Bulletin of Pure and Applied Science Physics, Vol.42D No.1, January-June 2023 P.6-34
Original Research Article
Print version ISSN 0970 6569
Online version ISSN 2320 3218
DOI: 10.48165/bpas.2023.42D.1.2
Available online at www.bpasjournals.com

Desingularization of Black Hole Space-Times

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Received on 21.01.2023
Revised on 15.02.2023
Approved on 15.05.2023
Accepted on 23.05.2023
Published on 17.06.2023

ABSTRACT The desingularization of a class of black hole space-times arising as solutions to the string equations is considered in connection with the consistency of the quantum theory and the description of nonperturbative states with quantum numbers from the particle spectrum. The geometry arising in an extreme limit of one of the singular solutions to the gravitational field equations is demonstrated to be a background of N = 2 string theory. The positivity of the masses in the particle spectrum is proven through quasilocal integrals near the resolved singularities in these limits of black hole space-times.

KEYWORDS Desingularization, Algebraic Singularities, Higher Dimensions, Quantum Consistency
PACS: 02.40.Xx, 04.20.Dw, 04.60.Cf

How to cite this article: Davis S. (2023). Desingularization of Black Hole Space-Times. *Bulletin of Pure and Applied Sciences- Physics,* 42D (1), 6-34.

1. INTRODUCTION

The universal gravitational attraction of matter may be regarded as the source of the implosion of stars which may reach a critical density such that a closed trapped surface is formed, which prevents the propagation of light beyond the horizon. The effect of gravity is the focussing of a geodesic congruence described by a differential equation of the expansion parameter with respect to arc length [1]. The singularity theorems of general relativity [2][3] predict the occurrence of caustics within a finite arc length and geodesic incompleteness of space-times with the stress tensor of the matter distribution satisfying the dominant energy condition. Furthermore, initial curvature singularities would arise in classical cosmological models derived from the gravitational action.

Quantum effects are considered to provide a possible resolution to the singularity problem. The low-energy limits of superstring and heterotic string theories are known to yield generalizations of the Ricci scalar Lagrangian to quadratic curvature terms [4]. Nonsin- gular classical solutions to the equations of a one-loop quadratic gravity theory derived from heterotic strings have been found over a range of the parameters [5]. The quantum cosmological wavefunction for this model may be evaluated and found to be regular in the limit of initial times with the inclusion of a scalar field [6].

A generalization of the geodesic congruence equation may be defined either through fluctuations of the three-metric or the quantum variations on the space of paths. The first method yields an additional term in the congruence equation [7], while the second technique provides a model yielding the simultaneous equations for the space-time metric and the geodesic [8]. Semiclassical solutions to these equations through the WKB approximation already can be shown to generate geodesic congruences that circumvent caustics [9].

However, the singularities in a manifold can be resolved classically. Any manifold de- fined by algebraic equations may be desingularized [10]. For a Whitney stratified space $X = \prod X_\alpha,$ there exists a smooth $\bar{X} = \prod \bar{X}_\alpha,\ X_\alpha \subset \bar{X}_\alpha,$ and a compact Lie group G, $G \overset{\varphi}{\to} Diff(\bar{X})$ such that $X = \bar{X}/G.$ An example of such a stratified space is $S^1/\sim$ where

let

$$\varphi : S^1 \to S^1,\ \rho : \mathbb{Z} \to Diff(S^1),$$

with

$$\rho(m) = \varphi^m.$$

Therefore, space-times which have the form of an algebraic variety may be desingularized. This method is feasible for black-hole space-times which have conical singularities or fixed points of a discrete group action. Examples include the three-dimensional black hole space-time describing an $SL(2;R)$ Wess-Zumino-Witten model [11] and the Milne orbifold [12]. It does not generically extend to space-times with curvature singularities.

The curvature singularities of a black hole space-time, which is a solution to the equations derived from an effective action, may be resolved in a nonsingular metrics which satisfies equations of a higher-dimensional theory. Nevertheless, it is demonstrated that the conformal invariance of the string theory is not restored in the higher-dimensional space-time. By contrast, the consistency of string theory on a manifold, that is a quotient of a nonsingular space by a discrete group, is valid both for the singular geometry and its cover.

A description of particle states through space-times with divergences in the curvature is problematic because infinities arise and the classical laws are no longer valid in the neighbourhoods of singularities. The geodesic congruences of the Kerr-Schild geometry have been found to model the dynamics of particles with a gyromagnetic ratio equal to 2, independently of the horizon, which would be consistent with the limit to an extreme black hole having zero horizon area [13]. The geometry must be replaced by surfaces with boundaries or algebraic singularities.

2. THE GEOMETRY NEAR A CAUSTIC

For any solution to the field equations of general relativity, caustics develop in geodesic congruences when the energy-momentum tensor has positive components within the future light-cone. Suppose that $\{x_m\}$ is a sequence of points that converges to p, $\{x_{mj}\}$ is a set of points on a neighbouring curve and the set of curves $\{\lambda(m, j)\}$ are defined such that x_{mj} is a limit point of $\{\lambda(m-1, j)_n\}$ and $\{\lambda(m, j-1)_n\}$ and a point of convergence of $\{\lambda(m, j)_n\}$. Let $\lambda'm$ belong to the subsequence $\{\lambda(m, m)_n\}$, which would converge to x_{mm} [14]. If $\lambda'm$ could miss the ball of radius $m^{-1}b$ centered at x_m, $B(x_m, m^{-1}b)$, xm' m would be required to be a limit point only. With a conformal rescaling of the distances that causes $\lambda'm$ to miss the region $B(x_{mj}, m^{-1}b)$, the sequence $\lambda(m, m)_n$ would not necessarily converge to x_{mm} or the point p. By this mechanism, geodesic congruences could avoid caustics.

These results may be confirmed by the geometry of caustics. For an achronal set S in a four-dimensional Lorentzian manifold, the time coordinate x^0 is a Lipschitz coordinate of the spatial coordinates $\{x^i\}$. Therefore, $\{x^0, x^i\}$ represent a coordinate system $S \cap U_a$, where $\{U_a, \varphi_a\}$ is an atlas on M , such that $\varphi_a : S \cap U_a \rightarrow R^4$. Therefore, the geometry of a caustic with a point of convergence on the achronal set can be conformally transformed to a Lorentzian geometry.

When the set is not achronal, x^0 is not necessarily a Lipschitz function of $\{x^i\}$ and curves, which are hyperbolic with the same values of x^i, but different values of x^0 are allowed. These congruences would be conformally transformed to the de Sitter geometry. With an additional Weyl transformation, there exist pinched geometries near a caustic that may be changed to resemble a constant curvature space such as de Sitter space.

It has been proven that, if the second fundamental form satisfies the inequality

$$S \le \frac{6\sqrt{2}}{3+2\sqrt{2}}(1+ \parallel \eta \parallel^2), \tag{2.1}$$

where η is the parallel mean curvature vector, the complete spacelike submanifold of a constant curvature space of mixed signature has no pinches and it is located in a totally geodesic Lorentzian submanifold [15].

In a conformal theory of gravity, it must be determined whether the second fundamental form satisfies this inequality or the extra contribution to the energy-momentum tensor affects the validity of the dominant energy condition in the space-time. It has been demonstrated that modified gravity theories with the field equations $G_{\mu\nu} + H_{\mu\nu} = 8\pi G T_{\mu\nu}$ would have the inequality

$$\left(G_{\mu\nu} + H_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} H\right) t^\mu t^\nu \ge 0 \tag{2.2}$$

for causal vectors t^μ derived from the strong energy condition, and the focussing of geodesics no longer follows directly [16]. The additional tensor in Weyl theory is

$$H_{\mu\nu} = 2\nabla_\rho \nabla_\sigma C^\rho{}_{\mu\nu}{}^\sigma + C^\rho{}_{\mu\nu}{}^\sigma R_{\rho\sigma}. \tag{2.3}$$

Before the minimal radius, the size of the spacelike surfaces of a de Sitter hyperboloid are decreasing as $r(t) \sim e^{-\lambda t}$. After the minimum radius, $r(t) \sim e^{\lambda t}$. A direct continuation of the first dependence yields $r_1(t) \sim e^{-\lambda t}$, while the transformation

$$r_1(t) \rightarrow r_2(t) = e^{\lambda t} \sim \frac{1}{r_1(t)} \tag{2.4}$$

is compatible with the T-dual description of space-time in string theory.

Geodesic incompleteness can result from the removal of one point from a manifold or factorization by a discrete symmetry. Since the curvature can have a well-defined limit on these manifolds, there exist extensions to another manifold without these singularities.

The problem of curvature singularities can be approached initially by placing bounds on the scalar curvature combinations and deriving an effective field theory which will satisfy these bounds. It may be demonstrated that the limited curvature hypothesis yields a de Sitter manifold in the isotropic model [17]. Two different space-times, de Sitter space and Minkowski space-time with a non-analytic solution for the logarithmic derivative of the scale factor, may result in general [18].

The non-analyticity may be related to the conical singularity of the light cone in Minkowski space-time. A study of conformal boundaries and singularities of space-times includes two types of desingularizations of geodesic congruences, related to topology of the conformal resolution [19] of the endpoint of a geodesic congruence.

For the caustics that can be transformed to a de Sitter region, the inequality satisfied by the second fundamental form will be valid, by contrast with the energy-momentum tensor. A set of allowed deformations may be defined from this inequality.

