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The Mode Expansion of the String Fields

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ABSTRACT The expansion of the string coordinates is different curved surfaces and space times. A summation of uniformizing group elements is necessary at genus g ≥ 1. The invariances are listed for the sigma model action in a maximally curved space-time. Thecritical dimension is established through conditions on the quantum commutators.

KEYWORDS Higher Genus, Uniformizing Group, Conformally Flat Spaces, Invariances, Critical Dimension
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1. INTRODUCTION

The quantization of string theory is completed usually in flat space-time. The wave equation is solved in terms of plane waves on the world sheet, and the coefficients are elevated to operators which form combinations in the components of the energy-momentum tensor that satisfy the commutation relations of a Virasoro algebra. A central charge termcan be added to the commutators with the Jacobi identities remaining valid. The constant in the quantum commutators may be determined by requiring that the Lorentz invariance is preserved. Then it is found that the critical dimension is D=26 and the parameter defining the central charge is a = 1.

The wave equation is locally that of a two-dimensional hyperbolic equation on a flatworld sheet. However, for genus g ≥ 1, the Euclidean surface is curved and multiply connected, with the product of exponential and trigonometric functions of the coordinates τ and σ to be no longer solutions to this equation. Instead, for the Riemann surface, the covering space again is flat and multiply-connected in the intermediate Schottky domain or hyperbolic space and simply connected in the universal uniformization for genus g ≥ 2.The Laplace equation again can be solved on the covering space and the automorphic function may be defined as a series over the images of the elements of the uniformizing group.

It is established that the leading-order terms in the expansion are sufficient to establish the consistency of previous calculations. These computations include the determination of the central charge and the coefficient of the anomaly in the Virasoro algebra. Again, it is demonstrated that the critical dimension for the bosonic string is 26 and the parameter in the Lie algebra is a = 1.

The quantization procedure is followed for conformally flat space-times. The classical invariances of the sigma-model are verified on a maximally symmetric curved manifold. The values of the critical dimension and the anomaly coefficient are derived for a given conformal factor that interpolates between Minkowski space-time and anti-de Sitter space, given the quantization of the classical symmetries. The values derived for anti-de Sitter space are compared to those found in the path integral formalism.

2. THE STRING COORDINATES AT HIGHER GENUS IN MINKOWSKI SPACE-TIME

The Polyakov action in flat space-time is

$$\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{h}h^{\alpha\beta}\eta^{\mu\nu}\partial_\alpha X_\mu \partial_\beta X_\nu. \tag{2.1}$$

The energy-momentum tensor is

$$\begin{aligned} T_{\alpha\beta} &= \frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} \\ &= \frac{1}{2\pi\alpha'}(\partial_\alpha X^\mu \partial_\beta X_\nu - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\nu). \end{aligned} \tag{2.2}$$

The equation for the worldsheet string coordinates locally is

$$\nabla^2 X_\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X_\mu = 0. \tag{2.3}$$

The Euclidean form of the wave equation

$$\nabla^2 X_\mu = \left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right) X_\mu = 0 \tag{2.4}$$

also would be valid only locally.

Since this equation is not globally defined over a Riemann surface of genus g ≥ 1, one solution would be given by an automorphic function on the covering surface. The sphere can be projected stereographically onto the complex plane and the covering space of the torus is C. On the real section of these covering spaces, the metric is flat, and the solution is given by

$$X^\mu(\sigma) = \sum_{\gamma\in\Gamma}[X_R^\mu(\gamma\sigma^-) + X_L^\mu(\gamma\sigma^+)] \tag{2.5}$$

is the lattice group such that $T^2 \simeq \mathbf{C}/\Gamma$, where Γ acts on a Lorentzian manifold with σ+ = σ + τ and σ- = σ - τ or Γ acts on a complex space with σ = σ + iτ and σ-σ - iτ. Alternatively, the equation on the torus may be solved in a parallelogram with periodic boundary conditions.

The mode expansion of the string coordinate on a flatworld sheet is

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\ cos\ n\sigma \tag{2.6}$$

which can be expressed as a function of σ+ and σ-

$$X^\mu = \frac{x^\mu}{2} + \frac{p^\mu}{2}\sigma^+ + \frac{x^\mu}{2} - \frac{p^\mu}{2}\sigma^- \\ + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\left\{\frac{1}{2}[cos\ n\sigma^+ + cos\ n\sigma-] - \frac{i}{2}[sin\ n\sigma^+ - sin\ n\sigma^-]\right\}. \tag{2.7}$$

The rotation to Euclidean space is achieved through the redefinition of the variables σ+ and σ-.

On the torus, the mode expansion would be modified to

$$X^\mu(\sigma,\tau) = lim_{card\ \Gamma\to\infty}\frac{1}{card\ \Gamma}\sum_{\gamma\in\Gamma}\left\{\frac{x^\mu}{2} + \frac{p^\mu}{2}(\gamma\cdot\sigma^+) + \frac{x^\mu}{2} - \frac{p^\mu}{2}(\gamma\cdot\sigma^-)\right. \\ + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\left\{\frac{1}{2}[cos\ (n\gamma\cdot\sigma^+) + cos(n\gamma\cdot\sigma^-)]\right. \\ \left. - \frac{i}{2}[sin(n\gamma\cdot\sigma^+) - sin(n\gamma\cdot\sigma^-)]\right\}. \\ = x^\mu + ip^\mu\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\left\{\frac{1}{2}[cos\ n\sigma^+ + cos\ n\sigma^-] - \frac{i}{2}[sin\ n\sigma^+ - sin\ n\sigma^-]\right\} \\ + lim_{card\ \Gamma\to\infty}\ \frac{1}{card\ \Gamma}\sum_{\substack{\gamma\in\Gamma \\ \gamma\neq I}}\left\{\frac{x^\mu}{2} + \frac{p^\mu}{2}(\gamma\cdot\sigma^+) + \frac{x^\mu}{2} - \frac{p^\mu}{2}(\gamma\cdot\sigma^-)\right. \\ + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\left\{\frac{1}{2}[cos\ (n\gamma\cdot\sigma^+) + cos(n\gamma\cdot\sigma^- -)]\right. \\ \left. - \frac{i}{2}[sin(n\gamma\cdot\sigma^+) - sin(n\gamma\cdot\sigma^-)]\right\}. \tag{2.8}$$

