthermore, it emphasizes the importance of understanding the potential landscape within crystalline solids. By accurately modelling the potential wells, we can predict and optimize the conditions for enhanced fusion.

In conclusion, crystalline solids with tetrahedral symmetry provide a promising framework for understanding and exploring the TSC model. The lattice's ability to create potential wells is crucial for the formation and stabilization of tetrahedral clusters, opening up new possibilities for low-energy nuclear research.

PT-SYMMETRIC POTENTIAL APPROXIMATION

To simplify the analysis of the screening potential experienced by the deuterons within the crystalline lattice, we can approximate it using a PT-symmetric potential. PT-symmetric quantum mechanics, pioneered by Carl Bender, allows for the existence of real eigenvalues even when the Hamiltonian is not Hermitian, provided it satisfies PT symmetry (parity and time-reversal symmetry).

A sinusoidal potential, which can be tailored to exhibit PT symmetry, offers a tractable model for simulating the crystalline environment. Consider a potential of the form:

$$V(x)=V_0.\sin(kx)+i.V_1.\cos(kx) \qquad (1)$$

where V_0 and V_1 are real constants, and k is the wave number. This potential is PT-symmetric because:

$$V(-x)=V_0.\sin(-kx)+i.V_1.\cos(-kx)=-V_0 .\sin(kx)+i.V_1.\cos(kx) \qquad (2)$$

$$V*(-x)=-V_0.\sin(kx)-i.V_1.\cos(kx)=V(-x) \qquad (3)$$

This potential can be used to model the periodic potential landscape experienced by the deuterons within the crystalline lattice. The imaginary component $iV_1\cos(kx)$ can represent the gain and loss effects that might arise from interactions with the lattice vibrations or other excitations.

CONNECTING TSC AND PT-SYMMETRIC QM

The application of PT-symmetric potentials to the crystalline environment surrounding the TSC offers several advantages:

1. Simplified Modeling: The sinusoidal potential provides a relatively simple model for the complex screening potential within the crystal.
2. Real Eigenvalues: PT-symmetric potentials can yield real eigenvalues, corresponding to stable energy states of the deuteron cluster.
3. Enhanced Wave Function Overlap: The specific form of the PT-symmetric potential can be tuned to enhance the overlap of the deuteron wave functions, thereby increasing the fusion probability.
4. Lattice Interactions: PT-symmetric interactions can model lattice effects, such as phonons and other excitations, that influence the fusion process.

ANALYZING THE PT-SYMMETRIC POTENTIAL AND ITS IMPLICATIONS FOR LOW-ENERGY FUSION

The potential described by equations (2) and (3) is:

$$V(x)=V_0.\sin(kx)+i.V_1.\cos(kx)$$

This potential possesses PT symmetry, meaning it remains invariant under simultaneous parity (P: x→−x) and time-reversal (T: i→−i) operations. Let's break down its properties and implications:

1. PT Symmetry:

- **Parity (P):** $V(-x)=-V_0.\sin(kx)+i.V_1.\cos(kx)$
- **Time-Reversal (T):** $V*(-x)=-V_0.\sin(kx)-i.V_1.\cos(kx)$
- **PT Symmetry:** $V(-x)=V*(-x)$

This symmetry ensures that the Hamiltonian, when constructed with this potential, can have real eigenvalues, which correspond to stable energy states. This is crucial for maintaining the stability of the tetrahedral deuteron cluster.

2. Physical Interpretation:

- The real part, V0sin(kx), represents a periodic potential, akin to the lattice potential in a crystal.
- The imaginary part, iV1cos(kx), introduces gain and loss effects, which can model interactions with lattice vibrations (phonons), defects, or other excitations. These interactions can influence the tunneling probability and, consequently, the fusion rate.

3. Schrödinger Equation:

The time-independent Schrödinger equation for a deuteron in this potential is:

$$-2m.\hbar^2dx^{2\cdot}d^2\psi(x)+V(x).\psi(x)=E\psi(x) \quad (4)$$

where:

- $\hbar$ is the reduced Planck constant.
- m is the deuteron mass.
- ψ(x) is the deuteron wave function.
- E is the energy eigenvalue.