3. THE CLASSICAL LIMIT OF A SINGULAR METRIC

The Schwarzschild-de Sitter metric is

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} + r^2 d\Omega^2 \tag{3.1}$$

Under the Wick rotation, $t \to i\tau$, the metric is

$$ds^2_{\tau=it} = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} + r^2 d\Omega^2. \tag{3.2}$$

There exists a range of radial coordinate r between r_+ and r_{++} such that

$$1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 > 0$$

if $9M^2\Lambda < 1$. Then the Wick rotation $t \to \tau = it$ yields a Euclidean section

$$ds^2_E = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.3}$$

The region $\frac{\Lambda}{3}r^3 - r + 2M < 0$ is determined by the roots of

$$r^3 - \frac{3}{\Lambda}r + \frac{6M}{\Lambda} = 0. \tag{3.4}$$

The solutions are

$$r_1 = \left[-\frac{3M}{\Lambda} + \frac{3M}{\Lambda}\sqrt{1 - \frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}} + \left[-\frac{3M}{\Lambda} - \frac{3M}{\Lambda}\sqrt{1 - \frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}$$

$$r_2 = -\frac{1}{2}\left\{\left[-\frac{3M}{\Lambda}+\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}+\left[-\frac{3M}{\Lambda}-\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}\right\}$$
$$+\frac{\sqrt{3}i}{2}\left\{\left[-\frac{3M}{\Lambda}+\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}-\left[-\frac{3M}{\Lambda}-\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}\right\}$$
$$r_3 = -\frac{1}{2}\left\{\left[-\frac{3M}{\Lambda}+\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}+\left[-\frac{3M}{\Lambda}-\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}\right\}$$
$$-\frac{\sqrt{3}i}{2}\left\{\left[-\frac{3M}{\Lambda}+\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}-\left[-\frac{3M}{\Lambda}-\frac{3M}{\Lambda}\sqrt{1-\frac{1}{9M^2\Lambda}}\right]^{\frac{1}{3}}\right\}. \quad (3.5)$$

In the extreme limit, when

$9M^2\Lambda < 1,\ Q < 0$

and the trigonometric form of the roots is

$$r_1 = 2\sqrt{-\frac{p}{3}}\ cos\ \frac{\theta}{3}$$
$$r_2 = 2\sqrt{-\frac{p}{3}}cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right) \quad (3.6)$$
$$r_3 = 2\sqrt{-\frac{p}{3}}cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right).$$

Then

$$r_1 = \frac{2}{\sqrt{\Lambda}}cos\left(\frac{1}{3}\ arc\ cos(-3M\sqrt{\Lambda})\right)$$
$$r_2 = \frac{2}{\sqrt{\Lambda}}cos\left(\frac{1}{3}\ arc\ cos(-3M\sqrt{\Lambda})+\frac{4\pi}{3}\right) \quad (3.7)$$
$$r_3 = \frac{2}{\sqrt{\Lambda}}cos\left(\frac{1}{3}\ arc\ cos(-3M\sqrt{\Lambda})+\frac{2\pi}{3}\right)$$

Two of the roots, r_1 and r_2, are positive, and the third root, r_3, is negative The phases in the cube roots of the form $e^{\frac{2\pi k}{3}i}$ have been given in Eq.(3.5), and it is not necessary to include a new phase in

$$\left[-\frac{3M}{\Lambda}+\frac{3M}{\Lambda}\sqrt{\frac{1}{9M^2\Lambda}-1}\right]^{\frac{1}{3}}.$$

In the extreme limit, when $9M^2\Lambda = 1$, the two positive roots are equal and the horizons coincide

At $r = r_2$,

$$\frac{d}{dr}\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\Big|_{r_2} = \frac{2M}{r_2^2}-\frac{2\Lambda}{3}r_2$$
$$= \frac{M\Lambda}{2}sec^2\left(\frac{1}{3}arc\ cos(-3M\sqrt{\Lambda})+\frac{4\pi}{3}\right) \quad (3.8)$$
$$-\frac{4\sqrt{\Lambda}}{3}cos\left(\frac{1}{3}\ arc\ cos(-3M\sqrt{\Lambda})+\frac{4\pi}{3}\right).$$

Since

$$r_2^3-\tfrac{3}{\Lambda}r_2+\tfrac{6M}{\Lambda}=0,$$

the gradient equals

$$\tfrac{1}{r_2^2}(1-\Lambda r_2^2)$$

which is positive for

$$r_2 < 2M \leq \tfrac{2}{3\sqrt{\Lambda}}.$$

Then

$$1-\tfrac{2M}{r}-\tfrac{\Lambda}{3}r^2$$

increases for $r_2 < r < 2M$ when $9M^2\Lambda < 1$, and it remains positive if $2M < r < r_1$, with the value of $\theta = arc\ cos(-3M\sqrt{\Lambda})$ belonging to the second quadrant.

Theorem 1: The extremal limit of the Euclidean Schwarzschild-de Sitter metric has the metric

$$\frac{1}{\Lambda}(d\vartheta^2+sin^2\vartheta d\varphi^2)+\tilde{r}^2(d\theta^2+sin^2\theta d\phi^2)$$

with the $S^2 \times S^2$ topology.

Proof: If $9M^2\Lambda = 1$,

$$1-\tfrac{2M}{r}-\tfrac{\Lambda}{3}r^2 = 1-\tfrac{2}{3\sqrt{\Lambda}r}-\tfrac{\Lambda}{3}r^2$$

which vanishes at $r=\tfrac{1}{\sqrt{\Lambda}}$.

First, let $9M^2\Lambda = 1-3\epsilon^2$

and suppose that

$cos\ \tilde{\vartheta}_1 = \frac{\sqrt{\Lambda}\tilde{r}-1}{\epsilon} + \frac{\epsilon}{6}$

and

$\tilde{\varphi}_1 = \sqrt{\Lambda}\epsilon\tau$ [20],

where $\tilde{r}$ is the radial coordinate. It follows that

$$\begin{aligned} d\tilde{\vartheta}_1^2 &= \frac{\Lambda}{\epsilon^2}\left[\frac{4}{3} - \frac{\Lambda\tilde{r}^2 - 2\sqrt{\Lambda}\tilde{r}+1}{\epsilon^2} - \frac{\epsilon^2}{36} - \frac{\sqrt{\Lambda}r}{3}\right]^{-1} d\tilde{r}^2 \\ &\underset{\epsilon\to 0}{\to} \frac{\Lambda}{\epsilon^2}\left[-\frac{\epsilon^2}{\Lambda^2 - 2\sqrt{\Lambda}\tilde{r}+1} + \mathcal{O}(\epsilon^4)\right] d\tilde{r}^2 \\ &= -\frac{\Lambda}{\Lambda\tilde{r}^2 - 2\sqrt{\Lambda}\tilde{r}+1} d\tilde{r}^2 \end{aligned} \tag{3.9}$$

$$\begin{aligned} sin^2\tilde{\vartheta}_1 d\tilde{\varphi}_1^2 &= \left[\frac{4}{3} - \frac{\Lambda\tilde{r}^2 - 2\sqrt{\Lambda}\tilde{r}+1}{\epsilon^2} + \frac{\epsilon^2}{36} - \frac{\sqrt{\Lambda}\tilde{r}}{3}\right]\Lambda\epsilon^2 d\tau^2 \\ &\underset{\epsilon\to 0}{\to} -(\Lambda\tilde{r}^2 - 2\sqrt{\Lambda}\tilde{r}+1)\Lambda d\tau^2. \end{aligned} \tag{3.10}$$

Therefore, two of the differential elements in the metric near the horizon of the extreme black hole geometry would appear with negative signs under this transformation. With angular variables $(\vartheta_1, \phi_1) = (i\tilde{\vartheta}_1, \tilde{\phi}_1)$, the signs would be reversed. Then

$$\begin{aligned} &\frac{1}{\Lambda}\left(d\vartheta_1^2 + sinh^2\vartheta_1 d\varphi_1^2) + \tilde{r}^2(d\theta^2 + sin^2\theta\ d\phi^2)\right) \\ &= (\Lambda\tilde{r}^2 + 2\sqrt{\Lambda}\tilde{r}+1)d\tau^2 + \frac{d\tilde{r}^2}{\Lambda\tilde{r}^2 - 2\sqrt{\Lambda}\tilde{r}+1} + \frac{\tilde{r}^2}{\Lambda}(d\theta^2 + sin^2\theta d\phi^2). \end{aligned} \tag{3.11}$$

The trigonometric function $cos\ \tilde{\vartheta}_1$ cannot be set equal to $\frac{\sqrt{\Lambda}\tilde{r}-1}{\epsilon}$ for $\epsilon \to 0$ unless

$\tilde{r} = \frac{1}{\sqrt{\Lambda}} + \kappa\epsilon,$

where

$-1 - \frac{\epsilon}{6} \le \sqrt{\Lambda}\kappa \le 1 - \frac{\epsilon}{6}$ for any real value of $\tilde{\vartheta}_1$. However, in this limit,

$$\frac{1}{\Lambda}(d\tilde{\vartheta}_1^2 + sin^2\tilde{\vartheta}_1 d\varphi_1^2) = \epsilon^2\left[1 - \left(\sqrt{\Lambda}\kappa + \frac{\epsilon}{6}\right)^2\right]d\tau^2 + \frac{1}{\epsilon^2}\left[1 - \left(\sqrt{\Lambda}\kappa + \frac{\epsilon}{6}\right)^2\right]^{-1} d\tilde{r}^2,$$

which is singular as ϵ tends to zero.