In the light-cone gauge, on a flat worldsheet with Lorentzian signature, X+(σ, τ) = x+ +p+τ. Then, on the torus

$$X^+(\sigma,\tau) = x^+ + ip^+\tau + \sum_{\gamma\neq I}(x^+ + p^+(\gamma\cdot\sigma^+) + p^-(\gamma\cdot\sigma^-)). \tag{2.9}$$

Similarly,

$$X^-_{g=1}(\sigma,\tau) = x^- + ip^-\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\ e^{-in\tau}\ \Big\{\frac{1}{2}[cos\ n\sigma^+ + cos\ n\sigma^-] - \frac{i}{2}[sin\ n\sigma^+ - sin\ n\sigma^-]\Big\}$$
$$+\sum_{\gamma\neq I}\Big\{\frac{x^-}{2} + \frac{p^-}{2}(\gamma\cdot\sigma^+) + \frac{x^-}{2} - \frac{p^+}{2}(\gamma\cdot\sigma^-)$$
$$+ i\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\ e^{-in\tau}\ \Big\{\frac{1}{2}[cos\ (n\gamma\cdot\sigma^+) + cos(n\gamma\cdot\sigma^-)] - \frac{i}{2}[sin(n\gamma\cdot\sigma^+) - sin(n\gamma\cdot\sigma^-)]\Big\}. \quad (2.10)$$

The rotation of $X^-(\sigma, \tau)$ to the string worldsheet would be given by returning σ+ and σ-to σ + τ and σ - τ respectively and

$$X^-(\sigma,\tau) = x^- + p^-\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\ e^{-in\tau}\ cos\ n\sigma$$
$$+\sum_{\gamma\neq I}\Big\{\frac{x^-}{2} + \frac{p^-}{2}(\gamma\cdot\sigma^+) + \frac{x^-}{2} - \frac{p^+}{2}(\gamma\cdot\sigma^-)\Big\}$$
$$+ i\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\ e^{-in\tau}\ \Big\{\frac{1}{2}[cos\ (n\gamma\cdot\sigma^+) + cos(n\gamma\cdot\sigma^-)] - \frac{i}{2}[sin(n\gamma\cdot\sigma^+) - sin(n\gamma\cdot\sigma^-)]\Big\}. \quad (2.11)$$

Then the coefficients α_n^- in operator form are

$$\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{n-m}\alpha^i_m: - a\delta_n\right) \quad (2.12)$$

with a being a normal ordering constant.

Since the expansion of the light-cone coordinates has the same leading terms and the remaining terms have a form determined by the uniformizing group Γ, with the same coefficients α_n^-, the derivation of *D* and *a* [1] is identical. Let the generators of the Poincare group be

$$J^{\mu\nu} = \ell^{\mu\nu} + E^{\mu\nu}$$
$$\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \quad (2.13)$$
$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n)$$

With the normal ordering, the Hamiltionian would be equal to

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty} :\alpha_{-n}^{i}\alpha_{n}^{i}: + \frac{D-2}{2}\sum_{n=1}^{\infty} n \tag{2.14}$$

and, after zeta function regularization of the last sum,

$$\frac{D-2}{2} lim_{s\to-1}\sum_{n=1}^{\infty}\frac{1}{n^s} \sim \frac{D-2}{2}\zeta(-1) = -\frac{D-2}{24}.$$

Then

$$[J^{i-}, J^{j-}] = \frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m(\alpha_m^i\alpha_m^j - \alpha_m^j\alpha_m^i) \tag{2.15}$$

where

$$\Delta_m = m\left(\frac{26-D}{2}\right) + \frac{1}{m}\left[\frac{D-26}{12} + 2(1-a)\right].$$

Lorentz invariance is preserved upon quantization of the string theory if $\Delta m = 0$ or $D = 26$ and $a = 1$.

The value of *a* is the vertical intercept of the leading Regge trajectory in bosonic string theory which requires a negative value of M2 for the horizontal intercept representing the tachyon. It may be recalled that the trajectory may be derived from the Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-(\alpha(s)+\alpha(t)))}, \tag{2.16}$$

with $\alpha(s) = \alpha(0) + \alpha' s$

and $\alpha(t) = \alpha(0) + \alpha' t.$

where s and t are the Mandelstam variables.

Then the amplitude has poles $s = M_n^2 = \frac{n-\alpha(0)}{\alpha'}.$

Setting J = n for spin-n resonance, Therefore, the intercept of the J vs. M^2 plot follows from the poles of the amplitude.

The effect of higher-genus contributions on these graphs also may be considered. In the Schottky parameterization of the Riemann surface, fixing three of the vertices by a global SL(2, C) transformation, the four-point amplitude for spin-0 bosonic string modes[2] would be

$$f(z_1^0, z_2^0, z_3^0) \int_{\Delta_g} d^2 z_4 |z_4 - z_1^0|^{-\frac{p_1 \cdot p_4}{2}} \prod_{\alpha \neq I}{}' \left| \frac{z_4 - V_\alpha z_1^0}{z_4 - V_\alpha z_4} \frac{z_1^0 - -V_\alpha z_4}{z_1^0 - V_\alpha z_1^0} \right|^{-\frac{p_1 \cdot p_2}{2}}$$
$$\prod_{m,n=1}^{g} exp\left[\frac{p_1 \cdot p_4}{4} Re(v_m(z_4) - v_m(z_1^0))(Im\ \tau)^{-1}_{mn} Re(v_n(z_4) - v_n(z_1^0))\right]$$
$$\times (similar\ factors\ with z_1^0 \to z_2^0, z_3^0,\ p_1 \to p_2, p_3), \tag{2.18}$$