MATHEMATICA CODE FOR SOLVING THE AFOREMENTIONED EQUATION CORRESPONDING TO CRYSTALLINE SOLID

Here's the Mathematica code to solve the Schrödinger equation numerically:

Mathematica (outline only)

```
Clear["Global`*"]; (* --- Parameters for the PT-symmetric Sinusoidal Potential in a Crystalline Solid --- *) (* Define the potential. For a crystalline solid, it's periodic. *) (* Let's assume a lattice constant 'a'. V[x] = V[x + a] *) (* The potential is PT-symmetric if V[x] = V*[-x] and V[x] is even, and V[x] is real *) (* or V[x] = V*[-x] holds for the complex part. *) (* For a sinusoidal potential, V[x] = V0 Sin[k x] + I V1 Cos[k x] is PT-symmetric if k is chosen such that *) (* k a = 2 Pi n for some integer n, meaning it's periodic over 'a'. *) a = 2*Pi; (* Lattice constant (unit cell size). Chosen to match kValue=1 *) kValue = 2*Pi/a; (* Wave number, chosen so that Sin[kValue*x] and Cos[kValue*x] are periodic with period 'a' *) V0Value = 1; (* Real potential strength *) V1Value = 0.5; (* Imaginary potential strength for PT-symmetry *) V[x_] := V0Value*Sin[kValue*x] + I*V1Value*Cos[kValue*x]; (* Other physical constants *) hbar = 1; (* Reduced Planck constant, for simplicity *) m = 1; (* Mass, for simplicity *) (* --- Solving the Schrödinger Equation with Bloch Boundary Conditions --- *) (* For a crystalline solid, we solve over one unit cell, say from 0 to 'a'. *) (* We need to apply Bloch's theorem: psi[x + a] = Exp[I*K*a] * psi[x] *) (* and psi'[x + a] = Exp[I*K*a] * psi'[x], where K is the Bloch wave vector (quasi-momentum). *) (* We'll iterate over K values to find the energy bands. *) (* Range for Bloch wave vector K (typically from -Pi/a to Pi/a, the first Brillouin zone) *) KValues = Range[-Pi/a, Pi/a, Pi/(10*a)]; (* Example: 21 K points in the first Brillouin zone *) Print["Calculating Energy Bands for PT-symmetric Sinusoidal Potential..."]; energyBands = Monitor[ Table[ (* Define the differential equation *) schrodingerEq = -hbar^2/(2*m)*psi''[x] + V[x]*psi[x] == E*psi[x]; (* Set up the boundary conditions based on Bloch's theorem *) (* We need to solve a generalized eigenvalue problem. *) (* NDEigenSystem is ideal for this if the boundary conditions can be expressed in its form. *) (* For more complex boundary conditions like Bloch's, we can use NDSolve's capabilities to find E for a given K. *) (* This approach uses NDSolve with a search for E that satisfies Bloch conditions. *) (* This is more involved and might require a root-finding approach or manual iteration for E. *) (* A common alternative for band structures is to use transfer matrices or plane wave expansion. *) (* Let's simplify by using NDEigenSystem on a finite interval with periodic-like conditions *) (* (This isn't *exactly* Bloch's theorem for arbitrary K, but a common approximation or a specific case). *) (* For a true Bloch approach, you'd typically look for values of E where the transfer matrix over one unit cell has eigenvalues of Exp[I K a] *) (* Here's a common numerical approach for periodic potentials: *) (* We define a trial solution and use NDSolve, then find E that
```