Instead, let

$sinh\ \tilde{\vartheta}_2 = \frac{\sqrt{\Lambda}\tilde{r}-1}{\epsilon} + \frac{\epsilon}{6}$

and

$$\tilde{\varphi}_2 = \sqrt{\Lambda}\epsilon\tau.$$

Then

$$\begin{aligned} d\vartheta_2^2 &= \frac{\Lambda}{\epsilon^2} sinh^2\vartheta_2^2 + 1]^{-1} dr^2 \\ &= \frac{\Lambda}{\epsilon^2}\left[\frac{(\sqrt{\Lambda}-1)^2}{\epsilon^2} - \frac{\epsilon^2}{36} + \frac{\sqrt{\Lambda}\tilde{r}-1}{3} + 1\right]^{-1} d\tilde{r}^2 \\ &\underset{\epsilon \to 0}{\rightarrow} \Lambda(\sqrt{\Lambda}\tilde{r}-1)^{-1} d\tilde{r}^2, \end{aligned} \tag{3.12}$$

$$\begin{aligned} sinh^2\tilde{\vartheta}_2 d\varphi_2^2 &\underset{\epsilon \to 0}{\rightarrow} \left[\frac{(\sqrt{\Lambda}\tilde{r}-1)^2}{\epsilon^2} + \mathcal{O}(1)\right]\Lambda\epsilon^2 d\tau^2 \\ &= \Lambda(\sqrt{\Lambda}\tilde{r}-1)^2 d\tau^2 \end{aligned} \tag{3.13}$$

and

$$\frac{1}{\Lambda}(d\vartheta_2^2 + sinh^2\tilde{\vartheta}_2 d\tilde{\varphi}_2^2) + \tilde{r}^2(d\theta^2 + sin^2\theta d\phi^2) \underset{\epsilon \to 0}{\rightarrow} (\sqrt{\Lambda}\tilde{r}-1)^2 d\tau^2 + \frac{d\tilde{r}^2}{(\sqrt{\Lambda}\tilde{r}-1)^2} + \tilde{r}^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.14}$$

This line element of the first two-dimensional component cannot be transformed into a spherical metric with an imaginary coordinate through the transformation

$$(\vartheta_2, \varphi_2) = (i\tilde{\vartheta}_2, \tilde{\varphi}_2)$$

unless the space-time has (2,2) signature in the coordinates

$$(\tilde{\vartheta}_2, \tilde{\varphi}_2, \theta, \phi),$$

which would be feasible only if τ and $\tilde{r}$ had been imaginary, requiring then a change in the sign of Λ for reality. Consequently, there exists another class of analytically continued Schwarzschild- de Sitter metrics which have an extreme limit with a nonsingular Euclidean metric. The extremal metric is derived when $\epsilon = 0$ and the limit $\epsilon \to 0$ will cause $cosh\ \vartheta_2$ to diverge for general values of the radial coordinate except if

$$r = \frac{1}{\sqrt{\Lambda}} + \kappa\epsilon + \mathcal{O}(\epsilon^2)$$

for bounded κ.

The limit $\theta_2 \to \infty$ for fixed r such that $\mathcal{O}(1) \leq \sqrt{\Lambda}r - 1 = \infty$ does not leave any available.

It is necessary to consider the next orders in ϵ. If $9M^2\Lambda = 1 - 3\epsilon^2,$

$$M = \frac{(1-3\epsilon^2)^{\frac{1}{2}}}{3\sqrt{\Lambda}} = \frac{1-\frac{3}{2}\epsilon^2+\mathcal{O}(\epsilon^4)}{3\sqrt{\Lambda}} \tag{3.15}$$

and

$$1-\frac{2M}{r}-\frac{\Lambda}{3}r^2 = 1-\frac{2}{3\sqrt{\Lambda}r}\left(1-\frac{3}{2}\epsilon^2+\mathcal{O}(\epsilon^4)\right)-\frac{\Lambda}{3}r^2. \tag{3.16}$$

Then the analytically continued Schwarzschild-de Sitter metric is

$$\begin{aligned} ds^2_{\tau=it} = &\left[1-\frac{2}{3\sqrt{\Lambda}r}\left(1-\frac{3}{2}\epsilon^2+\mathcal{O}(\epsilon^4)\right)-\frac{\Lambda}{3}r^2\right]d\tau^2 \\ &+\frac{dr^2}{1-\frac{2}{3\sqrt{\Lambda}r}\left(1-\frac{3}{2}\epsilon^2+\mathcal{O}(\epsilon^4)\right)-\frac{\Lambda}{3}r^2} \\ &+r^2(d\theta^2+sin^2\theta d\phi^2). \end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned} \frac{1}{\Lambda}(d\tilde{\vartheta}_2^2+sinh^2\tilde{\vartheta}_2 d\tilde{\varphi}_2^2)+\tilde{r}^2(d\theta^2+sin^2\theta d\phi^2) = &\left[(\sqrt{\Lambda}\tilde{r}-1)^2+\left(\frac{\sqrt{\Lambda}\tilde{r}}{3}-\frac{4}{3}\right)\epsilon^2+\frac{\epsilon^4}{36}\right]d\tau^2 \\ &+\left[(\sqrt{\Lambda}\tilde{r}-1)^2+\left(\frac{\sqrt{\Lambda}\tilde{r}}{3}-\frac{4}{3}\right)\epsilon^2+\frac{\epsilon^4}{36}\right]^{-1}d\tilde{r}^2 \\ &+\tilde{r}^2(d\theta^2+sin^2\theta d\phi^2). \end{aligned} \tag{3.18}$$

The minimum value of

$\frac{2}{3\sqrt{\Lambda}r}+\frac{\Lambda}{3}r^2,$

However, occurs at a root of

$$\frac{d}{dr}\left(\frac{2}{3\sqrt{\Lambda}r}+\frac{\Lambda}{3}r^2\right) = -\frac{2}{3\sqrt{\Lambda}r^2}+\frac{2\Lambda}{3}r = 0 \tag{3.19}$$

or $r=\frac{1}{\sqrt{\Lambda}}$, and it equals 1. Since all other r yield a larger value of this expression, it is necessary to find a range of r such that

$1-\frac{2}{3\sqrt{\Lambda}r}-\frac{\Lambda}{3}r^2 = -(\sqrt{\Lambda}\tilde{r}-1)^2.$

Then

$$\sqrt{\Lambda}\tilde{r}-1 = \left[\frac{2}{3\sqrt{\Lambda}r}+\frac{\Lambda}{3}r^2-1\right]^{\frac{1}{2}} \tag{3.20}$$

The first term in the radical will be considerably larger if

$r \ll \frac{2^{\frac{1}{3}}}{\sqrt{\Lambda}},$

when

$$\frac{\sqrt{\Lambda}\tilde{r}}{3} - \frac{4}{3} \approx \frac{1}{3}\left(\frac{2}{3\sqrt{\Lambda}r}\right)^{\frac{1}{2}} - 1. \quad (3.21)$$

Although this expression does not exactly match the coefficient of ϵ^2 in $ds^2_{\tau=it},$ it is nearly proportional to $\frac{1}{\sqrt{r}}$. Therefore, the hyperbolic line element yields a good approximation to the two-dimensional component of the Euclidean Schwarzschild-de Sitter metric only for $\epsilon \ll 1,$ which is equivalent to $\tilde{\vartheta}_2 \ll \frac{\pi}{4},$ where the functions $sin^2\tilde{\vartheta}_2$ and $sinh^2\vartheta_2$ nearly coincide.

When $\epsilon \neq 0,$ the geometry in the vicinity of the horizon is described by the line element

$$ds^2 = \frac{1}{\Lambda}(d\hat{\tau}^2 + d\hat{r}^2) + r^2(d\theta^2 + sin^2\theta d\phi^2)$$
$$\hat{\tau} = \sqrt{\Lambda}\epsilon\left[1 - \left(\sqrt{\Lambda}\kappa + \frac{\epsilon}{6}\right)^2\right]^{\frac{1}{2}}\tau \qquad \hat{r} = \frac{\sqrt{\Lambda}}{\epsilon}\left[1 - \left(\sqrt{\Lambda}\kappa + \frac{\epsilon}{6}\right)^2\right]^{-\frac{1}{2}}\tilde{r}, \quad (3.22)$$

where $\hat{\tau}$ and $\hat{r}$ cover the extended plane including infinity, which can be stereographically projected to a sphere. Then, it characterizes $S^2 \times S^2$ with the metric

$$ds^2 = \frac{1}{\Lambda}(d\vartheta^2 + sin^2\vartheta d\varphi^2) + r^2(d\theta^2 + sin^2\theta d\phi^2). \quad (3.23)$$

Nevertheless, this method reflects the scarcity of singular black hole metrics that can be analytically continued to a nonsingular space. Therefore, the relation between these two types of manifolds would not clearly allow both geometries to be models for elementary particle phenomena.

It has been established that N=2 string theory may be formulated on a flat space-time of (2,2) signature. The sigma model action

$$S_0 = \int \frac{d^2z}{\pi}(\partial x^1\bar{\partial}\bar{x}^1 - \partial x^2\bar{\partial}\bar{x}^2 + \bar{\partial}x^1\partial\bar{x}^1 - \bar{\partial}x^2\partial\bar{x}^2 + \bar{\psi}_R\cdot\bar{\partial}\psi_R + \bar{\psi}_L\cdot\partial\psi_L + \bar{F}\cdot F)$$

where F is auxiliary field component of an N = 2 superfield, and (x1, x2) are complex coordinates such that the metric is $ds^2 = dxd\bar{x}^1 - dx^2d\bar{x}^2$ [21].

Let $x^1 = x_{11} + ix_{12},\ \bar{x}^1 = x_{11} - ix_{12},\ x^2 = x_{21} + ix_{22}$ and $\bar{x}^2 = x_{21} - ix_{22}$ on the two spheres.