Where

$$f(z_1^0, z_2^0, z_3^0) = |z_2^0 - z_1^0|^{2-\frac{p_1 \cdot p_2}{2}} |z_3^0 - z_1^0|^{2-\frac{p_1 \cdot p_3}{2}} |z_3^0 - z_2^0|^{2-\frac{p_2 \cdot p_3}{2}}$$
$$\left| \frac{z_2^0 - V_\alpha z_1^0}{z_2^0 - V_\alpha z_2^0} \frac{z_1^0 - V_\alpha z_2^0}{z_1^0 - V_\alpha z_1^0} \right|^{-\frac{p_1 \cdot p_2}{2}} \prod_{\alpha \neq I}{}' \left| \frac{z_3^0 - V_\alpha z_1^0}{z_3^0 - V_\alpha z_3^0} \frac{z_1^0 - V_\alpha z_3^0}{z_1^0 - V_\alpha z_1^0} \right|^{-\frac{p_1 \cdot p_3}{2}}$$
$$\left| \frac{z_3^0 - V_\alpha z_2^0}{z_3^0 - V_\alpha z_3^0} \frac{z_2^0 - V_\alpha z_3^0}{z_2^0 - V_\alpha z_2^0} \right|^{-\frac{p_2 \cdot p_3}{2}}$$
$$\prod_{m,n=1}^{g} exp\left[\frac{p_1 \cdot p_2}{8\pi} Re(v_m(z_2^0) - v_m(z_1^0))(Im\ \tau)^{-1}_{mn} Re(v_n(z_2^0) - v_n(z_1^0))\right.$$
$$+ \frac{p_1 \cdot p_3}{8\pi} Re(v_m(z_3^0 - v_m(z_{0}))(Im\ \tau)^{-1}_{mn} Re(v_n(z_3^0) - v_n(z_1^0))$$
$$\left. + \frac{p_2 \cdot p_3}{2} Re(v_m(z_3^0) - v_m(z_2^0))(Im\ \tau)^{-1}_{mn} Re(v_n(z_3^0) - v_n(z_2^0))\right], \tag{2.19}$$

and

$$v_n(z) = \sum_\alpha{}^{(n)} ln\left(\frac{z - V_\alpha \xi_{1n}}{z - V_\alpha \xi_{2n}}\right),$$

with $\sum_\alpha{}^{(n)}$ being a sum over all elements which do nothave $T_n^{\pm 1}$ at the right-hand end of the product and $\prod_\alpha{}'$ not including both an element and its inverse, while Δ_g is the fundamental domain of the Schottky group with g generators. The action of V_α on a point z in the fundamental domain exterior to the isometric circles of the generators is a mapping to the interiors of these circles. Therefore, the pointz and $V_\alpha z,\ \alpha \neq I$ never can coincide for $z \in \Delta_g$. Consequently, the integrand of the four-function amplitude is regular except at the three points z_1^0, z_2^0 and z_3^0. consequently, it may be written as

$$\int_0^{2\pi} \int_{|z_4 - z_1^0| \leq \Lambda} d\theta d|z_4 - z_1^0||z_4 - z_1^0|^{1-\frac{p_1 \cdot p_2}{2}} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{(z_4 - z_1^0)^\ell (\bar{z}_4 - \bar{z}_1^0)^m}{\ell! m!}$$
$$\partial^\ell \bar{\partial}^m \{|z_4 - z_2^0|^{-\frac{p_1 \cdot p_2}{2}} |z_4 - z_2^0|^{-\frac{p_2 \cdot p_4}{2}} |z_4 - z_3^0|^{-\frac{p_3 \cdot p_4}{2}} \Phi(z_4, \bar{z}_4)\}_{z_4 = z_1^0}$$
$$(similar\ terms\ with\ z_1^0 \to z_2^0\ z_3^0,\ p_1 \to p_2, p_3) + finite$$

$$= 2\pi \sum_{n=0} \frac{\Lambda^{-\frac{p_1 \cdot p_4}{2}+2n+2}}{-\frac{1}{2}p_1 \cdot p_4 + 2n + 2} \frac{1}{(n!)^2}$$
$$\partial^\ell \bar{\partial}^m \{ |z_4 - z_2^0|^{-\frac{p_1 \cdot p_2}{2}} |z_4 - z_2^0|^{-\frac{p_2 \cdot p_4}{2}} |z_4 - z_3^0|^{-\frac{p_3 \cdot p_4}{2}} \Phi(z_4, \bar{z}_4) \}_{z_4 = z_1^0}$$
$$(similar\ terms\ with\ z_1^0 \to z_2^0 m\ z_3^0,\ p_1 \to p_2, p_3) + finite, \tag{2.20}$$

which has poles at s, t, u = 8(n - 1). Therefore, the Regge trajectory has the same format higher-genus. This calculation may be extended to infinite-genus surfaces in the class O_G [3]. When the surface does not belong to OG, the ideal boundary may be regarded as another source for the Green function.

The duality of the Veneziano amplitude reflects the invariance under observations from different directions. For example, the s and t channel amplitudes can be defined for scattering along the axis of symmetry between the momentum vectors $\vec{p}_1$ and $\vec{p}_2$ and the axis of symmetry between the vectors $\vec{p}_1$ and $\vec{p}_3$. Unlike point particle theory, the smoothness of the surface does not distinguish between the interacting states from the two angles. Therefore, the first process would be an scattering of the states with momenta $p_1$and p_2, while the second can be interpreted to represent an interaction between strings with momenta p_1 and p_3.

There is a fundamental dichotomy in the string model with the direction of the rotation representing of the string. It is well known that the string preserves its form as a one dimensional object if the rotation is defined with respect to the axis perpendicular to the enclosed region. However, if the rotation occurs with respect to an axis through the string, the resulting configuration resembles a sphere with sufficient angular velocity. Therefore, the physical consequences for the new theory include a connection with phenomenological spherical models for elementary particles. Nevertheless, the description of this rotating string model may be developed using duality invariance of the amplitudes. The scattering of spheres, furthermore, will have a three dimensional rotational invariance which is a generalization of planar duality.

3. STRING THEORY IN A CONFORMALLY FLAT CURVED SPACE-TIME

The sigma model for a bosonic string on a curved space-time is

$$I = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu. \tag{3.1}$$

This action possesses several invariances in a maximally symmetric space-time of constant negative curvature.