satisfies the Bloch condition *) (* This is typically done by looking for the eigenvalues of a matrix derived from the discretized Hamiltonian. *) (* For simplicity and to fit the NDSolve paradigm somewhat, let's consider the problem on one unit cell. *) (* We will search for E such that psi[a] == Exp[I K a] psi[0] and psi'[a] == Exp[I K a] psi'[0] *) (* This is a numerical root-finding problem for E. Let's try to set up an objective function. *) (* For illustrative purposes, and because NDEigenSystem has simpler BCs, I'll show a simpler case where we find eigenvalues for a *periodic* system (K=0 or K=Pi/a equivalent) or a single-well bound state if we were to treat it as a finite system. *) (* To find Bloch waves, one common technique is to search for eigenvalues E for which a fundamental set of solutions {phi1[x], phi2[x]} (e.g., phi1[0]=1,phi1'[0]=0; phi2[0]=0,phi2'[0]=1) satisfies the Bloch condition at x=a. This leads to a determinant equation for E. *) (* Method using `WhenEvent` to find eigenvalues that satisfy Bloch conditions: *) (* This is still complex with `NDSolve`. A more robust way for bands is a matrix method. *) (* Let's illustrate with a direct NDEigenSystem for periodic potential (K=0) for a basis. *) (* This will give discrete eigenvalues, which for a periodic system correspond to band edges (K=0 or K=Pi/a). *) (* For a true band structure, you'd solve for psi[a] and psi'[a] in terms of psi[0] and psi'[0] *) (* using NDSolve, then set up a 2x2 matrix M such that {psi[a], psi'[a]} = M . {psi[0], psi'[0]}. *) (* The eigenvalues of M are Exp[I K a]. You then solve for E given the K. *) (* Re-evaluating the best approach within NDSolve/NDEigenSystem for Bloch waves. *) (* `NDEigenSystem` can handle periodic boundary conditions, which is a specific case of Bloch's theorem (K=0 or K=Pi/a). *) (* For a generic K, `NDSolve` is used to propagate, then a secular equation is solved. *) (* Let's use a simpler NDEigenSystem approach to find energy levels within one unit cell that would correspond to band edges (e.g., Gamma point K=0). *) If[Abs[K] == 0, (* Gamma point: K=0, so psi[a]=psi[0] and psi'[a]=psi'[0] *) {vals, funcs} = NDEigenSystem[{-hbar^2/(2*m)*psi''[x] + V[x]*psi[x], DirichletCondition[psi[x] == psi[x + a], x == 0 || x == a]}, psi[x], {x, 0, a}, 5, (* Requesting 5 eigenvalues/functions *) Method -> {"SpatialDiscretization" -> {"FiniteElement", {"MeshOptions" -> MaxCellMeasure -> a/50}}}, AccuracyGoal -> 4]; If[Length[vals] > 0, {K, Sort[Re[vals]]}, (* Only store real part of eigenvalues for plotting *) {K, {}}], (* For K != 0, we need to implement Bloch conditions directly. This is more involved. *) (* A common method is to use a shooting method or find the determinant of a boundary condition matrix. *) (* For demonstration, let's keep it simple and just show the K=0 case. *) (* For other K, the energies will typically split and form bands. *) (* As an alternative for a general K, one would set up a matrix whose determinant must be zero. *) (* This involves two independent solutions of the Schrödinger equation for a given E and K. *) (* Let psi1[x] be solution with psi1[0]=1, psi1'[0]=0 and psi2[x] with psi2[0]=0, psi2'[0]=1 *) (* The condition is: Det[{{psi1[a] - Exp[I K a], psi2[a]}, {psi1'[a], psi2'[a] - Exp[I K a]}}] == 0 *) (* This requires finding roots E for each K. *) (* Given the constraints, I will provide a framework that aims for band structure, *) (* but a full, robust solution for arbitrary K is quite involved for a general potential *) (* within NDSolve directly without a numerical matrix method. *) (* Let's provide a placeholder for a more advanced Bloch calculation: *) (* A common way is to solve for a given E and then check if the Bloch conditions are met *) (* by propagating the solution across the unit cell. *) (* This requires finding roots of an implicit function, which is not straightforward with NDSolve alone. *) (* For a full band structure, the Transfer Matrix Method is often used for 1D. *) (* For a given E, propagate {psi[0], psi'[0]} to {psi[a], psi'[a]} using NDSolve. *) (* This gives a transfer matrix T(E) = {{m11, m12}, {m21, m22}}. *) (* Eigenvalues of T(E) are Exp[I K a]. So, we solve for E when eigenvalues are on unit circle. *) (* Or, Trace[T(E)] = 2 Cos[K a]. So, find E for each K from this equation. *) (* Let's implement the Trace[T(E)] method for finding E for each K. *) (* This involves an `NDSolve` call inside a `FindRoot` or `Solve` for E. *) (* Define a function to calculate the transfer matrix for a given E *) transferMatrix[energy_] := Module[{sol, psi0, psiP0, psiA, psiPA}, sol = NDSolveValue[{-hbar^2/(2*m)*psi''[x] + V[x]*psi[x] == energy*psi[x], psi[0] == #[[1]], psi'[0] == #[[2]]}, {psi[a], psi'[a]}, {x, 0, a}] &; (* Construct the transfer matrix by solving for two linearly independent initial conditions *) {{sol[{1, 0}][[1]], sol[{0, 1}][[1]]}, {sol[{1, 0}][[2]], sol[{0, 1}][[2]]}}]; (* Define the secular equation based on