After analytic continuation of the metric on $\mathbb{R}^2$ to the signature (++), the stereographic projection onto each sphere is

$$\begin{aligned} g_{11}(x_{11}, x_{12}) &= \left(\frac{2x_{11}}{x_{11}^2 + x_{12}^2 + 1}, \frac{2x_{12}}{x_{11}^2 + x_{12}^2 + 1}, \frac{x_{11}^2 + x_{12}^2 - 1}{x_{11}^2 + x_{12}^2 + 1}\right) \\ g_{12}(x_{11}, x_{12}) &= \left(\frac{2x_{11}}{x_{11}^2 + x_{12}^2 + 1}, \frac{2x_{12}}{x_{11}^2 + x_{12}^2 + 1}, -\frac{x_{11}^2 + x_{12}^2 - 1}{x_{11}^2 + x_{12}^2 + 1}\right) \\ g_{21}(x_{21}, x_{22}) &= \left(\frac{2x_{21}}{x_{21}^2 + x_{22}^2 + 1}, \frac{2x_{22}}{x_{21}^2 + x_{22}^2 + 1}, \frac{x_{21}^2 + x_{22}^2 - 1}{x_{21}^2 + x_{22}^2 + 1}\right) \\ g_{22}(x_{21}, x_{22}) &= \left(\frac{2x_{21}}{x_{21}^2 + x_{22}^2 + 1}, \frac{2x_{22}}{x_{21}^2 + x_{22}^2 + 1}, \frac{x_{21}^2 + x_{22}^2 - 1}{x_{21}^2 + x_{22}^2 + 1}\right). \end{aligned} \tag{3.24}$$

The polar coordinates for the first sphere of unit radius are given by

$$\begin{aligned} \frac{2x_{11}}{x_{11}^2 + x_{12}^1 + 1} &= sin\ \theta_1\ cos\ \phi_1 \\ \frac{2x_{12}}{x_{11}^2 + x_{12}^2 + 1} &= sin\ \theta_1\ sin\ \phi_1 \\ \frac{x_{11}^2 + x_{12}^1 - 1}{x_{11}^2 + x_{12}^2 + 1} &= cos\ \theta_1 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} sin^2\theta_1 &= \frac{4(x_{11}^2 + x_{12}^2)}{(x_{11}^2 + x_{12}^2 + 1)^2} \\ tan\ \phi_1 &= \frac{x_{12}}{x_{11}}. \end{aligned} \tag{3.26}$$

Therefore,

$$\begin{aligned} d\theta_1 &= \frac{2}{\sqrt{x_{11}^2 + x_{12}^2}((x_1^2 + x_2^2)^2 - 1)}(x_{11}dx_{11} + x_{12}dx_{12}) \\ d\phi_1 &= \frac{1}{x_{11}^2 + x_{12}^2}(-x_{12}dx_{11} + x_{11}dx_{12}) \end{aligned} \tag{3.27}$$

and

$$\frac{\partial(x_{11}, x_{12})}{\partial(\theta, \phi)} = \begin{pmatrix} \frac{x_{11}[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}} & \frac{x_{12}[(x_{11}^2 + x_{12})^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}} \\ -x_{12} & x_{11} \end{pmatrix}. \tag{3.28}$$

It follows that

$$\begin{aligned} \partial_z x^1 &= \partial_z \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}}(x_{11} + ix_{12})\right] + \partial_z \phi_1(-x_{12} + ix_{11}) \\ \partial_{\bar{z}} x^1 &= \partial_{\bar{z}} \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}}(x_{11} + ix_{12})\right] + \partial_{\bar{z}} \phi_1(-x_{12} + ix_{11}) \\ \partial_z \bar{x}^1 &= \partial_z \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}}(x_{11} - ix_{12})\right] + \partial_z \phi_1(-x_{12} - ix_{11}) \\ \partial_{\bar{z}} \bar{x}^1 &= \partial_{\bar{z}} \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}}(x_{11} - ix_{12})\right] + \partial_{\bar{z}} \phi_1(-x_{12} - ix_{11}). \end{aligned} \tag{3.29}$$

On the second sphere with polar coordinates (θ_2, φ_2),

$$
\begin{aligned}
\partial_z x^2 &= \partial_z \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} + i x_{22}) \right] + \partial_z \phi_2 (-x_{22} + i x_{21}) \\
\partial_{\bar{z}} x^2 &= \partial_{\bar{z}} \theta_2 \left\{ \frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} + i x_{22}) \right] + \partial_{\bar{z}} \phi_2 (-x_{22} + i x_{21}) \\
\partial_z \bar{x}^2 &= \partial_z \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} - i x_{22}) \right] + \partial_z \phi_2 (-x_{22} - i x_{21}) \\
\partial_{\bar{z}} \bar{x}^2 &= \partial_{\bar{z}} \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} - i x_{22}) \right] + \partial_{\bar{z}} \phi_2 (-x_{22} - i x_{21})
\end{aligned}
\tag{3.30}
$$

The S2 × S2 string action equals

$$
\begin{aligned}
S_{S^2 \times S^2} = \int \frac{d^2 z}{\pi} \Bigg\{ & \Bigg\{ \partial_z \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}} (x_{11} + i x_{12}) \right] + \partial_z \phi_1 (-x_{12} + i x_{11}) \Bigg\} \\
& \Bigg\{ \partial_{\bar{z}} \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}} (x_{11} - i x_{12}) \right] + \partial_{\bar{z}} \phi_1 (-x_{12} - i x_{11}) \Bigg\} \\
& + \Bigg\{ \partial_z \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} + i x_{22}) \right] + \partial_z \phi_2 (-x_{22} + i x_{21}) \Bigg\} \\
& \Bigg\{ \partial_{\bar{z}} \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} - i x_{22}) \right] + \partial_{\bar{z}} \phi_2 (-x_{22} - i x_{21}) \Bigg\} \\
& + \Bigg\{ \partial_{\bar{z}} \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}} (x_{11} + i x_{12}) \right] + \partial_{\bar{z}} \phi_1 (-x_{12} + i x_{11}) \Bigg\} \\
& \Bigg\{ \partial_z \theta_1 \left[\frac{[(x_{11}^2 + x_{12}^2)^2 - 1]}{2\sqrt{x_{11}^2 + x_{12}^2}} (x_{11} - i x_{12}) \right] + \partial_z \phi_1 (-x_{12} - i x_{11}) \Bigg\} \\
& + \Bigg\{ \partial_{\bar{z}} \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} + i x_{22}) \right] + \partial_{\bar{z}} \phi_2 (-x_{22} + i x_{21}) \Bigg\} \\
& \Bigg\{ \partial_z \theta_2 \left[\frac{[(x_{21}^2 + x_{22}^2)^2 - 1]}{2\sqrt{x_{21}^2 + x_{22}^2}} (x_{21} - i x_{22}) \right] + \partial_z \phi_2 (-x_{22} - i x_{21}) \Bigg\} \\
& \bar{\psi}_R(\theta_1, \phi_1, \theta_2, \phi_2) \partial_{\bar{z}} \psi_R(\theta_1, \phi_1, \theta_2, \phi_2) \\
& \bar{\psi}_L(\theta_1, \phi_1, \theta_2, \phi_2) \partial_z \psi_L(\theta_1, \phi_1, \theta_2, \phi_2) \\
& \bar{F}(\theta_1, \phi_1, \theta_2, \phi_2) F(\theta_1, \phi_1, \theta_2, \phi_2) \Bigg\}
\end{aligned}
\tag{3.31}
$$

The stereographic projection of the sphere to R^2 and the finiteness of derivatives of the coordinates in the vicinity of the pole mapped to infinity is sufficient for regularity of the extension of the action to the space $S^2 \times S^2$. When the entire set of fermion and auxiliary field terms can be expressed as a sum of functions of coordinates (x_{11}, x_{12}) and (x_{21}, x_{22}) respectively, the action would equal the sum of products of volume factors with integrals defined over each of the two-spheres. The only potential singularity occurs at $x_1 = x_2 = 0$. However, the product

$\frac{x_{11}+ix_{12}}{\sqrt{x_{11}^2+x_{12}^2}} \cdot \frac{x_{11}-ix_{12}}{\sqrt{x_{11}^2+x_{12}^2}}$

equals 1, and the limit of the integral is nonsingular.

4. DESINGULARIZATION IN HIGHER DIMENSIONS

The existence of a classical desingularization method for algebraically singular spacetimes in general relativity is sufficient to remove divergences in the metric before the inclusion of quantum effects. This technique may have the consequence that a field theory can be formulated consistently on the corresponding nonsingular space-time at the quantum level. The singularity can be removed in string theory through a replacement by a singularity in the dilaton expectation value. It is known that any singular space-time can be approximated by a conformal field theory which is nonanalytic in the dilaton expectation value and nonsingular in the metric defined by the coupling [22].

The field equations of four-dimensional action of the bosonic sector of a theory with a dilaton and a gauge potential [23]

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R - 2(\nabla\phi)^2 - e^{-2a\phi}F_2^2] \tag{4.1}$$

have the singular metric

$$ds^2 = -\left(1-\frac{\mu}{r}\right)^{\frac{2}{1+a^2}} dt^2 + \left(1-\frac{\mu}{r}\right)^{-\frac{2}{1+a^2}} dr^2 + \left(1-\frac{\mu}{r}\right)^{\frac{2a^2}{1+a^2}} r^2 d\Omega^2 \tag{4.2}$$

as a solution, where μ is defined by

$e^{\alpha\phi} = \left(1-\frac{\mu}{r}\right)^{-\frac{a^2}{1+a^2}}$.

The action may be derived from a higher-dimensional action

$$I_{4+n} = \int d^{4+n}\sqrt{-g}[R - F_2^2], \tag{4.3}$$

which has equations for the metric with the nonsingular solution

$$ds_{4+n}^2 = e^{2(a-a^{-1})\phi(x)}d\vec{y}\cdot d\vec{y} + e^{2a\phi(x)}g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{4.4}$$

if $a = \sqrt{\frac{n}{n+2}}$.

The solution in higher dimensions which reduces to the black hole metric in four dimensions [23] is

$$\begin{aligned} ds^2 = (n+1)^2\mu^2\left[\omega^2(-dt^2 + d\vec{y}\cdot d\vec{y}) + (1-[(n+1)\mu\omega)^{n+1})^{-4}\left(\frac{d\omega}{\omega}\right)^2\right] \\ + (1-[(n+1)\mu\omega]^{n+1})^{-2}\mu^2 d\Omega^2 \end{aligned} \tag{4.5}$$

and it can be extrapolated to r < μ and negative ω for even n.

