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Relativistic Quantum Scalar Fields

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ABSTRACT A new equation for the relativistic scalar field is derived from the correspondence with the classical commutator and covariance. It has the same form as the Klein Gordon equation with different coefficients.

KEYWORDS Commutator, Energy, Covariance, Wave Equation

1. INTRODUCTION

The wave equation in nonrelativistic quantum mechanics is generalized to relativistic field theory through the relation between the momentum, mass and energy that holds for all velocities. Since the correspondence with the classical commutator has been found to necessitate the introduction of a new term in the momentum operator, a similar term in the Hamiltonian would be required by covariance. A choice for the term will be determined by Hermiticity and reality of the coefficients.

With the new Hamiltonian, a relativistic wave equation is formulated in §2. It is a matrix equation, where difference between the multiplication by the scalar multiple of the identity matrix and by a complex number is preserved. Such an equation is unconventional for a scalar field, since the interpretation of the matrix elements is not specified. Nevertheless, the trace of the matrices may be evaluated separately from the complex number part of the equation to yield a new equation. This equation has a form similar to the Klein-Gordon equation with adjusted coefficients in the differential equation. Division by a constant yields an operator that begins with the - $(\partial/\partial t^{2)}$ of a four-dimensional Laplacian and then continues with the spatial Laplacian multiplied by a factor of 5/3, while the mass is reduced by a factor of $\sqrt{13}$.

The quantization of this model is developed in §3. It is proven that the modes are no longer purely oscillatory. Instead, there is a multipicative factor that describes an exponentially decreasing or expanding mode. The quantum commutators of the field and the conjugate momentum require a numerical factor in the commutator of the annihilation and creation operatos. Then, the commutator is evaluated, and the momentum-space propagator is found. It has a form which differs from the standard propagator by the values of the coefficients. It remains to be established if there are experimental effects connected to this change in the coefficients. Typically, the quantum theory of matter and radiation is described by fermion fields and photons. Nevertheless, there are some scalar fields that arise in phenomenological models such as that of the pion and the sigma-model.

2. THE EXISTENCE OF A COVARIANT ADDITION TO THE HAMILTONIAN

The commutator of the position and momentum operators in $[q, p] = i\hbar$ is the central relation in a fundamentally different formulation of physical dynamics. The variables which describe motion in classical mechanics have been found to be generalized from real quantities to operators with complex coefficients. The relation to physical experiment follows from the condition that all observables are represented by Hermitian operators with real eigenvalues. When the eigenvalue spectra of these operators are discrete, only those values will be measured with probabilities between 0 and 1. The classical laws of physics must be found in the limit of vanishing of the quantum parameter. It would be found then from the above commutator that the spatial coordinate and the momenta must commute. This result is compatible with the real number form of these kinematic variables, However, there is another type of commutator, the Poisson bracket, that is formed from the functional derivatives with respect to the the phase space coordinates (*qi*, *pj*). The advantage of this bracket is the existence of a symplectic metric on the phase space and another formulation of the Hamiltonian equations of motion. The Poisson bracket of the spatial coordinates and the momenta would be $\{qi, pj\} = \delta ij$ [1]. Since it is real and non-zero, a generalization of the quantum commutator is necessary for its derivation in the classical limit. The addition of the new term must be consistent with Hermiticity for the eigenvalues to be real. When the adjoint is evaluated, derivatives act on the other wavefunction in an inner product within the class of square integrable functions, and, from equivalence, given the boundary condition of vanishing at infinity, with the negative of the derivative acting on the original function, Hermiticity of the operator $\kappa_0\gamma^i\partial_i$ - $i\hbar\partial i$ may be verified, where

$$\left\{\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right\}.$$

The commutator of the position and momentum variables is altered to $[xi, pj] = (\kappa_0\gamma^i - i\hbar)\delta_{ij}$. By considering variations from the mean and tracing of the inequality, the uncertainty relation is generalized to

$$\Delta x_i \Delta p_j \geq \frac{\sqrt{4\kappa_0^2+\hbar^2}}{2}\delta_{ij}$$

for scalar wavefunctions [2] and

$$\Delta x_i \Delta p_j \geq \frac{\sqrt{\kappa_0^2+\hbar^2}}{2}\delta_{ij}$$

for spinor wavefunctions.