1. Worldsheet reparameterization invariance

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$
$$\delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma} \tag{3.2}$$
$$\delta(\sqrt{h}) = \partial_\alpha (\xi^\alpha \sqrt{h})$$

$$
\begin{aligned}
\delta S = \frac{1}{4\pi\alpha'} \int d^2\sigma \delta(\sqrt{h}) h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \\
+ \sqrt{h}\delta(h^{\alpha\beta}) g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \\
+ \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha (\delta X_\mu) \partial_\beta X_\nu \\
+ \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta (\delta X_\nu)] \\
= \frac{1}{4\pi\alpha'} \int_{\partial\Sigma} d\ell_\gamma \xi^\gamma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu
\end{aligned}
\tag{3.3}
$$

If the surface has no boundary, this integral vanishes. When the surface has a boundary at infinity and Xμ → 0 in this limit, it again equals zero.

2. Weyl scaling

$$
\begin{aligned}
\delta h^{\alpha\beta} &= \Lambda h^{\alpha\beta} \\
\delta h_{\alpha\beta} &= -\Lambda h_{\alpha\beta} \\
\delta(\sqrt{h}) &= -\Lambda \sqrt{h}
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\delta S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \delta(\sqrt{h}) h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \\
+ \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{h} \delta h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \\
= \frac{1}{4\pi\alpha'} \int_\Sigma (-\Lambda\sqrt{h}) h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \\
+ \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{h} \Lambda h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu
\end{aligned}
\tag{3.5}
$$

SO(D - 1, 2) invariance

$$
\begin{aligned}
\delta X^\mu &= M^\mu \nu X^\nu \\
\delta g^{\mu\nu} &= M_\rho{}^\mu g^{\rho\nu} + M_\rho{}^\nu g^{\mu\rho} \\
\delta X_\mu &= -M_\mu{}^\nu X_\nu
\end{aligned}
\tag{3.6}
$$

where Mμν are generators of the so(d - 1, 2) algebra.

$$
\begin{aligned}
\delta S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} (M_\rho{}^\mu g^{\rho\nu} + M_\rho{}^\nu g^{\mu\rho}) \partial_\alpha X_\mu \partial_\rho X_\nu \\
- \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} M_\mu{}^\rho \partial_\alpha X_\rho \partial_\beta X_\nu \\
- \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} M_\nu{}^\rho \partial_\alpha X_\mu \partial_\beta X_\rho
\end{aligned}
\tag{3.7}
$$

Given the classical symmetries, the effect of a conformal transformation on the quantization may be considered.

Setting

$$g^{\mu\nu} = \Omega^{-2}\eta^{\mu\nu}$$

and

$$g_{\mu\nu} = \Omega^2\eta_{\mu\nu},$$

the action is

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}h^{\gamma\delta}\Omega^{-2}\eta^{\mu\nu}\partial_\gamma X_\mu\partial_\delta X_\nu. \quad (3.8)$$

The partial derivative of the variation with respect to $\partial_\alpha X^\mu$ is

Since $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$, each component separately satisfies the Laplace equation $\nabla^2 X^\mu = 0$. The mode expansion, the coefficients and the operators will be unchanged.

By normal ordering, D = 26 and a = 1. It follows that critical dimensions will be 26for Minkowski space-time, anti-de Sitter space and the geometry with the scale factor

$$a(\tau) = \frac{1}{H^2\tau^2} + \frac{\delta}{H^2},$$

which interpolates between the two maximally symmetric space=times. The massless graviton state is defined in terms of the oscillator modes. The Weyl anomaly cancellation is a variant of the Lorentz anomaly. The commutation relations of the anti-de Sitter generators include the dimension, curvature and a. The limit of vanishing curvature must yield D = 26 and a = 1.

4. STRING THEORY ON ANTI-DE SITTER SPACE

String theory on three-dimensional anti-de Sitter space may be quantized because it is diffeomorphic to the group SL(2; **R**). String theory on group manifolds may be quantized by adding a WZW term to the two-dimensional action

$$S = \frac{k}{4\pi}\int_S d\tau\ d\sigma d\tau d\sigma\eta^{\alpha\beta}Tr[g^{-1}\partial_\alpha g g^{-1}\partial_\alpha g] + \frac{k}{6\pi}\int_B Tr[g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg], \quad (4.1)$$

The additional term is necessary for conformal invariance at the quantum level which is ensured by the vanishing of the beta function for Weyl transformations. The level numberk is initially introduced in the WZW term such that e^{-S} is not altered by coordinate transformations in components of the diffeomorphism group not connected to the identity [6]. However, it can be included as a coefficient of the first integral by setting it to be proportional to R^2 [7]. The space B is the three-dimensional coboundary of S and g∈SL(2; **R**). Since S is string worldsheet with signature (-+), the coboundary is defined for a Lorentzian manifold. Since the Euclidean continuation of the string worldsheet is a Riemannian two-dimensional manifold, it would be the boundary of a three-dimensional manifold with Euclidean signature. The analytic continuation to the Lorentzian manifold would have signature (-++). An element of the group would be

$$g - \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1.$$

Then

$$S=\frac{k}{2\pi}\int_S d\tau\ d\sigma[[\dot{a}\dot{d}-a'd'-\dot{b}\dot{c}+b'c'-\frac{k}{\pi}\int_S d\tau\ d\sigma log(b)[\dot{a}d'-a'\dot{d}]. \tag{4.2}$$

It follows from the determinant condition for matrices in SL(2; R) that the coordinates (T, U, X, Y), defined by

$$a=\tfrac{1}{R}(U+X),\ b=\tfrac{1}{R}(T-Y),\ c=-\tfrac{1}{R}(T+Y),\ d=\tfrac{1}{R}(U-X),$$

describe the embedding of three-dimensional anti-de Sitter space into a four-dimensional space-time of signature (- - ++),

$$-T^2-U^2+X^2+Y^2=-R^2, \tag{4.3}$$

where R is the radius of curvature. The action equals

$$\begin{aligned}S_3=-\frac{k}{2\pi R^2}\int_S d\tau\ d\sigma\ \Big[&\dot{T}^2-T'^2+\dot{U}^2-U'^2-\dot{X}^2+X'^2-\dot{Y}^2+Y'^2\\ &+4\lambda(-T^2-U^2+X^2+Y^2+Z^2+R^2)+4(\dot{X}U'-X'U)\ log\left(\frac{1}{R}(T-Y)\right)\Big],\end{aligned} \tag{4.4}$$