The strength of the singularity in the four-dimensional metric is

$\frac{1}{\tilde{r}^{\frac{2}{1+2a^2}}}$, where

$\tilde{r} = 1 - \frac{\mu}{r}$,

and the curvature similarly diverges in the limit r → μ. The curvature of the (4 + n)-dimensional metric is well-defined in this limit. The nature of the desingularization necessary for a removal of a polynomial divergence in the scalar curvature, therefore, has been identified. The presence of the scalar field may be verified as a result of desingularization of the metric in a higher dimensions. It is known that the energy-momentum tensor of Klein-Gordon theory

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\partial^\mu \phi \partial_\mu \phi g_{\mu\nu} - \frac{1}{2} m^2 \phi^2 g_{\mu\nu} \tag{4.6}$$

does not satisfy the strong energy condition [24] because

$$\begin{aligned} T_{\mu\nu} t^\mu t^\nu &= (t^\mu \partial_\mu \phi)^2 - \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi)^2 t^\nu t_\nu + \frac{1}{2} m^2 \phi^2 t^\nu t_\nu \\ -\frac{1}{d-2} g_{\mu\nu} T t^\mu t^\nu &= -\frac{1}{d-2} g_{\mu\nu} \left[\left(1 - \frac{1}{2} d\right)(\partial^\mu \phi \partial_\mu \phi) - \frac{d}{2} m^2 \phi^2 \right] t^\mu t^\nu \\ &= \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi) t^\nu t_\nu + \frac{d}{d-2} m^2 \phi^2 t^\nu t_\nu \end{aligned} \tag{4.7}$$

$$\left(T_{\mu\nu} - \frac{1}{d-2} g_{\mu\nu} T\right) t^\mu t^\nu = (t^\mu \partial_\mu \phi)^2 + \frac{1}{d-2} m^2 \phi^2 t^\mu t_\mu$$

is negative for sufficiently large $|t^\mu t_\mu|$ with the signature being (- + ++).

5. QUANTUM STRING THEORY ON A CURVED GEOMETRY

String theory amplitudes are evaluated on a class of backgrounds with a high degree of symmetry such as flat space and group manifolds. The quantum consistency of a string theory, therefore, is not immediately related to the singular nature of the space-time. However, pp wave metrics are relevant in this connection [25], because these space-times may have singularities of a non-scalar type.

The existence of a classical desingularization method for singular space-times in general relativity is sufficient to remove divergences in the metric before the inclusion of quantum effects. By this technique a field theory consequently can be formulated consistently on the corresponding nonsingular space-time at the quantum level.

The consistency of string dynamics on plane-wave space-times follows from vanishing of scalar curvature combinations in the β-function which allows conformal invariance to be preserved at the quantum level. Parallelizable solutions to the equations of the supergravity theories with torsion given by the antisymmetric field strength for the field $B_{\mu\nu}$ would be α′-exact [26].

There are classes of space-times with divergent curvature components which are backgrounds of perturbatively finite string theories. Two examples are the singular planewave metrics in Brinkmann coordinates and the Penrose limit of

$M^{(0)}_{p+2} \times S^{D-p-2}$ [27].

Although it has been suggested that the quantum theory of the propagating string is valid in these manifolds in the Rosen coordinates [28], there are infinities in several of the physical observables [29]. A change of coordinates cannot remove scalar curvature singularities, even though singularities in certain components of the Riemann tensor may be eliminated. Since the scalar curvature combinations vanish in a plane-wave space-time, these will remain zero in all coordinates, while divergences in quantities dependent on the components of a Riemann curvature tensor can be removed by a change of coordinates. The physical effects in general relativity must be unaffected by such transformations, and inconsistencies in the quantum string theory would result.

There remains the viability of string theory in the desingularized space-time in higher dimensions. An evaluation of the scalar curvature combinations reveals that singularities in the components of the Riemann tensor may be removed, while the scalar curvature combinations receive a non-zero contribution.

Theorem 2: The beta function of the string sigma model is nonvanishing in the desingularized space-time in higher dimensions.

Proof: The metric (4.5) reduces to that of

$$(AdS)_{n+2} \times S^2,\ (n+1)^2\mu^2\left[\omega^2(-dt^2 + d\vec{y}\cdot\ d\vec{y}) + \left(\frac{d\omega}{\omega}\right)^2\right] + \mu^2 d\Omega_2^2,$$

When

$$\omega \to 0\ [23],$$

because the line element on the hyperboloid

$$X_0^2 - \vec{Y}\cdot\vec{Y} + X_1^2 - X_2^2 = 1 \text{ in } n+3$$

dimensions, with

$$X_- = X_1 - X_2 = (n+1)\mu\omega, X_0 = (n+1)\mu t\omega$$

and

$$\vec{Y} = (n+1)\mu\vec{y}\omega,\ X_- = X_1 - X_2 = (n+1)\mu$$

And

$$X_+ = X_1 + X_2 = \frac{(n+1)^2\mu^2 - X_0^2 + \vec{Y}\cdot\vec{Y}}{X_-} = (n+1)\mu\left(\frac{1-\omega^2(|\vec{y}|^2 - t^2)}{\omega}\right),$$

equals

$$ds^2 = -dX_0^2 + d\vec{Y}\cdot d\vec{Y} - dX_+ dX_-.$$

Given that radius of curvature of the hyperboloid and the cosmological constant are related by

$$a^{-1} = \sqrt{-\frac{n+1}{\Lambda}}.$$

Then

$$\Lambda = -(n+1)a^2 = -(n+1)^{-1}\mu^{-2}$$

and

$$\begin{aligned} R_{\mu\nu}^{(n+2)} &= -(n+1)^{-1}\mu^{-2}g_{\mu\nu}^{(n+2)} \\ R_{\mu\lambda\rho\sigma}^{(n+2)}R_\nu{}^{\lambda\rho\sigma\,(n+2)} &= 2(n+1)a^4 g_{\mu\nu}^{(n+2)} = 2(n+1)^{-3}\mu^{-4}g_{\mu\nu}^{(n+2)}. \end{aligned} \qquad (5.1)$$

The beta function of the heterotic string without the dilaton and antisymmetric tensor field to two loops, where the variation with respect to the metric of the Gauss-Bonnet action is evaluated in the first-order formalism

$$\beta_{MN} = \alpha' \left(R_{MN} - \tfrac{1}{2}Rg_{MN}\right) + \tfrac{1}{4}\alpha'^2 \cdot 2\left[RR_{MN} - 4R^{RS}R_{MRNS} + R_{MRST}R_N{}^{RST}\right] + ...,$$

would equal

$$\begin{aligned}\beta^{(n+2)}_{\mu\nu}(\omega \sim 0) &= \Bigg[\left(-(n+1)^{-1} - \frac{1}{2}\left(2 - \frac{n+2}{n+1}\right)\right)\mu^{-2}\alpha' \\ &\quad + \left(2(n+1)^{-3} + \frac{3}{2}(n+1)^{-2} - \frac{1}{2}(n+1)^{-1}\right)\mu^{-4}\alpha'^2 + ...\Bigg] g^{(n+2)}_{\mu\nu} \\ &= \Bigg[-\frac{1}{2}\frac{n+2}{n+1}\mu^{-2}\alpha' + \left(2(n+1)^{-3} + \frac{3}{2}(n+1)^{-2} - \frac{1}{2}(n+1)^{-1}\right)\mu^{-4}\alpha'^2 \\ &\quad + ...\Bigg]\, g^{(n+2)}_{\mu\nu} \\ \beta^{(2)}_{ij}(\omega \sim 0) &= \left[\left(1 - \frac{1}{2}\left(2 - \frac{n+2}{n+1}\right)\right)\mu^{-2}\alpha' - \frac{1}{2}\left(1 + \frac{1}{n+1}\right)\mu^{-4}\alpha'^2 + ...\right] g^{(2)}_{ij} \\ &= \left[\frac{1}{2}\frac{n+2}{n+1}\mu^{-2}\alpha' - \frac{1}{2}\frac{n+2}{n+1}\mu^{-4}\alpha'^2 + ...\right] g^{(2)}_{ij}.\end{aligned} \tag{5.2}$$

The Riemann tensor have mixed components between the (n+2)-dimensional space and the two-sphere when ω 6= 0, and the beta functions will be similarly nonvanishing. Therefore, it follows that the β-function will no longer vanish at higher orders in σ-model perturbation theory and conformal invariance is not preserved. Consequently, the quantum theory of the superstring is less viable in these Lagrangian desingularized space-times.

The contrast between the desingularization of space-times with curvature singularities through higher-dimensional actions and the resolution of singularities in algebraic varieties is reflected in the consistency of the quantum theory. It may noted that there exist manifolds which can be desingularized such that the string theory becomes consistent. An example is the K3 manifold which may be regarded as a quotient of the torus T^4. Consistency of the string theory on the flat manifold $M^6 \times T^4$ follows from existence of a connection with vanishing curvature. Furthermore, there is a duality between Type II string theory on $M^4 \times K3 \times T^2$ and heterotic string theory on $M^4 \times T^6$ [30]. The preservation of conformal invariance at the quantum level in the latter model is sufficient to establish consistency of the initial theory. The intersection matrix of the K3 manifold includes E_8. The moduli space of two-dimensional instantons of the string theory on K3 is characterized by a second homology group that is dual to that of a cohomology group of a curve of genus g ≥ 2 [31].

When the genus tends to infinity, and the ideal boundary introduces a nonsmooth structure in an embedding four-manifold, the intersection matrix is that of

$$mE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Where n is odd, which can be the basis for the effects of exceptional group symmetry in elementary particle physics [32]

The mass, charge and angular momentum of Schwarzschild, Reissner-Nordstrom and Kerr black hole space-times suggest a model of particles based on such geometries. Since particles can travel in space-time without introducing singularities, however, and the ADM mass is defined by an integral at spatial infinty given by the asymptotic dependence of the metric components. The geometry within the event horizon can be replaced by a nonsingular submanifold without affecting the integrals for the conserved quantities. The quasilocal mass requires the solution to projected twistor equations on a surface of non-zero radius, which allows the removal of the singularity.