The standard lower bound of $\hbar/2$ arises in the limit $\kappa_0 \rightarrow 0$. Experimental evidence indicates that the bounds can be increased to for integer spin fields.

The nonrelativistic form of the Hamiltonian is $-i\hbar\,(\partial/\partial t)$. Even though the classical Poisson brackets are defined for functions of the spatial position and momentum variables, the condition of covariance in a relativistic field theory requires an additional term in the Hamiltonian if the momentum is generalized. The choice $\kappa_0\gamma^0\partial_0 - i\hbar\partial_i$ is not Hermitian and $i\kappa_0\gamma^0\partial_0$ - $i\hbar\partial_i$ does not have any real coefficients with the gamma matrix [3]

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Therefore, another modification is necessary to satisfy both conditions.

The Clifford algebra is defined by the relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}I \qquad (2.1)$$

where $\eta_{\mu\nu} = diag(+ - - -)$ is the Lorentz metric. This equality is valid for the set of gamma matrices $\gamma^\mu = (\gamma^0, \gamma^i)$. It is equally valid for another set of matrices (γ^5, γ^i) since $\gamma_5^2 = 1$and $\{\gamma^i, \gamma_5\} = 0$. Consider then the operator $\kappa_0\gamma^5 (\partial/\partial t) - i\hbar(\partial/\partial t)$. The adjoint is

$$\begin{aligned}\left(\kappa_0\gamma_5\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}\right)^\dagger &= \kappa_0\gamma_5^\dagger\frac{\overleftarrow{\partial}}{\partial t} + i\hbar\frac{\overleftarrow{\partial}}{\partial t} \\ &= \kappa_0\gamma_0\gamma_5\gamma_0\frac{\overleftarrow{\partial}}{\partial t} + i\hbar\frac{\overleftarrow{\partial}}{\partial t} \\ &= -\kappa_0\gamma_5\frac{\overleftarrow{\partial}}{\partial t} + i\hbar\frac{\overleftarrow{\partial}}{\partial t}\end{aligned} \qquad (2.2)$$

In the space of square integrable function

$$\int_{-\infty}^{\infty} \bar{\chi}\ dt\ \left(-\kappa_0\gamma_5 \frac{\overleftarrow{\partial}}{\partial t} + i\hbar \frac{\overleftarrow{\partial}}{\partial t}\right)\eta = \int_{-\infty}^{\infty}\ dt\ \bar{\chi}\eta + \int_{-\infty}^{\infty}\ dt\ \bar{\chi}\left(\kappa_0\gamma_5\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}\right)\eta$$
$$= \int_{-\infty}^{\infty}\ dt\ \bar{\chi}\left(\kappa_0\gamma_5\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}\right)\eta \qquad (2.3)$$

since the wave functions χ and η vanish at $t = -\infty$ and $t = \infty$. Then the operator in Eq. (2.2) is equivalent in the space of square integrable functions over the interval $(-\infty, \infty)$ to $\kappa_0\gamma^5\ (\partial/\partial t)\ - i\hbar(\partial/\partial t)$.