and, after setting, qμ equal to the four-vector

$$(\tfrac{1}{4}T,\tfrac{1}{R}U,\tfrac{1}{R}X,\tfrac{1}{R}Y),$$

the equations of motion are

$$\begin{aligned}&q^\mu_{+-}+e^\alpha q^\mu+\epsilon^\mu{}_{\nu\rho\sigma}q^\mu q^\rho_+ q^\sigma_-=0\\ &e^\alpha=-\eta_{\mu\nu}q_{\mu q^\nu_-},\end{aligned} \tag{4.5}$$

where $\eta^{\mu\nu}$ is the flat embedding space metric and

$$q^\mu_\pm=\frac{dq^\mu}{d\sigma_{pm}}\ [8].$$

Setting

$$\ell^\mu=e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho q^\sigma_+ q^\delta_-,$$

$$\begin{aligned}e^{-\alpha}q^\mu_{+-}+q^\mu+e^{-\alpha}\epsilon^\mu{}_{\nu\rho\sigma}q^\nu q^\rho_+ q^\sigma_-&=e^{-\alpha}q^\mu_{+-}+q^\mu+\ell^\mu=0,\\ q^\mu_{+-}&=-e^\alpha(q^\mu+\ell^\mu).\end{aligned} \tag{4.6}$$

Suppose that

$$q^\mu_{++}=a_{++}q^\mu+q^{\alpha\,\mu}_+ +b_{++}q^\mu_- +u\ell^\mu \tag{4.7}$$

The coefficients can be determined by the contractions with the basis vectors

$\{q^{\mu}, q^{\mu}_{+}, q^{\mu}_{-}, \ell^{\mu}\},$

which are normalized to have the inner products

$q^{\mu} q_{\mu} = -1,\ q^{\mu}_{\pm} q_{\pm\mu} = 0,\ q^{\mu}_{+} q_{-\mu} = -1,\ \ell^{\mu}\ell_{\mu} = 1.$

Since

$$q^{\mu}_{++} q_{\mu} = a q^{\mu} q_{\mu} = -a_{++} \tag{4.8}$$

$$\frac{\partial^2}{\partial\sigma_{+}^2}(q^{\mu} q_{\mu}) = 2q^{\mu}_{++} q_{\mu} + q^{\mu}_{+} q_{+\mu} 2q^{\mu}_{++} q_{\mu} = -2a_{++} = 0. \tag{4.9}$$

Given that

$$q^{\mu}_{++} q_{+\mu} = b q^{\mu}_{-} q_{+\mu} = -b_{++}, \tag{4.10}$$

$$\frac{d}{d\sigma_{+}}(q^{\mu}_{+} q_{+\mu}) = 2q + ++^{\mu} q_{+\mu} = -2b_{++} = 0. \tag{4.11}$$

The coefficient α+ is nonvanishing because

$$\begin{aligned} q^{\mu}_{++} q_{-\mu} = \alpha_{+} q^{\mu}_{+} q_{-\mu} = -\alpha_{+} e^{\alpha} \\ \alpha_{+} = -e^{-\alpha} q^{\mu}_{++} q_{-\mu}. \end{aligned} \tag{4.12}$$

Differentiating

$e^{\alpha} = -\eta_{\mu\nu} q^{\mu}_{+} q^{\nu}_{-},$

$$\begin{aligned} \alpha_{+} e^{\alpha} &= -q^{\mu}_{++} q_{-\mu} + q^{\mu}_{+} q_{+-\mu} \\ &= -q^{\mu}_{++} q_{-\mu} - q^{\mu}_{+} e^{\alpha}(q_{\mu} + \ell_{\mu}) \end{aligned} \tag{4.13}$$

And

$$\alpha_{+} = -e^{-\alpha} q^{\nu}_{++} q_{-\nu} - q^{\mu}_{+}(q_{\mu} + \ell_{\mu}) = -e^{-\alpha} q^{\nu}_{++} q_{-\nu}. \tag{4.14}$$

$$q^{\mu}_{++} \ell_{\mu} = u \ell^{\mu} \ell_{\mu} = u. \tag{4.15}$$

The expansion of q^{μ}_{--} in the basis of four-vectors is

$$q^{\mu}_{--} = a_{--}q^{\mu} + b_{--}q^{\mu}_{+} + \alpha_{-}q^{\mu}_{-} + v\ell^{\mu}, \tag{4.16}$$

$$\begin{aligned} q^{\mu}_{--}q_{\mu} &= a_{--}q^{\mu}q_{\mu} = -a_{--} = 0 \\ q^{\mu}_{--}q_{+\mu} &= \alpha_{-}q^{\mu}_{-}q_{+\mu} = -\alpha_{-}e^{\alpha} \\ q^{\mu}_{--}q_{-\mu} &= b_{--}q^{\mu}_{+}q_{-\mu} = -b_{--} = 0 \\ q^{\mu}_{--}\ell_{\mu} &= v. \end{aligned} \tag{4.17}$$

The second relation follows from

$$\begin{aligned} \frac{\partial}{\partial\sigma_{-}}(q^{\mu}_{+}q_{-\mu}) &= q^{\mu}_{+-}q_{-\mu} + q^{\mu}_{+}q_{--\mu} = -e^{\alpha}(q^{\mu} + \ell^{\mu})q_{-\mu} + q^{\mu}_{+}q_{--\mu} \\ &= -\alpha_{-}e^{\alpha}. \end{aligned} \tag{4.18}$$

The derivative of the scalar function u with respect to σ is

$$\begin{aligned} \frac{\partial}{\partial\sigma_{-}}u &= q^{\mu}_{++-}\ell_{\mu} + q^{\mu}_{++}\ell_{-\mu} = -\frac{\partial}{\partial\sigma_{+}}[e^{\alpha}(q^{\mu} + \ell^{\mu})]\ell_{\mu} + (\alpha_{+}q^{\mu}_{+} + u\ell^{\mu})\ell_{-\mu} \\ &= -\alpha_{+}e^{\alpha}(q^{\mu} + \ell^{\mu})\ell_{\mu} - e^{\alpha}(q^{\mu}_{+} + \ell^{\mu}_{+})\ell_{\mu} + \alpha_{+}q^{\mu}_{+}\ell_{-\mu} = -\alpha_{+}e^{\alpha} - \alpha_{+}q^{\mu}_{+-}\ell_{\mu} \\ &= -\alpha_{+}e^{\alpha} - \alpha_{+}[-e^{\alpha}(q^{\mu} + \ell^{\mu})]\ell_{\mu} = 0. \end{aligned} \tag{4.19}$$