Bloch's theorem for a given K *) (* Trace[TransferMatrix[E]] == 2 Cos[K a] *) secularEquation[energy_, K_] := Trace[transferMatrix[energy]] - 2*Cos[K*a]; (* Find the energy eigenvalues for the current K *) (* We need initial guesses for FindRoot. Let's try to find a few low-lying bands. *) (* This can be tricky. A range of initial guesses might be needed. *) (* For simplicity, let's assume we are looking for the lowest few bands. *) (* Initial guess range for E. This is crucial and problem-dependent. *) eGuessRange = {-2, 2}; (* Find roots of the secular equation. We'll use `FindRoot` and iterate. *) foundEnergies = {}; For[eTrial = eGuessRange[[1]], eTrial <= eGuessRange[[2]], eTrial += 0.2, Quiet[(* Suppress warnings for now if FindRoot fails *) res = FindRoot[secularEquation[Evalue, K] == 0, {Evalue, eTrial}, Method -> "Newton", AccuracyGoal -> 5, PrecisionGoal -> 5, MaxIterations -> 100]; If[Head[res] === Rule, (* Check if FindRoot returned a solution *) newE = Evalue / . res; (* Ensure the energy is real and not a duplicate *) If[Abs[Im[newE]] < 10^-5 && !MemberQ[foundEnergies, Round[Re[newE], 0.01]], AppendTo[foundEnergies, Round[Re[newE], 0.01]] (* Round to avoid floating point duplicates *)]]]]; {K, Sort[foundEnergies]}], {K, KValues} , "Calculating for K=" <> ToString[K]], "Finished calculations."]; (* --- Plotting the Energy Bands (E vs K) --- *) ListPlot[Flatten[Table[{K, EVal}, {K, energyBands[[All, 1]]}, {EVal, energyBands[[All, 2]]}], 1], Joined -> False, (* Don't join points if bands are disconnected *) PlotStyle -> PointSize[Medium], AxesLabel -> {"K", "Energy (E)"}, PlotLabel -> "Energy Bands for PT-symmetric Sinusoidal Potential", GridLines -> Automatic, PlotRange -> All] Print["Calculated Energy Bands (K, Energies): ", Column[energyBands]]; (* --- Further Exploration: Plotting a Wavefunction for a Specific E and K (Optional) --- *) (* To plot a Bloch wavefunction, you'd pick a specific K and one of the calculated E values. *) (* Then, solve the Schrödinger equation with that E and K, using the Bloch boundary conditions. *) (* This would involve setting up `NDSolve` with the correct initial conditions derived from the Bloch condition. *) (* Example: Let's pick the lowest energy at K=0 (if found) and plot its real and imaginary parts. *) (* This is complex because we need to find the correct initial conditions (psi[0], psi'[0]) *) (* that yield a solution satisfying psi[a] == Exp[I K a] psi[0] etc. *) (* For the lowest energy at K=0: *) lowestK0Energy = Select[energyBands, #[[1]] == 0 &][[1, 2, 1]]; (* Assumes K=0 exists and has energies *) If[NumberQ[lowestK0Energy], Print["Plotting wavefunction for K=0, E=", lowestK0Energy]; (* To find the actual wavefunction, we need to solve the homogeneous equation *) (* and ensure the Bloch conditions are met. This means finding the correct psi[0], psi'[0]. *) (* For K=0, psi[a]=psi[0] and psi'[a]=psi'[0]. *) (* We need to find initial conditions {psi0, psiP0} such that: *) (* m11 psi0 + m12 psiP0 = psi0 *) (* m21 psi0 + m22 psiP0 = psiP0 *) (* This is an eigenvalue problem for the transfer matrix T - I. Find null space. *) tm = transferMatrix[lowestK0Energy]; identity = IdentityMatrix[2]; (* Find the null space of (tm - identity) *) (* {psi0, psiP0} is the eigenvector with eigenvalue 1. *) eigenvecs = Eigenvectors[tm, -1]; (* Get eigenvectors for eigenvalue 1 *) If[Length[eigenvecs] > 0, initialConditions = eigenvecs[[1]]; (* Pick one non-trivial eigenvector *) (* Normalize the wavefunction, e.g., psi[0]=1 *) initialConditions = initialConditions / initialConditions[[1]] /. {Indeterminate -> 1, ComplexInfinity -> 1}; (* Handle potential division by zero *) (* Solve the Schrödinger equation with these initial conditions *) psiSolution = NDSolveValue[{-hbar^2/(2*m)*psi"[x] + V[x]*psi[x] == lowestK0Energy*psi[x], psi[0] == initialConditions[[1]], psi'[0] == initialConditions[[2]]}, psi, {x, 0, a}]; Plot[{Re[psiSolution[x]], Im[psiSolution[x]], Re[V[x]]}, {x, 0, a}, PlotLegends -> {"Re[psi]", "Im[psi]", "Re[V]"}, PlotLabel -> StringJoin["Wavefunction for K=0, E=", ToString[lowestK0Energy]], PlotRange -> All, GridLines -> Automatic] , Print["Could not find initial conditions for K=0, E=", lowestK0Energy]] , Print["K=0 energy not found or invalid."]];

ESTIMATING FUSION RATE

To estimate the fusion rate, we need to consider the tunnelling probability through the potential barrier. This involves:

1. **Calculating the Wave Function:** The Mathematica code provides the wave function ψ(x).

2. **Determining Tunneling Probability:** The tunneling probability can be estimated by analyzing the wave function's behavior in the classically forbidden region.
3. **Applying Gamow Factor:** The Gamow factor, which depends on the Coulomb barrier and the effective potential, can be used to estimate the fusion rate.

   However, to precisely calculate the fusion rate requires a more complex model that considers:

- The Coulomb barrier between deuterons.
- The overlap integral of the deuteron wave functions.
- The nuclear reaction cross-section.
- The effects of the lattice.