The Euclidean rotation of the BTZ black hole is the quotient H^3/Γ, where $\Gamma = q\mathbf{Z}$ with q being the generator of the discrete group. Solutions to the vacuum field equations in three dimensions are characterized by the conical angle determining the energy for a space-time [33]. The correlation function of a bosonic field theory on the boundary, computed through the path integral [34]

$$g(z,1) = log\left(|q|^{\frac{B_2\left(\frac{log|z|}{log\ |q|}\right)}{2}}|1-z|\prod_{n=1}^{\infty}|1-q^n z||1-q^n z^{-1}|\right)$$

is well-defined except if z → 0 and z → 1, and it may be given by a formula in terms of the lengths of geodesics in the handle body [35]

$$g(z,1) = -\frac{1}{2}\ell(\gamma_0)B_2\left(\frac{\ell_{\gamma_0}(\bar{z},\bar{1})}{\ell(\gamma_0)}\right) + \sum_{n\geq 0}\ell_{\gamma_1}(\bar{0},\bar{z}_n) + \sum_{n\geq 1}\ell_{\gamma_1}(\bar{0},\tilde{z}_n). \qquad (5.3)$$

Since

$$\ell(\gamma_0)B_2\left(\frac{\ell_{\gamma_0}(\bar{z},\bar{1})}{\ell(\gamma_0)}\right) = \frac{\ell^2_{\gamma_0}(\bar{z},\bar{1})}{\ell(\gamma_0)} - \ell_{\gamma_0}(\bar{z},\bar{1}) + \frac{1}{6}\ell(\gamma_0) \qquad (5.4)$$

The "Entry Name" is you has a minimum value of

$$\left(\frac{2}{\sqrt{6}} - 1\right)\ell_{\gamma_0}(\bar{z},\bar{1})\ <\ 0$$

When

$$\ell(\gamma_0)\ =\ \sqrt{6}\ell_{\gamma_0}(\bar{z},\bar{1}),$$

The contribution of the geodesic in fundamental domain of the discrete group is positive only

when

$$\ell_{\gamma_0} \in [(3-\sqrt{3})\ell_{\gamma_0}(\bar{z},\bar{1}), (3+\sqrt{3}\ell_{\gamma_0}(\bar{z},\bar{1})].$$

There are extremal black holes solutions to the heterotic string equations that have

no event horizons [36]. The metric in the coordinates (t,ρ,θ,ϕ) is

$$ds^2\ =\ -\Delta^{-\frac{1}{2}}dt^2\ +\ \Delta^{\frac{1}{2}}\rho^{-1}d\rho^2 + \Delta^{-\frac{1}{2}}\rho(d\theta^2 + sin^2\theta d\phi^2),$$

Where

$$\Delta = \rho^2 + 2m\rho\ cosh\ \alpha + m^2\ [37].$$

The nature of this singularity may be changed by the inclusion of higher-derivative terms, since the space-times then have an event horizon [38]. These terms arise as quantum effects in the sigma model. If the sigma model coupling α′ is replaced by ϵ, the method yields a classical resolution of the existence of observable singularities.

Although these metrics may be characterized by electric and magnetic charges and similar rotating solutions exist with angular momentum, the identification with the elementary particles does not necessarily follow because the number of accessible states increases as e^S, with an extra contribution from the worldsheet metric. The entropy of a black hole with an event horizon of area A equals 1/4A. The area of the event horizon of the Schwarzschild metric is

$$\pi r_H^2 = \pi(2M)^2 = 4\pi M^2.$$

Instead, the extremal black holes with zero horizon area are modified by string worldsheet corrections such that area of the sphere in the limit of vanishing radial coordinate and identified Euclidean time coordinate is non-zero. The area of the stretched horizon is found to be proportional to the mass parameter [39], which matches the degeneracy of the string states given initially by given initially by

$$d_{string} \sim e^{2\pi\sqrt{\kappa_{str}}m},$$

where m is the mass level.

The counting of the tower of BPS states in the compactification of the heterotic string over T 6 is given by the partition function

$$Z(\beta) = 16\sum_N d_N e^{-\beta N}. \tag{5.5}$$

where N = w|n| where w is the winding number in the direction x5 and |n| is the quantized momentum. Since the sum equals

$\frac{1}{\Delta(q)} = \frac{1}{\eta(q)^{24}},$

With

$q = e^{-\beta},$

$$\begin{aligned} d_N &= \frac{1}{2\pi i}\int_C \frac{dq}{\Delta(q)}\frac{1}{q^{N+1}} \\ &= \frac{1}{2\pi i}\int_C \frac{d(e^{-\beta})}{\Delta(e^{-\beta})}\frac{1}{e^{-\beta(N+1)}} \end{aligned} \tag{5.6}$$

where the contour in the q plane is the union of the infinite line, with the exception of a semicircle around the origin, and a semicircular arc at infinity.

Theorem 3: The entropy at level N of the tower of BPS states in the compactification of the heterotic string over the six-torus is

$$S = 4\pi\sqrt{N} - 7\ ln\ N - ln\ 2\pi + \frac{1}{N^2} + \frac{1}{2N^4} + \ldots .$$

Proof: In the limit $\beta \to 0,\ \Delta\left(e^{-\frac{4\pi^2}{\beta}}\right) \sim e^{-\frac{4\pi^2}{\beta}},$

and, by duality, it follows from

$$\Delta(e^{-\beta}) = \left(\frac{\beta}{2\pi}\right)^{-12} \Delta\left(e^{-\frac{4\pi^2}{\beta}}\right)$$

That

$$\begin{aligned} d_N &= -\frac{1}{2\pi i}\int_{C'} d\beta e^{\beta N}\left(\frac{\beta}{2\pi}\right)^{12}\frac{1}{\Delta(e^{-\frac{4\pi^2}{\beta}})} \\ &\approx -\frac{1}{2\pi i}\int_{C'} d\beta \left(\frac{\beta}{2\pi}\right)^{12} e^{\beta N + \frac{4\pi^2}{\beta}}. \end{aligned} \tag{5.7}$$

where C′ is a u-shaped contour of the form

$$(-\infty + i\pi, \infty + i\pi) \cup (\infty + i\pi, \infty) \cup (\infty, -\infty),$$

since the contribution of the segment

$$(-\infty, -\infty + i\pi)$$

vanishes. Substituting -β for β, the integral along the real line is

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\beta \left(\frac{\beta}{2\pi}\right)^{12} e^{-\beta N - \frac{4\pi^2}{\beta}}. \tag{5.8}$$

Along the real line, there is a minimum at $\frac{2\pi}{\sqrt{N}}$. The singularity of the contour integral in the q-plane is located at q = 0, while the β → 0 limit occurs at q = 1. The contour integral around q = 0 introduces a phase shift with respect to the integral in the neighbourhood of q = 1 of ei π2 . Consequently, introducing the relative phase shift factor

$$e^{-i\frac{\pi}{2}},$$

let $\beta = \frac{2\pi}{\sqrt{N}} + \tilde{\beta}$ for infinitesimal variations $\tilde{\beta}$. Then

$$\begin{aligned} d_N &= \frac{-e^{-i\frac{\pi}{2}}}{2\pi i}\int_{-\epsilon}^{0} d\tilde{\beta}\left(\frac{1}{\sqrt{N}} + \frac{\tilde{\beta}}{2\pi}\right)^{12} e^{4\pi\sqrt{N} + \tilde{\beta}(N - \frac{1}{N})} \\ &\approx \frac{1}{2\pi}\frac{1}{N^6} e^{4\pi\sqrt{N}} \frac{1 - e^{-\epsilon(N - \frac{1}{N})}}{N - \frac{1}{N}} \\ &\approx \frac{1}{2\pi}\frac{1}{N^7 - N^5} e^{4\pi\sqrt{N}} \end{aligned} \tag{5.9}$$

when $N \gg \frac{1}{\epsilon} > 1.$

The entropy would equal

$$\begin{aligned} S = ln\ d_N &\approx ln\left(\frac{1}{2\pi}\frac{1}{N^7 - N^5} e^{4\pi\sqrt{N}}\right) \\ &= 4\pi\sqrt{N} - 5ln\ N - ln(N^2 - 1) - ln\ 2\pi \\ &= 4\pi\sqrt{N} - 7\ ln\ N - ln\ 2\pi + \frac{1}{N^2} + \frac{1}{2N^4} + \dots . \end{aligned} \tag{5.10}$$

The winding number and the units of momentum can be interchanged under duality yielding a critical value of

$$N = n^2 \text{ at } w = |n|.$$

Since the quantized momentum yields the mass,

$$S \sim 4\pi n \propto m. \tag{5.11}$$

It is certainly not possible to derive m^2 because the string state is wound over only one direction in the torus T^2 in $T^4 \times T^2$ [38] since integration in the fifth dimension gives the four-dimensional effective action with extreme black hole solutions.

The metric of the electrically charged black hole solution to the equations of the effective action of the heterotic string compactified on a torus [39] is given by

$$\begin{aligned} ds^2 &= -K^{-\frac{1}{2}}\rho dt^2 + K^{\frac{1}{2}}\rho^{-1}d\rho^2 + K^{\frac{1}{2}}\rho(d\theta^2 + sin^2\theta d\phi^2) \\ K &= \rho^2 + 2m_0\rho cosh\ \alpha + m_0^2 \end{aligned} \tag{5.12}$$

The string metric in the vicinity of horizon equals

$$\begin{aligned} ds^2_{E\ string} &= K^{-\frac{1}{2}}\rho g^2 \ -K^{\frac{1}{2}}\rho dt^2 + K^{\frac{1}{2}}\rho^{-1}d\rho^2 + K^{\frac{1}{2}}\rho(d\theta^2 + sin^2\theta d\phi^2)\Big) \\ &= -K^{-1}\rho^2 g^2 dt^2 + d\rho^2 + \rho^2 g^2(d\theta^2 + sin^2\theta d\phi^2). \\ &\simeq -\frac{\rho^2}{m_0^2}g^2dt^2 + g^2d\rho^2 + g^2\rho^(d\theta^2 + sin^2\theta d\phi^2). \end{aligned} \tag{5.13}$$

In the variables $\bar{\rho} = g\rho$ and

$\bar{t} = \frac{t}{m_0}, \bar{\rho} = \frac{\bar{r}^2}{4}, \bar{g}_{\mu\nu}^{black\ hole} = e^{\bar{\Phi}}G_{\mu\nu}^{string},$

Where

$\bar{\Phi} = \Phi - ln\left(\frac{g}{m_0}\right),$

$d\bar{s}_E^2 \sim \frac{1}{4}\bar{r}^2 d\bar{\tau}^2 + d\bar{r}^2 + \frac{1}{4}\bar{r}^2(d\theta^2 + sin^2\phi^2)$ [37].