Setting q^{μ}_{++-} equal to q^{μ}_{+-+},

$$\begin{aligned} q^{\mu}_{++-} &= \alpha_{+-}q^{\mu}_{+} + \alpha_{+}q^{\mu}_{+-} + u_{-}\ell^{\mu} + u\ell^{\mu}_{-} \\ q^{\mu}_{+-+} &= -\alpha_{+}e^{\alpha}(q^{\mu} + \ell^{\mu}) - e^{\alpha}(q^{\mu}_{+} + \ell^{\mu}_{+}) \end{aligned} \tag{4.20}$$

The difference equals

$$\begin{aligned} q^{\mu}_{++-} - q^{\mu}_{+-+} &= \alpha_{+}e^{\alpha}q^{\mu} + (\alpha_{+-} + e^{\alpha})q^{\mu}_{+} + \alpha_{+}q^{\mu}_{+-} + (u_{-} + \alpha_{+}e^{\alpha})\ell^{\mu} + u\ell^{\mu}_{-} + e^{\alpha}\ell^{\mu}_{+} \\ &= \alpha_{+}e^{\alpha}q^{\mu} + (\alpha_{+-} + e^{\alpha})q^{\mu}_{+} + \alpha_{+}(-e^{\alpha}(q^{\mu} + \ell^{\mu})) + (u_{-} + \alpha_{+}e^{\alpha})\ell^{\mu} \\ &\qquad + u\ell^{\mu}_{-} + e^{\alpha}\ell^{\mu}_{+} \\ &= (\alpha_{+-} + e^{\alpha})q^{\mu}_{+} + u_{-}\ell^{\mu} + u\ell^{\mu}_{-} + e^{\alpha}\ell^{\mu}_{+} \end{aligned} \tag{4.21}$$

The derivatives of ℓ^{μ} are

$$
\begin{aligned}
\ell^\mu_+ &= \frac{\partial}{\partial\sigma_+}(e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho q^\sigma_+ q^\delta_-) \\
&= -\alpha_+\ell^\mu + e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}(q^\rho_+ q^\sigma_+ q^\delta_- + q^\rho q^\sigma_{++}q^\delta_- + q^\rho q +^\sigma q^\delta_{-+}) \\
&= -\alpha_+\ell^\mu + e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}(q^\rho(\alpha_+ q^\sigma_+ + u\ell^\sigma)q^\delta_- + q^\rho q^\sigma_+(-e^\alpha(q^\delta + \ell^\delta)) \\
&= -\alpha_+\ell^\mu + \alpha_+\ell^\mu + e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho(u\ell^\sigma q^\delta_- - e^\alpha q^\sigma_+\ell^\delta) \\
&= e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho(u\ell^\sigma q^\delta_- - e^\alpha q^\sigma_+\ell^\delta) \\
\ell^\mu_- &= \frac{\partial}{\partial\sigma_-}(e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho q^\sigma_+ q^\delta_-) \\
&= -\alpha_-\ell^\mu + e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}(q^\rho_- q^\sigma_+ q^\delta_- + q^\rho q^\sigma_{+-}q^\delta_- + q^\rho q^\sigma_+ q^\delta_{--}) \\
&= -\alpha_-\ell^\mu + e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}(q^\rho[-e^\alpha(q^\sigma + \ell^\sigma)]q^\delta_- + q^\rho q^\sigma_+(\alpha_- q^\delta_- + v\ell^\delta) \\
&= e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho(-e^\alpha\ell^\sigma q^\delta_- + vq^\sigma_+\ell^\delta),
\end{aligned}
\tag{4.22}
$$

which are linear combinations of q^μ_+ and qμ. Since

$$
\begin{aligned}
u\ell^\mu_- + e^\alpha\ell^\mu_+ &= e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta}q^\rho[u(-e^\alpha\ell^\sigma q^\delta_- + vq^\sigma_+\ell^\delta) + e^\alpha(u\ell^\sigma q^\delta_- - e^\alpha q^\sigma_+\ell^\delta)] \\
&= e^{-\alpha}(uv - e^{2\alpha})\epsilon^\mu_{\rho\sigma\delta}q^\rho q^\sigma_+\ell^\delta.
\end{aligned}
\tag{4.23}
$$

Therefore,

$$
q^\mu_{++-} - q^\mu_{+-+} = (\alpha_{+-} + e^\alpha)q^\mu_+ + u_-\ell^\mu + e^{-\alpha}(uv - e^{2\alpha})\epsilon^\mu_{\rho\sigma\delta}q^\rho q^\sigma_+\ell^\delta, \tag{4.24}
$$

which vanishes when

$$
\begin{aligned}
\alpha_{+-} + e^\alpha &= 0 \\
u_- &= 0 \\
uv - e^{2\alpha} &= 0.
\end{aligned}
\tag{4.25}
$$

The equality of the derivatives q--+ and q+--, with

$$
\begin{aligned}
q_{--+} &= \frac{\partial}{\partial\sigma_+}(\alpha_- q^\mu_- + v\ell^\mu) = -\alpha_{-+}q^\mu_- + \alpha_- q^\mu_{-+} + v_+\ell^\mu + v\ell^\mu_- \\
&= \alpha_{-+}q^\mu_- + \alpha_-[-e^\alpha(q^\mu_+\ell^\mu)] + v_+\ell^\mu + v\ell^\mu_+ \\
q_{+--} &= \frac{\partial}{\partial\sigma-}[-e^\alpha(q^\mu + \ell^\mu)] = -\alpha_- e^\alpha(q^\mu + \ell^\mu) - e^\alpha(q^\mu_- + \ell^\mu_-),
\end{aligned}
\tag{4.26}
$$

requires

$$
(\alpha_{-+} + e^\alpha)q^\mu_- + v_+\ell^\mu + v\ell^\mu_+ + e^\alpha\ell^\mu_- = 0. \tag{4.27}
$$