3. THE RELATIVISTIC WAVE EQUATION FOR A SCALAR FIELD

The following operators will be used in a covariant formulation of relativistic field theory.
Let

$$\tilde{p}_i = \kappa_0\gamma^i\frac{\partial}{\partial x_i} - i\hbar\frac{\partial}{\partial x_i}$$
$$\tilde{E} = \kappa_0\gamma_5\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}. \qquad (3.1)$$

The relativistic relation between the energy, momentum and mass is

$$-\tilde{E}^2 + \sum_i (\tilde{p})_i^2 + m^2 = 0. \qquad (3.2)$$

Since

$$\tilde{E}^2 = \left(\kappa_0\gamma_5\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}\right)^{-}$$
$$= \kappa_0(\gamma_5)^2\frac{\partial^2}{\partial t^2} - 2i\kappa_0\hbar\gamma_5\frac{\partial^2}{\partial t^2} - \hbar^2\frac{\partial^2}{\partial t^2} \qquad (3.3)$$
$$= (\kappa_0^2 I - \hbar^2)\frac{\partial^2}{\partial t^2} - 2i\kappa_0\hbar\gamma_5\frac{\partial^2}{\partial t^2}$$

And

$$\sum_i \tilde{p}_i^2 = \sum_i \left(\kappa_0\gamma^i\frac{\partial}{\partial x_i} - i\hbar\frac{\partial}{\partial x_i}\right)^2$$
$$= \sum_i \left(\kappa_0^2(\gamma^i)^2\frac{\partial^2}{\partial x_i^2} - 2i\kappa_0\hbar\frac{\partial^2}{\partial x_i^2} - \hbar^2\frac{\partial^2}{\partial x_i^2}\right) \qquad (3.4)$$
$$= \sum_i \left[\left(-\kappa_0^2 I - \hbar^2\right)\frac{\partial^2}{\partial x_i^2} - 2i\kappa_0\hbar\gamma^i\frac{\partial^2}{\partial x_i^2}\right],$$

the matrix generalization of the Klein-Gordon equaiton is

$$-(\kappa_0^2 I - \hbar^2)\frac{\partial^2\phi}{\partial t^2} + 2i\kappa_0\hbar\gamma_5\frac{\partial^2\phi}{\partial t^2}$$
$$- \sum_i \left[\left(\kappa_0^2 I + \hbar^2\right)\frac{\partial^2\phi}{\partial x_i^2} + 2i\kappa_0\hbar\gamma^i\frac{\partial^2\phi}{\partial x_i^2}\right] + m^2\phi = 0. \qquad (3.5)$$

For a scalar field, the action of matrix is defined only after the trace. The original form of the term with the coefficient $-\hbar^2$ did not include a matrix, and it is unnecessary to evaluate the trace. When the traces of the matrices are evaluated for the scalar field, the resulting equation is

$$-(4\kappa_0^2 - \hbar^2)\frac{\partial^2 \phi}{\partial t^2} - \sum_i (4\kappa_0^2 + \hbar^2)\frac{\partial^2 \phi}{\partial x_i^2} + m^2\phi = 0. \tag{3.6}$$

The limit $\kappa_0 \rightarrow 0$, with $\hbar$ set equal to 1, yields the Klein-Gordon equation

$$(-\kappa_0 + m^2)\phi = 0. \tag{3.7}$$

When $\kappa_0 = \hbar$,

$$-3\hbar^2 \frac{\partial^2 \phi}{\partial t^2} - 5\hbar^2 \sum_i \frac{\partial^2 \phi}{\partial x_i^2} + m^2\phi = 0. \tag{3.8}$$

Setting ~ equal to one,

$$-3\left(\frac{\partial^2 \phi}{\partial t^2} + \frac{5}{3}\sum_i \frac{\partial^2 \phi}{\partial x_i^2}\right) + m^2\phi = 0 \tag{3.9}$$

And

$$-\left(\frac{\partial^2 \phi}{\partial t^2} + \frac{5}{3}\sum_i \frac{\partial^2 \phi}{\partial x_i^2}\right) + \left(\frac{m}{\sqrt{3}}\right)^2 \phi = 0. \tag{3.10}$$

4. QUANTIZATION OF EXPONENTIAL MODES

The action for the scalar field consistent with the correspondence principle which gives rise to this equation is

$$S_{corr.sc.} = \int d^4x \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{5}{6}\sum_i \left(\frac{\partial \phi}{\partial x_i}\right)^2 + \frac{1}{6}m^2\phi^2\right] \tag{4.1}$$