FALACO SOLITONS AND ENHANCED TUNNELLING; ESTIMATING LOW-ENERGY FUSION RATES IN PT-SYMMETRIC POTENTIALS

The prospect of enhancing tunnelling probability in low-energy nuclear reactions (LENR) through PT-symmetric potentials offers an intriguing avenue for exploration. As previously discussed, the PT-symmetric potential, defined by $V(x)=V_0.\sin(kx)+i.V_1.\cos(kx)$, can influence the wave function's behaviour, particularly in regions where tunnelling is critical. The imaginary component of this potential, iV1.cos(kx), plays a crucial role in this influence, potentially increasing the overlap of deuteron wave functions and, consequently, boosting the fusion rate. This enhancement, combined with the Gamow factor, directly impacts the fusion rate, which is proportional to the square of the wave function overlap. By strategically adjusting the parameters V_0, V_1, and k, we can investigate conditions that optimize this fusion rate.

However, let's introduce a further layer of complexity and potential enhancement by considering the possibility of tunnelling facilitated by a mechanism we'll term "Falaco solitons." (cf. R.M. Kiehn). These hypothetical solitons, analogous to nonlinear wave packets, could propagate through the crystalline lattice, carrying the deuterons and significantly increasing their tunnelling probability. The Falaco soliton concept suggests that the deuterons, rather than tunnelling as individual particles, could tunnel coherently as a soliton-like entity, drastically increasing the tunnelling probability.

CONSIDERING FALACO SOLITONS INTO TUNNELLING PROBABILITY ESTIMATION

To estimate the tunnelling probability with the Falaco soliton mechanism, we need to modify our approach [3-4]. Instead of calculating the tunnelling probability for a single deuteron, we must consider the soliton's wave function and its interaction with the PT-symmetric potential.

Here's a conceptual outline of how we can approach this:

1. Soliton Wave Function: We need to model the Falaco soliton's wave function, which will likely be a nonlinear solution to a modified Schrödinger equation.
2. Soliton-Potential Interaction: We must analyse the interaction between the soliton and the PT-symmetric potential, considering the soliton's collective behaviour.
3. Tunnelling Probability Calculation: The tunnelling probability will be determined by the soliton's transmission through the potential barrier, considering its shape and the soliton's properties.
4. Fusion Rate Estimation**:** The fusion rate will still be proportional to the square of the soliton's wave function overlap and the Gamow factor, but now considering the soliton's properties.

MATHEMATICA CODE FOR ESTIMATING TUNNELLING PROBABILITY OF LOW ENERGY FUSION BY ASSUMING FALACO SOLITON MECHANISM

Since we are introducing a hypothetical soliton, we will need to create a model soliton wave function. For the purpose of this example, we will model a nonlinear wave packet using a Gaussian shape with a nonlinear phase term.

```
Clear["Global`*"]; (* Define the PT-symmetric
potential *) V[x_, V0_, V1_, k_] := V0*Sin[k*x] +
I*V1*Cos[k*x]; (* Define the Falaco soliton wave
function (Gaussian with nonlinear phase) *)
solitonWave[x_, A_, sigma_, alpha_, kSoliton_] :=
A*Exp[-x^2/(2*sigma^2)]*Exp[I*(kSoliton*x +
alpha*Abs[x]^2)]; (* Define parameters *) hbar =
1; (* Reduced Planck constant, for simplicity *) m
```

```
= 1; (* Soliton mass, for simplicity *) V0Value = 1;
(* Real potential strength *) V1Value = 0.5; (*
Imaginary potential strength *) kValue = 1; (*
Wave number for potential *) ASoliton = 1; (*
Soliton amplitude *) sigmaSoliton = 2; (* Soliton
width *) alphaSoliton = 0.2; (* Nonlinear phase
term *) kSolitonValue = 1; (* Soliton wave number
*) (* Define the Schrödinger equation with the
soliton wave function *) schrodingerEquation = -
hbar^2/(2*m)*D[solitonWave[x, ASoliton,
sigmaSoliton, alphaSoliton, kSolitonValue], {x,
2}] + V[x, V0Value, V1Value,
kValue]*solitonWave[x, ASoliton, sigmaSoliton,
alphaSoliton, kSolitonValue]; (* Numerical
solution of the Schrodinger Equation, to examine
the energy *) energySoliton =
ParametricNDSolveValue[ {schrodingerEquation
== E*solitonWave[x, ASoliton, sigmaSoliton,
alphaSoliton, kSolitonValue], solitonWave[0,
ASoliton, sigmaSoliton, alphaSoliton,
kSolitonValue] == ASoliton,
Derivative[1][solitonWave][0, ASoliton,
sigmaSoliton, alphaSoliton, kSolitonValue] ==
I*kSolitonValue*ASoliton}, E, {x, -10, 10}, {E}]; (*
Plot the soliton wave function and potential *)
Plot[{Re[solitonWave[x, ASoliton, sigmaSoliton,
alphaSoliton, kSolitonValue]], Im[solitonWave[x,
ASoliton, sigmaSoliton, alphaSoliton,
kSolitonValue]], Re[V[x, V0Value, V1Value,
kValue]]}, {x, -10, 10}, PlotLegends ->
{"Re[Soliton]", "Im[Soliton]", "Re[V]"}, PlotRange
-> All] Print["Soliton Energy: ", energySoliton]; (*
Tunneling probability estimation (simplified) *)
tunnelingProbability = Abs[solitonWave[10,
ASoliton, sigmaSoliton, alphaSoliton,
kSolitonValue]/ solitonWave[-10, ASoliton,
sigmaSoliton, alphaSoliton, kSolitonValue]]^2;
Print["Estimated Tunneling Probability: ",
tunnelingProbability];
```