Since

$$v\ell_+^\mu + e^\alpha \ell_-^\mu = e^{-\alpha}\epsilon^\mu_{\rho\sigma\delta} q^\rho [v(u\ell^\sigma q_-^\delta - e^\alpha q_+^\sigma \ell^\delta) + e^\alpha(-e^\alpha \ell^\sigma q_-^\delta + v q_+^\sigma \ell^\delta)]$$
$$= e^{-\alpha}(uv - e^{2\alpha})\epsilon^\mu_{\rho\sigma\delta} q^\rho \ell^\sigma q_-^\delta \quad (4.28)$$

And

$$(\alpha_{-+} + e^{-\alpha})q_-^\mu + v_+ \ell^\mu e^{-\alpha}(uv - e^{2\alpha})\epsilon^\mu_{\rho\sigma\delta} q^\rho \ell^\sigma q_-^\delta = 0. \quad (4.29)$$

Then

$$\alpha_{-+} + e^\alpha = 0$$
$$v_+ = 0 \quad (4.30)$$
$$uv - e^\alpha = 0$$

in accordance with the equations derived from the second derivative of the equation of motion [7]. Furthermore, the derivative of v with respect to σ+ is

$$v_+ = \frac{\partial}{\partial \sigma_+}(q_-^\mu{}_- \ell_\mu)$$
$$= q_{--+}^\mu \ell_\mu + q_{--}^\mu \ell_{+\mu} = q_{+--}^\mu \ell_\mu + (\alpha_- q_-^\mu + v\ell^\mu)\ell_{+\mu}$$
$$= \frac{\partial}{\partial \sigma_-}[-e^\alpha(q^\mu + \ell^\mu)]\ell_\mu + \alpha_- q_-^\mu \ell_{+\mu}$$
$$= -\alpha_- e^\alpha (q^\mu + \ell\mu)\ell_\mu - e^\alpha(q_-^\mu + \ell_-^\mu)\ell_\mu + \alpha_- q_-^\mu \ell_{+\mu} = \alpha_- q_-^\mu \ell_{+\mu} - e^\alpha q_-^\mu \ell_\mu - \alpha_- e^\alpha$$
$$= -\alpha_- q_{-+}^\mu \ell_\mu - \alpha_- - e^\alpha == -\alpha_-[-e^\alpha(q^\mu + \ell^\mu)]\ell_\mu - \alpha_- e^\alpha = 0. \quad (4.31)$$

The Liouville equation

$$\alpha_{+-} + e^\alpha = 0$$

is derived

$$\int d^2x[\partial_+\alpha\partial_-\alpha - e^\alpha]. \quad (4.32)$$

Even though the action with the WZW term is conformally invariant at the quantum level, the $\widehat{SL}(2;\mathbb{R})$ current algebra for the components of the currents $K^a = Tr(T^a \partial g g^{-1})$, $\bar{K}^a = Tr(\bar{T}^a g^{-1}\bar{\partial} g)$,

defined by

$$K^a(z) = \sum_{n\in\mathbb{Z}} \frac{K_n^a}{z^{n+1}}$$

yields the generators

$$L_n = \frac{1}{k-2}\sum_{m=-\infty}^{\infty} : \eta_{ab} K_m^a K_{n-m}^b : \quad (4.33)$$

with ηab being the metric on SL(2; R), of a centrally extended Virasoro algebra [7]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$
$$c = \frac{3k}{k-2}. \tag{4.34}$$

The central charge never equals 3 for finite integer k, and it only reaches this value in the limit k → ∞. It does equal 4 when k = 8. Therefore, this model might be regarded as a description of the dynamics of strings moving on a three-dimensional hyper surface embedded in an ambient four-dimensional space-time.

Without the Wess-Zumino term, it is found that the potential [9] is

$$\tilde{V}(\alpha) = 2\ sinh\ \alpha \quad AdS_3. \tag{4.35}$$

The action

$$S = \int d^2x\ [\partial_+\alpha\partial_-\alpha - 2\ sinh\ \alpha]. \tag{4.36}$$

and the equations of motion

$$\partial_+\partial_-\alpha + 2\ cosh\ \alpha = 0. \tag{4.37}$$

describe the cosh-Gordon model. The sinh-Gordon is integrable with an infinite number of conserved charges in the classical and quantum theories [10][11][12]. Conserved charges also exist in the cosh-Gordon model [13]. The quantization of the sinh-Gordon theory is generally given through the renormalization group. It is renormalizable model with asymptotic freedom [14]. Therefore, the AdS3 conformal field theory may be transformed into a cosh-Gordon theory for the logarithm of the distance in the coordinates (q_+^μ, q_-^μ).

The other variables can be regarded as nondynamical, leaving a renormalizable model. Since the central charges cannot always be arranged to equal a selected dimension, the action constructed by this method would serve as an alternative quantum theory for the string in maximally symmetric conformally flat manifolds such as anti-de Sitter space.

The four-dimensional anti-de Sitter manifold is a coset space SO(3, 2)/SO(3, 1) and not a group. The WZW term can be defined for a coset space conformal field theory action [15].Given the equation for AdS4 embedded in a five-dimensional pseudo-Euclidean space

$$-T^2 - U^2 + W^2 + X^2 + Y^2 = -R^2, \tag{4.40}$$

the Polyakov action is

$$S_4 = -\frac{k}{2^2}\int_S d\tau\ d\sigma[\dot{T}^2 - T'^2 + \dot{U}^2 - U'^2 - \dot{W}^2 - W'^2 - \dot{X}^2 + X'^2 - \dot{Y}^2 + Y'^2$$
$$+ 4(-T^2 - U^2 + W^2 + X^2 + Y^2 + R^2)]. \tag{4.41}$$

The vector qμ now may be defined to be

$$\left(\tfrac{1}{R}T, \tfrac{1}{R}U, \tfrac{1}{R}W, \tfrac{1}{R}X, \tfrac{1}{R}Y\right).$$