The factor of 1/6 is reminiscent of the conformal coupling of scalar fields to gravity through the term $1/6(R\varphi^2)$, with the Ricci scalar having a role similar to m^2. The quantization proceeds by setting

$$\phi = \int \frac{d^3p}{(2\pi)^3 2E}[a(\vec{p})e^{\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x} - \frac{E}{\sqrt{3}}t} + a^\dagger(\vec{p})e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x} + \frac{E}{\sqrt{3}}t}]. \tag{4.2}$$

Then

$$\frac{\partial \phi}{\partial t} = \int \frac{d^3p}{(2\pi)^3 2E}\left[-\frac{E}{\sqrt{3}}a(\vec{p})e^{\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x} + \frac{E}{\sqrt{3}}t} + \frac{E}{\sqrt{3}}a^\dagger(\vec{p})e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x} + \frac{E}{\sqrt{3}}t}\right] \tag{4.3}$$

and

$$\frac{\partial^2\phi}{\partial t^2} = \int \frac{d^3p}{(2\pi)^3 2E}\Bigg[\left(-\frac{E}{\sqrt{3}}\right)\left(-\frac{E}{\sqrt{3}}\right)a(\vec{p})e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}+\frac{E}{\sqrt{3}}t} + \left(\frac{E}{\sqrt{3}}\right)\left(\frac{E}{\sqrt{3}}\right)a^\dagger(\vec{p})e^{\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}-\frac{E}{\sqrt{3}}t}\Bigg]. \quad (4.4)$$
$$= \frac{E^2}{3}\phi$$

The spatial derivative is

$$\frac{\partial\phi}{\partial x_i} = \int \frac{d^3p}{(2\pi)^3 2E}\left[\frac{ip_i}{\sqrt{5}}a(\vec{p})e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}-\frac{E}{\sqrt{3}}t} - \frac{ip_i}{\sqrt{5}}a^\dagger(\vec{p})e^{\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}-\frac{E}{\sqrt{3}}t}\right] \quad (4.5)$$

and

$$\sum_i \frac{\partial^2\phi}{\partial x_i^2} = -\frac{|\vec{p}|^2}{5}\phi. \quad (4.6)$$

The scalar field equation is satisfied because

$$-\left(\frac{\partial^2\phi}{\partial t}^2 + \frac{5}{3}\sum_i\frac{\partial^2\phi}{\partial x_i^2}\right) + \left(\frac{m}{\sqrt{3}}\right)^2\phi = -\frac{E^2}{3}\phi + \frac{5}{3}\frac{|\vec{p}|^2}{5}\phi + \frac{m^2}{3}\phi = (-E^2+|\vec{p}|^2+m^2)\phi = 0. \quad (4.7)$$

There are exponentially decaying and increasing modes in the expansion of φ. The measure has been selected to be Lorentz invariant since this property remains valid, regardless of the Lagrangian.

The canonical momentum variable in the theory is

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}. \quad (4.8)$$

The Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \pi_\phi\dot{\phi} - \mathcal{L}\\ &= \left(\frac{\partial\phi}{\partial t}\right)^2 - \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{5}{6}\sum_i\left(\frac{\partial\phi}{\partial x_i}\right)^2 - \frac{1}{6}m^2\phi^2\right]\\ &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{5}{6}\sum_i\left(\frac{\partial\phi}{\partial x_i}\right)^2 - \frac{1}{6}m^2\phi^2\end{aligned} \quad (4.9)$$