Identifying the Euclidean time coordinate $\bar{\tau} = -i\bar{t}$ with a period 4π to remove a coordinate singularity $\bar{r} = 0$, at the metric is that of a direct product of a cylinder with a sphere having topology $S^1 \times \mathbb{R}^1 \times S^2$.

Compactification of the radial direction yields a space with topology $T^2 \times S^2$.

The extremal limit of the Reissner-Nordstrom metric is

$$ds^2 = -\left(1 - \frac{Q}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{Q}{r}\right)^2} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.14}$$

Setting

$r = \left(1 + \frac{Q}{\bar{r}}\right)\bar{r} = \bar{r} + Q \text{ and } dr = d\bar{r},$

$$ds^2 = -\left(1+\frac{Q}{r-Q}\right)^{-2} dt^2 + \left(1+\frac{Q}{r-Q}\right)^2 dr^2 + r^2(d\theta^2 + sin^2\theta\ d\phi^2)$$
$$= -\left(1+\frac{Q}{\bar{r}}\right)^{-2} dt^2 + \left(1+\frac{Q}{\bar{r}}\right)^2 d\bar{r}^2 + r^2(d\theta^2 + sin^2\theta\ d\phi^2) \quad (5.15)$$
$$= -\left(1+\frac{Q}{\bar{r}}\right)^{-2} dt^2 + \left(1+\frac{Q}{\bar{r}}\right)^2 \left(d\bar{r}^2 + \bar{r}^2(d\theta^2 + sin^2\theta\ d\phi^2\right).$$

The horizon at r = Q is located in the isotropic coorinates at $\bar{r} = 0,$ where

$$ds^2 \approx -\frac{\bar{r}^2}{Q^2} dt^2 + \frac{Q^2}{\bar{r}^2} d\bar{r}^2 + Q^2(d\theta^2 + sin^2\theta\ d\phi^2). \quad (5.16)$$

Replacing $\frac{\bar{r}^2}{Q^2}$ by $1+\frac{\tilde{r}^2}{Q^2}$ and $d\bar{r}^2$ by $d\tilde{r}^2,$ the metric is that of $(AdS)^2 \times S^2$ and the Euclidean section of this space is H2 ×S2. Factoring by a discrete group gives Σg ×S2 for some g ≥ 2.

The topology of the event horizon surrounding a singularity in four dimensions is that of the two-sphere. In the extreme limit, the area of the event horizon tends to zero and it must approximate to a increasing degree S^2, and therefore, the geometry in the vicinity of the point singularity would be conical. Consequently, the singularities become algebraic for extreme black hole geometries. If the conical singularity belongs to a light cone, the location of the particle $\tilde{r}^2$ may be replaced by this cone, which is the identification made in twistor theory. There exists a twistor description of the extreme black hole geometry which is a background of the N=2 string model.

Therefore, it may be expected that a particle spectrum may be predicted by this method. A massive particle might curve the geometry and deform the conical metric. Positive-energy theorems for asymptotic matter distributions are not necessarily valid on an approximately conical metric. However, the local extreme black hole metric is embedded in a manifold that tends to asymptotically Minkowski space-time. The existence of a global positive energy theorem would suggest that the positivity of masses of particles in a region with an approximately conical metric then may be proven for quasilocal integrals [40]. The metric near a conical singularity is $ds^2 = d\rho^2 + \alpha^2\rho^2 d\phi^2,$ which differs from the flat metric in polar coordinates by the factor of α2.

Redefining $\phi \to \frac{\phi}{\alpha}$ restores the metric, although there is a deficit angle of $2\pi\left(1-\frac{1}{\alpha}\right).$

The four-dimensional metric $ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\tilde{\phi}^2$
Will admit the maximal number of Killing vectors, which may be expressed as bilinears of Killing spinors. The introduction of mass would cause the metric to be

$$ds^2 = -\left(1-\frac{2m}{\rho}\right) dt^2 + dz^2 + \frac{d\rho^2}{1-\frac{2m}{\rho}} + \rho^2\tilde{\phi}^2.$$

Writing the Killing spinors in the four dimensions, or equivalently the conformal Killing spinors of a hypersurface Σ with the metric

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\tilde{\phi}^2,$$

in terms of twistors on the timelike surface, the conserved quantities defined over a cross-sectional surface S equal

$$J_{ab} = \frac{1}{8\pi}\int_{\tilde{S}} \hat{\phi}_{ABCD}\tilde{\omega}_a^C\tilde{\omega}_b^D d\tilde{S}^{AB} \quad (5.17)$$

Where $\hat{\phi}_{ABCD} = \Omega^{-1}\tilde{\Psi}_{ABCD}$ is the spinor equivalent of the rescaled Weyl tensor [41], and the conformal factor is $\Omega \sim \frac{1}{\rho}$.

Given that $\Psi_{\mu\nu\lambda\sigma}n^\nu n^\sigma = \frac{2m}{\rho^3}e_{\mu\lambda}$, with $e_{\mu\lambda}$ being a tensor normalized by $e_{tt} = -1$, and $dS^\lambda = \rho^2 d\Omega t^\mu$, where n^ν is a normal vector to Σ, $d\Omega$ is a spherical element and tμ is normal vector to S in Σ

$$\hat{E}_{\nu\lambda}\tilde{k}^\lambda d\tilde{S}^\mu = \rho^3\frac{2m}{\rho^3}e_{\mu\lambda}\tilde{k}^\lambda\rho^{-2}\rho^2 d\Omega\tilde{t}^\mu = 2me_{\mu\lambda}\tilde{k}^\lambda\tilde{t}^\mu d\Omega.$$

For the Killing vector kλ representing the energy, this integral would equal

$$\frac{1}{8\pi}\frac{(2m)4\pi}{\alpha} = \frac{m}{\alpha}.$$

Letting α → 1 for a extremal value of the parameters in the black hole space-time, the entire spherical area at a fixed radius gives

$$lim_{\alpha\to 1}\frac{m}{\alpha} = m.$$

The surface integrals would be invariant under time translations along Σ, and represent the quasilocal mass and other conserved quantities.

The physical states on a Riemann surface would be constructed in the Hilbert space on the boundary. In a closed string theory, interactions with open strings are required for the insertion of Dirichlet boundaries, while the ideal boundary of a noncompact surfaces exists for surfaces of infinite-genus. It follows that electric charges can be created on the boundaries of infinite-genus surfaces arising from the extreme limit of the Reissner-Nordstrom metric, while the dyonic charges of the black hole solution in the compactified string theory occurs only with the introduction of open strings.

The contradiction in the formula for the entropy derived from black hole metrics in string theory [39] can be resolved generally for those four-manifolds that are fibre bundles locally diffeomorphic to products of Riemann surfaces. The counting of the resulting tensor product of physical states prepared on each surface would be given by an exponential of the mass, given that it is considerably larger than the spacing between mass levels. The formulation of a Hilbert space on the boundary of a Riemann surface has been given [42], and the identification of particle states may be extended to cusps and conical singularities. The singularity of a black hole space-time ordinarily would not be modelled by this geometry.

However, if it is a fixed point of a discrete group action on a nonsingular manifold, there may exist a representation by a fibre bundle in this category.

The existence of supersymmetry may be considered for dyonic black hole solutions to the equations of the heterotic string over T^6 having the metrics

$$ds^2 = -\left(\prod_{i=1}^{n} H_i^{-2r_i}\right)\left(1-\frac{2r_0}{\rho}\right)dt^2 + \left(\prod_{i=1}^{n} H_i^{-2r_i}\right)^{-1}\left[\left(1-\frac{2r_0}{\rho}\right)^{-1}d\rho^2 + \rho^2 d\Omega\right] \quad (5.18)$$

where

$\sum_i r_i = 1,\ H_i = 1 + \frac{h_i}{\rho}$
and
$h_i = -r_0 + \sqrt{r_0^2 + q_i^2},$
with ρ being the spatial radial coordinate, r0 defined by the distance between two horizons and {qi} representing U(1) charges [43]. Since the expansion in inverse powers of ρ gives

$$g_{tt} \sim 1 - \frac{2m}{\rho}\prod_{i=1}^{n} H_i^{-2r_i} = \prod_{i=1}^{n}\left(1 + \frac{\sqrt{r_0^2+q_i^2} - r_0}{\rho}\right)^{-2r_i}$$
$$\sim \prod_{i=1}^{n}\left(1 - \frac{2r_i(\sqrt{r_0^2+q_i^2} - r_0)}{\rho} + \ ...\right)$$
$$\sim 1 - 2\sum_{i=1}^{n}\frac{r_i(\sqrt{r_0^2+q_i^2} - r_0)}{\rho} + ..., \qquad (5.19)$$

the mass can be set equal to
$m = \sum_{i=1}^{n}(r_i\sqrt{r_0^2+q_i^2} - r_0).$

Theorem 4: The range of values of the mass m in the extremal dyonic black hole metric matches the skew eigenvalues of the central charge matrix in the supergravity theory.

Proof:

When
$n = 1,\ m = \sqrt{r_0^2 + q^2} - r_0$

And

$$r_0 = \frac{m^2 - q^2}{2m} \qquad (5.20)$$

If $r_0 = 0$, m = |q|, which is the central charge in N = 2 supergravity. The Bogomolnyi bound is saturated and the extremal metric admits supersymmetry.