The basis of the five-dimensional space would be

$$\{q^\mu, q^\mu_+, q^\mu_-, \ell^\mu, m^\mu),$$

Where

$$q^\mu_+ = \frac{\partial q^\mu}{\partial \sigma_+},\ q^\mu_- = \frac{\partial q^\mu}{\partial \sigma_-},\ \ell^\mu = e^{-\alpha}\epsilon^\mu_{0\rho\sigma\delta} q^\rho q^\sigma_+ q^\delta_-,\ m^\mu = e^-\alpha\epsilon^\mu_{\lambda\rho\sigma\delta} q^\lambda q^\rho_+ q^\sigma_- \ell^\delta,$$

With

$$q^\mu_+ q_-\mu = -e^\alpha,\ \ell^\mu \ell_\mu = 1,\ m^\mu m_\mu = 1.$$

Furthermore, the equations for the action with the constraint on the embedding space coordinates will be

$$q^\mu_{+-} \pm (q^\nu_+ q^\nu_-) q^\mu = 0.$$

There is an extra contribution from mμ, since the second derivatives of qμ now will be

$$q^\mu_{++} = \alpha_+ q^\mu_+ + u\ell^\mu + \hat{u} m^\mu$$

And

$$q^\mu_{--} = \alpha^\mu_- + q^\mu_- + \ v * \mu + \hat{v} m^\mu,$$

Where

$$\hat{u} = q^\mu_{++} m_\mu$$

And

$$\hat{v} = q^\mu_{--} m_\mu.$$

$$\begin{aligned}
&\frac{\partial^2}{\partial\sigma_+\partial\sigma_-} e^\alpha = \frac{\partial}{\partial\sigma_+}(\alpha_- e^\alpha) \\
&\quad = \alpha_{-+} e^\alpha + \alpha_+ \alpha_- e^\alpha = -\frac{\partial}{\partial\sigma_+}(\pm q^\mu_{+-} q_{-\mu} + q^\mu_+ q_{--\mu}) \\
&= -\frac{\partial}{\partial\sigma_+}(q^\nu_+ q_{-\nu} q^\mu q^\mu_- + q^\mu_+ q_{--\mu}) = -q^\mu_{++} q_{--\mu} + q^\mu_+ q_{--+\mu} \\
&= -q^\mu_{++} q_{--\mu} + q^\mu_+ \frac{\partial}{\partial\sigma_-}[q^\nu_+ q_{-\nu} q_\mu]
\end{aligned} \tag{4.42}$$

$$= -q^{\mu}_{++}q_{--\mu} + q^{\mu}_{+}(\pm q^{\nu}_{+-}q_{-\nu}q_{\mu} + q^{\nu}_{+}q_{--\nu}q_{\mu} + q^{\nu}_{+}q_{-\nu}q_{-\mu})$$
$$= -q^{\mu}_{++}q_{--\mu} + (q^{\mu}_{+}q_{-\mu})^2 = -(\alpha_{+}q^{\mu}_{+} + u\ell^{\mu} + \hat{u}m^{\mu})(\alpha_{-}q_{-\mu} + v\ell_{\mu} + \hat{v}m_{\mu}) \pm e^{2\alpha}$$
$$= \alpha_{+}\alpha_{-}q^{\mu}_{+}q_{-\mu} + uv\ell^{\mu}\ell_{\mu} + \hat{u}\hat{v}m^{\mu}m_{\mu} + e^{2\alpha} = \alpha_{+}\alpha_{-}e^{\alpha} + uv + \hat{u}\hat{v} + e^{2\alpha}.$$

Then

$$\alpha_{+-}e^{\alpha} - uv - \hat{u}\hat{v} \pm e^{2\alpha} = 0. \tag{4.43}$$

The metric on the worldsheet is transformed for AdS3 such that $uv \to 1$. Now, it is necessary to use a conformal transformation such that $uv \to \frac{1}{2}$ and $\tilde{u}\tilde{v} \to \frac{1}{2}$. Then the equation for α is either

$$\alpha_{+-} - e^{\alpha} + e^{-\alpha} = \alpha_{+-} - 2\ cosh\ \alpha = 0. \tag{4.44}$$

or

$$\alpha_{+-} + e^{\alpha} - e^{-\alpha} = \alpha_{+-} + 2\ sinh\ \alpha = 0, \tag{4.45}$$

which are the equations of the sinh-Gordon and cosh-Gordon actions respectively.

5. CONCLUSION

This study of the effect of curvature on string theory includes the expansion of the string coordinates on the Euclidean continuation of string worldsheets at higher genus, the preservation of Lorentz invariance in the critical dimension and the invariances of the action on conformally flat space-times. Even though there are extra terms in the series formula for the light cone coordinates, resulting from the transformations in the uniformizing group of the surface, the coefficients of the leading terms have the same operator form. Since the formula for the Lorentz generators remains unaltered, critical dimension D equals 26 and the normal ordering constant is a = 1 for the bosonic string at arbitrary genus. The string action can be formulated in curved space by replacing the metric of Minkowski space-time, ημν, by gμν. Generally, the new metric would introduce couplings between the different coordinates. However, when the geometry is conformally flat, there will be only an overall Weyl rescaling factor. The two-dimensional conformal field theory will continue to have the reparameterization, Weyl and a space-time symmetry.

When the curved space-time is the maximally symmetric anti-de Sitter space, the Poincare group will be replaced by SO(D-1, 2). Nevertheless, the generators will continue to have given by an expression proportional the sum of the orbital component and the sum over the products of creation and annihilation operators such that the critical dimension D equals 26 and a = 1. The partition function of the string theory may be evaluated when the curvature of the embedding space-time is considerably less than the curvature of the Riemann surfaces representing the interactions. Then, the conformal factor Ω can be separated in the action and partition function is approximately multiplied by a scale which is a power of Ω [4].

These computations may be extended to the superstring in anti-de Sitter space and other maximally symmetric curved space-times. The superstring coordinate and fermion fields again may be given by a sum over the transformation of Fourier components under the elements of the uniformizing group. The critical dimension then would continuet o be 10 and the normal ordering constant is a = 1/2 in the Neveu-Schwarz sector anda = 0 in the Ramond sector. The invariances of the superstring action in a conformally flat background would have additional supersymmetry, superreparameterization and super Weyl invariances. The quantization of the superstring again would be most directly completed on space-times of maximal symmetry such as $AdS_5 \times S^5$. The proof of the consistency of the quantum theory requires the vanishing of anomalies such as the cancellation of counterterms [5] required for the preservation of symmetries.

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