Result of simulation:

$$\text{Soliton Energy: "ParametricNDSolveValue}[\{e^{-\frac{x^2}{8}+i(x+0.2\text{Abs}[x]^2)}((0. + 0.5i)\text{Cos}[x] + \text{Sin}[x]) + \frac{1}{2}(-e^{-\frac{x^2}{8}+i(x+0.2\text{Abs}[x]^2)}(-\frac{x}{4} + i(1 + 0.4\text{Abs}[x]\text{Abs}'[x]))^2 - e^{-\frac{x^2}{8}+i(x+0.2\text{Abs}[x]^2)}(-\frac{1}{4} + i(0.4\text{Abs}'[x]^2 + 0.4\text{Abs}[x]\text{Abs}''[x]))) =$$

$$= e^{1-\frac{x^2}{8}+i(x+0.2\text{Abs}[x]^2)}, \text{True}, \text{solitonWave}'[0,1,2,0.2,1] == i\}, e, \{x, -10,10\}, \{e\}]\text{Soliton Energy:}$$

"Estimated Tunneling Probability: 1"

FURTHER QUESTION

A particular question to ask is whether the approach considered here of PT-symmetric potential is approximately close to known nucleus-nucleus potentials, such as Woods-Saxon potential (see for instance: Jaqaman & Makjian, also Romaniega *et al*, ref. [9-12]).

We explore this question as follows, for instance by assuming near the centre of nucleus, of around 5-6 fm radii.

The goal is to see if, for reasonable parameters, our PT-symmetric potential can approximate the shape of a Woods-Saxon potential in a specific region (e.g., near the center of a nucleus, as suggested by Akito Takahashi's work).

Here's the plan:

1. Define the Woods-Saxon Potential: This is a well-known phenomenological potential used in nuclear physics.
2. Define the PT-Symmetric Sinusoidal Potential: We'll use the form: V_0*Sin[k*x] + I*V_1*Cos[k*x].
3. Choose Parameters: Select realistic parameters for the Woods-Saxon potential (radius, diffuseness, depth). For the PT-symmetric potential, we shall try to adjust V_0, V_1, and k to best fit the real part of the Woods-Saxon potential in the specified region. The imaginary part is a new feature of the PT-symmetric potential, so we can visualize its impact separately.
4. Plot and Compare: Visualize both potentials on the same graph, focusing on the real parts, within the specified radial range (e.g., 0 to 6 nm, though nuclear radii are typically in femtometers, so we'll use consistent units). We can also plot the imaginary part of the PT-symmetric potential.
5. Quantitative Comparison (Optional but good): We could use a fitting routine (e.g., NonlinearModelFit) to find the "best" parameters for the PT-symmetric potential to approximate the Woods-Saxon, but for visual comparison, manual adjustment is often a good start.

Mathematica code (outline only)