When n = 2 and $r_i = 1/2$,

$$m = \frac{1}{2}\left(\sqrt{r_0^2+q_1^2} + \sqrt{r_0^2+q_2^2} - 2r_0\right) \qquad (5.21)$$

and

$$\left[(m+r_0)^2 - \frac{1}{4}[(r_0^2+q_1^2)+(r_0^2+q_2^2)]\right]^2 = \frac{1}{4}(r_0^2+q_1^2)(r_0^2+q_2^2). \qquad (5.22)$$

An expansion of this equation and the cancellation of terms yields a cubic equation in r0

$$2mr_0^3 + 5m^2r_0^2 + 4m^3r_0 + m^4 - mr_0(q_1^2+q_2^2) - \frac{1}{2}m^2(q_1^2+q_2^2) = -\frac{1}{16}(q_1^2 - q_2)^2. \qquad (5.23)$$

If $r_0 = 0$,

$$m^4 - \frac{1}{2}m^2(q_1^2 + q_2^2) + \frac{1}{16}(q_1^2 - q_2^2) = 0 \tag{5.24}$$

And

$$m = \begin{cases} \frac{1}{2}(q_1 + q_2) \\ \frac{1}{2}(q_1 - q_2) \end{cases}. \tag{5.25}$$

matching the skew eigenvalues

$$\begin{aligned} |z_1| &= \frac{1}{2}|q_1 + q_2| \\ |z_2| &= \frac{1}{2}|q_1 - q_2|. \end{aligned} \tag{5.26}$$

of the central charge matrix in N = 4 supergravity.

Let n = 3 and $r_i = 1/3$. The formula for the mass is

$$m = \frac{1}{3}\left(\sqrt{r_0^2 + q_1^2} + \sqrt{r_0^2 + q_2^2} + \sqrt{r_0^2 + q_3^2} - 3r_0\right). \tag{5.27}$$

The first charge q_1 will be chosen to have the maximum value. Transposing the first radical,

$$9(m + r_0)^2 + r_0^2 + q_1^2 - 6(m + r_0)\sqrt{r_0^2 + q_1^2} = 2r_0^2 + q_2^2 + q_3^2 + 2\sqrt{r_0^2 + q_2^2}\sqrt{r_0^2 + q_3^2}. \tag{5.28}$$

Then

$$\begin{aligned} &[(9(m + r_0)^2 + q_1^2 - q_2^2 - q_3^2 - r_0^2)^2 - 36(m + r_0)^2(r_0^2 + q_1)^2 - 4(r_0^2 + q_2^2)(r_0^2 + q_3^2)]^2 \\ &\qquad = 576(m + r_0)^2(r_0^2 + q_1^2)(r_0^2 + q_2^2)(r_0^2 + q_3^2) \end{aligned} \tag{5.29}$$

yielding a seventh-order equation for r_0. Setting r_0 equal to zero in Eq.(5.26)

$$9m^2 - 6q_1 m + q_1^2 - (q_2 + q_3)^2 = 0 \tag{5.30}$$

which has the solutions

$$m = \frac{1}{3}[q_1 \pm (q_2 + q_3)] \tag{5.31}$$

Transposing the first two radicals,

$$\begin{aligned} &9(m + r_0)^2 + 2r_0^2 + q_1^2 + q_2^2 - 6(m + r_0)\sqrt{r_0^2 + q_1^2} + \sqrt{r_0^2 + q_2^2}\Big) + 2\sqrt{r_0^2 + q_1^2}\sqrt{r_0^2 + q_2^2} \\ &\qquad = r_0^2 + q_3^2. \end{aligned} \tag{5.32}$$

When $r_0 = 0$,

$$9m^2 - 6(q_1 + q_2)m + (q_1 + q_2)^2 - q_3^2 = 0 \tag{5.33}$$

And

$$m = \frac{1}{3}[q_1 + q_2 \pm q_3]. \tag{5.34}$$

The range of the values of the mass is

$$\begin{aligned} &\frac{1}{3}[q_1 + q_2 + q_3] \\ &\frac{1}{3}[q_1 - q_2 - q_3] \\ &\frac{1}{3}[q_1 + q_2 - q_3] \end{aligned} \tag{5.35}$$

while the skew eigenvalues of the central charge matrix of N=2 supergravity are

$$\begin{aligned} |z|_1 &= \frac{1}{3}|q_1 + q_2 + q_3| \\ |z|_2 &= \frac{1}{3}|q_1 - (q_2 + q_3)|. \end{aligned} \tag{5.36}$$

Let n = 4 and $r_i = 1/4$. Then

$$m = \frac{1}{4}\left(\sqrt{r_0^2 + q_1^2} + \sqrt{r_0^2 + q_2^2} + \sqrt{r_0^2 + q_3^2} + \sqrt{r_0^2 + q_4^2} - r_0\right) \tag{5.37}$$

And

$$\begin{aligned} 16(m + r_0)^2 + (q_1^2 + q_2^2 - q_3^2 - q_4^2) - 8(m + r_0)\left(\sqrt{r_0^2 + q_1^2} + \sqrt{r_0^2 + q_2^2}\right) \\ = 2\left(\sqrt{r_0^2 + q_3^2}\sqrt{r_0^2 + q_4^2} - \sqrt{r_0^2 + q_1^2}\sqrt{r_0^2 + q_1^2}\right). \end{aligned} \tag{5.38}$$

When $r_0 = 0$,

$$16m^2 + (q_1^2 + q_2^2 - q_3^2 - q_4^2) - 8m(q_3 + q_4) = 2(q_3 q_4 - q_1 q_2) \tag{5.39}$$

And

$$m = \frac{1}{4}[q_1 + q_2 \pm (q_3 + q_4)]. \tag{5.40}$$

Pairing the charges (q_1q_3) and (q_2q_4), the solution is

$$m = \frac{1}{4}[q_1 + q_3 \pm (q_2 + q_4)]. \tag{5.41}$$

Finally, pairing of the charges (q_1q_4) and (q_2q_3) gives

$$m = \frac{1}{4}[q_1 + q_4 \pm (q_2 + q_3)]. \tag{5.42}$$

The range of values of the mass is

$$\begin{aligned} &\frac{1}{4}[q_1 + q_2 + q_3 + q_4] \\ &\frac{1}{4}[q_1 - q_2 + q_3 - q_4] \\ &\frac{1}{4}[q_1 + q_2 - q_3 - q_4] \\ &\frac{1}{4}[q_1 - q_2 - q_3 + q_4] \end{aligned} \tag{5.43}$$

which matches the skew eigenvalues of the central charge matrix in N = 8 supergravity

$$
\begin{aligned}
|z_1| &= \frac{1}{4}|q_1 + q_2 + q_3 + q_4| \\
|z_2| &= \frac{1}{4}|q_1 - q_2 + q_3 - q_4| \\
|z_3| &= \frac{1}{4}|q_1 + q_2 - q_3 - q_4| \\
|z_4| &= \frac{1}{4}|q_1 - q_2 - q_3 + q_4|.
\end{aligned} \tag{5.44}
$$

The solution for m allows a change of sign in the coefficients of qi. It does not have to

equal $\sum_{i=1}^{n} \frac{1}{n}|q_i|$.
By the initial formula for

$$m = \sum_{i=1}^{n} \frac{1}{n}\left(\sqrt{r_0^2 + q_i^2} - r_0\right),$$

the absolute value signs follow from the positive sign for the square root. When either sign can be selected for the radicals, a range of values results for the mass. Since this range coincides with the skew eigenvalues of the central charge matrix, the Bogomolnyi bound would be saturated and supersymmetry is preserved. The existence of Killing spinors on these extremal geometries may be verified to be independent of the signs of the charges $\{q_i\}$.

6. CONCLUSION

The existence of geometric models of elementary particles is considered to be viable. The extremal form of black hole metrics has been found to be required by the consistency of string propagation. A large class of extremal metrics admit supersymmetry, and there exist geometries with two Killing spinors allowing the formulation of N=2 supeconformal field theories. The proof of the fourth theorem demonstrates that the supersymmetric theory can be constructed on a generic extremal black hole metric. The polynomial superpotential is characteristic of the ADE classification of singularities [44]. These Lie groups can be reduced to phenomenological gauge groups [45].

The masses of supersymmetric soliton states in super-Yang-Mills theory and string theory are known to be given by a formula that is compatible with the integer multiple rule [46]. Consequently, the electric and magnetic charges of the tensor product also must be identified with the charges of the particle states. These quantum numbers will label the physical states that are defined at the boundaries of the surfaces arising for four-dimensional manifold with Euclidean sections that are locally diffeomorphic to the products of two surfaces.

Point particle field theories formulated on Minkowski space-time have been experimentally verified and represent the standard models in high energy physics. The quantum consistency of a field theory on a manifold on algebraic singularities must be supplemented by a set of conditions, which would include non-negativity of the mass spectrum and an effective flat space propagation. The positive energy theorem has been proven for asymptotically anti-de Sitter space-times [47], from which the positivity of masses of particles may be deduced [40]. and a locally maximally symmetric curved region also can provide a description of the effect of a massive particle on the manifold. Furthermore, a shift in mass scales in the Hamiltonian in anti-de Sitter space-time cancels the tachyonic ground state energy of a bosonic string rendering the propagation in a Minkowski space-time [48].

The existence of gravitational instantons representing virtual black hole pair creation may be verified through a calculation of the ciontribution to the cosmic background radiation temperature [49]. Nevertheless, unitarity is preserved in particle scattering experiments, characteristic of Minkowski space-time. Consequently, it is necessary to prove that unitarity continues to be valid under quantum fluctuations of the metric.

It is evident also from the extremal limit of the Euclidean Scwharzschild metric that a (2,2) signature arises. Since N = 2 string theory is formulated on backgrounds with this signature [21], it may be conjectured that there exist backgrounds geometries that are extremal black hole space-times. Consequently, the geometrical model also would provide a unified description of the generation of nonperturbative particle states from a fundamental string theory. The space $S^2 \times S^2$ with a (2,2) signature, in §3, has been demonstrated to be a background for the quantization of the N = 2 string theory. A region with the $S^2 \times S^2$ metric also may be regarded as a quantum gravitational bubble [50] related to the virtual black hole creation in the summation over gravitational instantons. The problem of unitarity then could be resolved entirely within a quantum mechanically consistent N = 2 string theory.

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