Clear["Global`*"]; (* --- 1. Define the Woods-Saxon Potential --- *) (* Woods-Saxon potential parameters *) (* Typical nuclear parameters: R in fm, a in fm, V0 in MeV *) (* Since we're asked about 5-6 nm, we need to adjust the scale. *) (* Let's assume the "nucleus" in the cluster context has a larger effective radius. *) (* Woods-Saxon Potential: V_WS(r) = -V0 / (1 + Exp[(r - R) / a]) *) (* V_WS is typically negative for an attractive potential *) WS_V0 = 50; (* Depth of the potential well (e.g., in arbitrary units like MeV or eV) *) WS_R = 5; (* Radius of the potential (e.g., in nm, to match 5-6 nm discussion) *) WS_a = 0.5; (* Diffuseness parameter (e.g., in nm) *) WoodsSaxon[r_, v0_, R_, a_] := -v0 / (1 + Exp[(r - R) / a]); (* --- 2. Define the PT-Symmetric Sinusoidal Potential --- *) (* V_PT(x) = V0_PT * Sin[k_PT * x] + I * V1_PT * Cos[k_PT * x] *) (* Parameters for the PT-symmetric potential *) (* We will try to adjust these to visually match the Woods-Saxon potential *) PT_V0 = 25; (* Real part strength *) PT_V1 = 10; (* Imaginary part strength *) PT_k = Pi / 3; (* Wave number. Adjust to control oscillation frequency over the range *) PTSymmetricSinusoidal[x_, v0_PT_, v1_PT_, k_PT_] := v0_PT*Sin[k_PT*x] + I*v1_PT*Cos[k_PT*x]; (* --- 3. Explore the comparison range --- *) (* As suggested, near the center of the nucleus, around 5-6 nm radii. *) (* This implies we are looking at the effective potential for the cluster. *) (* Let's plot from 0 to 10 nm to see the full shape. *) comparisonRange = {x, 0, 10}; (* x in nm *) centerRange = {x, 3, 7}; (* Focus region for comparison, around 5-6 nm *) (* --- 4. Plot and Compare --- *) (* Generate data for the potentials *) wsData = Table[{x, WoodsSaxon[x, WS_V0, WS_R, WS_a]}, {x, 0, 10, 0.05}]; ptRealData = Table[{x, Re[PTSymmetricSinusoidal[x, PT_V0, PT_V1, PT_k]]}, {x, 0, 10, 0.05}]; ptImagData = Table[{x, Im[PTSymmetricSinusoidal[x, PT_V0, PT_V1, PT_k]]}, {x, 0, 10, 0.05}]; (* Plotting the real parts of the potentials *) plotReal = Plot[{WoodsSaxon[x, WS_V0, WS_R, WS_a], Re[PTSymmetricSinusoidal[x, PT_V0, PT_V1, PT_k]]}, comparisonRange, PlotLegends -> {"Woods-Saxon Potential", "Real Part of PT-symmetric Sinusoidal Potential"}, PlotStyle -> {Directive[Thick, Blue], Directive[Dashed, Red]}, PlotLabel -> "Comparison of Real Potential Shapes", AxesLabel -> {"Radius (nm)", "Potential"}, GridLines -> Automatic, PlotRange -> All]; (* Plotting the imaginary part of the PT-symmetric potential *) plotImaginary = Plot[Im[PTSymmetricSinusoidal[x, PT_V0, PT_V1, PT_k]], comparisonRange, PlotLegends -> {"Imaginary Part of PT-symmetric Sinusoidal Potential"}, PlotStyle -> {Directive[Thick, Green]}, PlotLabel -> "Imaginary Part of PT-symmetric Sinusoidal Potential", AxesLabel -> {"Radius (nm)", "Potential (Imaginary)"}, GridLines -> Automatic, PlotRange -> All]; (* Combine plots for a comprehensive view *) combinedPlot = Show[plotReal, Plot[Im[PTSymmetricSinusoidal[x, PT_V0, PT_V1, PT_k]], comparisonRange, PlotStyle -> {Directive[Thick, Green, Dotted]}, PlotLegends -> {"", "", "Imaginary Part of PT-symmetric Sinusoidal Potential"} (* Add legend manually *)], PlotLabel -> "Comparison of PT-symmetric Sinusoidal and Woods-Saxon Potentials", PlotLegends -> {"Woods-Saxon Potential", "Re[PT-symmetric Sinusoidal]", "Im[PT-symmetric Sinusoidal]"}]; Print["Visual Comparison of Potential Shapes:"]; Print[combinedPlot]; (* --- Optional: Focusing on the "Center of Nucleus" region (5-6 nm) --- *) focusedPlot = Plot[{WoodsSaxon[x, WS_V0, WS_R, WS_a], Re[PTSymmetricSinusoidal[x, PT_V0, PT_V1, PT_k]]}, centerRange, PlotLegends -> {"Woods-Saxon Potential", "Real Part of PT-symmetric Sinusoidal Potential"}, PlotStyle -> {Directive[Thick, Blue], Directive[Dashed, Red]}, PlotLabel -> "Comparison in Focused Region (3-7 nm)", AxesLabel -> {"Radius (nm)", "Potential"}, GridLines -> Automatic, PlotRange -> All]; Print["Focused Visual Comparison around 5-6 nm:"]; Print[focusedPlot]; (* --- Discussion --- *) Print["\nDiscussion:"]; Print["The Woods-Saxon potential describes a realistic attractive potential well for nucleus-nucleus interaction."]; Print["Its parameters (depth, radius, diffuseness) can be adjusted to model various systems."]; Print[""]; Print["The PT-symmetric sinusoidal potential (Re[V_PT] = V0_PT*Sin[k_PT*x], Im[V_PT] = V1_PT*Cos[k_PT*x])"]; Print["is fundamentally different in shape. It oscillates and has an imaginary component."]; Print[""]; Print["When comparing the *real part* of the PT-symmetric sinusoidal potential to the Woods-Saxon potential:"]; Print["- The PT-symmetric potential is inherently oscillatory. It cannot accurately reproduce the smooth, single-well shape of the

Woods-Saxon potential across a broad range."]; Print["- However, in a *very localized region*, such as around a specific radius (e.g., 5-6 nm as specified), one might be able to tune 'PT_V0', 'PT_k' to make a *segment* of its oscillation roughly match the slope or a part of the Woods-Saxon potential curve."]; Print["- The provided plots show that the real part of the sinusoidal potential can, at best, mimic a *small portion* of the Woods-Saxon potential's shape (e.g., its rising edge or a flat region) depending on 'k' and 'V0_PT'."]; Print["- The imaginary part of the PT-symmetric potential (V1_PT*Cos[k_PT*x]) introduces a non-hermitian term, which is the defining characteristic of PT-symmetry in this context. This part has no direct counterpart in the standard real Woods-Saxon potential and would represent gain/loss in the quantum system."]; Print[""]; Print["Conclusion:"]; Print["The PT-symmetric sinusoidal potential, as defined here, is not generally 'approximately close' to a Woods-Saxon potential in terms of its overall shape for nucleus-nucleus interactions."]; Print["Its oscillatory nature and inherent imaginary component make it fundamentally different from the smooth, real, attractive well of the Woods-Saxon potential."]; Print[""]; Print["If the intention is to model a *local effect* or a *small deviation* from a Woods-Saxon potential using PT-symmetry, then fine-tuning the PT-symmetric potential parameters to match a specific feature or region could be explored further, possibly using optimization algorithms for parameter fitting."]; Print["However, as a general replacement for a Woods-Saxon potential for describing the fundamental attractive nucleus-nucleus interaction over a broad range, it is not a suitable approximation."]; Print[""]; Print["The PT-symmetric potential might be more appropriate for describing specific phenomena like resonant states, open quantum systems, or effective potentials in a many-body environment where gain and loss mechanisms are present, rather than the static average interaction between two nuclei."];

IMPLICATIONS AND FUTURE DIRECTIONS

This conceptual framework suggests that the TSC model may be a specific instance of a broader phenomenon occurring within crystalline solids with tetrahedral symmetry. Approximating the screening potential using PT-symmetric potentials provides a valuable tool for analysing the dynamics of the deuteron clusters.

Future research should focus on:

- Developing more refined PT-symmetric potential models that accurately represent the crystalline environment, that incorporate the Coulomb barrier and lattice effects.
- Investigating the influence of different lattice structures and excitations on the formation and stability of tetrahedral clusters.
- Performing numerical simulations to calculate the wave function overlap and tunnelling probability.
- Performing numerical simulations to validate the theoretical predictions, including experimental validation of the existence of the PT-symmetric screening potential.
- The Falaco soliton model is a hypothetical concept, and further research is necessary to validate its existence and properties [3-4].
- The Mathematica code provided is a simplified example and requires further refinement to precisely model the soliton's behaviour and tunnelling probability.
- The Gamow factor needs to be incorporated into the fusion rate estimation for a more accurate result.
- It can be expected that a much improved result of the above new model of CMNS can be obtained by introducing laser focusing on the tetrahedral crystalline solid in question, i.e. in order to trigger phase transition induced by laser-matter interaction, see for instance ref. [7-8].

By combining theoretical analysis with numerical simulations, we can gain a deeper understanding of the role of PT-symmetric potentials in low-energy nuclear reactions.

CONCLUDING REMARK

By integrating Takahashi's TSC model with the principles of PT-symmetric quantum mechanics, we can develop a more comprehensive understanding of LENR in crystalline solids.

The present article is a follow-up to previous article from us, which first described this argument (Infinite Energy Magazine, 2008). By integrating the Falaco soliton concept with the PT-symmetric potential framework, we can explore new possibilities for enhancing tunneling probability and achieving higher fusion rates in LENR.

This approach can be considered as new avenues for exploring and potentially harnessing low-energy nuclear processes.

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Version 1.0: 6th April 2025
Version 1.1: 7th April 2025
Version 1.2: 28th Jul. 2025
VC, FS

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[2] *Note*: R. M. Kiehn - „The Falaco Effect, A Topological Soliton,‟ Talk at 1987 Dynamics. Days, Austin, Texas, Jan. (1987). See also, Victor Christianto & F. Smarandache. Remark on Falaco Soliton as a Tunneling Mechanism in a Navier Stokes Universe. *SciNexuses* Vol. 1 (2024) 151–159. Also in UNM Digital Repository. url: Remark on Falaco Soliton as a Tunneling Mechanism in a Navier-Stokes Universe