It is not positive-definite and the system can be unstable. The time-dependence of a variable in classical mechanics is given by $\partial g/\ \partial t = [g,\ H]_{P.B.}$. In nonrelativistic quantum mechanics, $i\hbar(\partial g/\partial t) = [g,\ H]$. The generalization of the energy operator to $\kappa_0\gamma^5\ (\partial/\partial t)\ - i\hbar(\partial/\partial t)$ yields

$$\begin{aligned}\left[g(\vec{x},t),\kappa_0\gamma_5\frac{\partial}{\partial t}-i\hbar\frac{\partial}{\partial t}\right]f(\vec{x},t)&=g(\vec{x},t)\left(\kappa_0\gamma_5\frac{\partial f(\vec{x},t)}{\partial t}-i\hbar\frac{\partial f(\vec{x},t)}{\partial t}\right)\\&\quad-\left(\kappa_0\gamma_5\frac{\partial}{\partial t}-i\hbar\frac{\partial}{\partial t}\right)(g(\vec{x},t)f(\vec{x},t))\\&=-\left(\kappa_0\gamma_5\frac{\partial g(\vec{x},t)}{\partial t}-i\hbar\frac{\partial g(\vec{x},t)}{\partial t}\right)f(\vec{x},t)\\&=(-\kappa_0\gamma_5+i\hbar)\frac{\partial g(\vec{x},t)}{\partial t}.\end{aligned} \tag{4.10}$$

The analogue of the Poisson bracket in the quantum theory of the scalar field is conventionally

$$[\phi(\vec{x},t),\pi_\phi(\vec{x}',t)]=i\hbar\delta(\vec{x}-\vec{x}') \tag{4.11}$$

When an additional term is included in the commutator of the time and energy coordinates in quantum mechanics, the equivalent matrix generalization of this commutator is

$$[\phi(\vec{x},t),\pi_\phi(\vec{x}',t)]=(-\kappa_0\gamma_5+i\hbar)\delta(\vec{x}-\vec{x}') \tag{4.12}$$

which is consistent with Eq.(4.9) for the time-dependence of a variable. The trace of the matrix terms for the scalar field again yields

$$[\phi(\vec{x},t),\pi_\phi(\vec{x}',t)]_{tr}=i\hbar\delta(\vec{x}-\vec{x}') \tag{4.13}$$

Then, in units with $\hbar = 1$,

$$\begin{aligned}[\phi(\vec{x},t),\dot{\phi}(\vec{x}',t)]_{tr}&=\int\frac{d^3p}{(2\pi)^3 2E}\int\frac{d^3p'}{(2\pi)^3(2E')}\\&\quad\left\{\left[a(\vec{p})e^{\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}-\frac{E}{\sqrt{3}}t}+a^\dagger(\vec{p})e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}+\frac{E}{\sqrt{3}}t},\right.\right.\\&\quad\left.\left.-\frac{E'}{\sqrt{3}}a(\vec{p}')e^{\frac{i}{\sqrt{5}}\vec{p}'\cdot\vec{x}'-\frac{E'}{\sqrt{3}}t}+\frac{E'}{\sqrt{3}}a^\dagger(\vec{p}')e^{-\frac{i}{\sqrt{5}}\vec{p}'\cdot\vec{x}'+\frac{E'}{\sqrt{3}}t}\right]\right\}\\&=\int\frac{d^3p}{(2\pi)^3 2E}\int\frac{d^3p'}{(2\pi)^3 2E'}\\&\quad\left\{-\frac{E'}{\sqrt{3}}[a^\dagger(\vec{p}),a(\vec{p}')]e^{\frac{i}{\sqrt{5}}(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{x}')+\frac{1}{\sqrt{3}}(E'-E)t}\right.\\&\quad\left.+\frac{E'}{\sqrt{3}}[a(\vec{p}),a^\dagger(\vec{p}')]e^{-\frac{i}{\sqrt{5}}(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{x}')+\frac{1}{\sqrt{3}}(E-E')t}\right\}\end{aligned} \tag{4.14}$$

with

$[a(\vec{p}),a(\vec{p}')]=[a^\dagger(\vec{p}),a^\dagger(\vec{p}')]=0.$

If the expansion of the scalar field consists only of exponentially decaying modes, the commutator

$[\phi(\vec{x},t),\pi_\phi(\vec{x}',t)]$

will vanish when standard commutators between annihilation and creation operators are used.

Let

$[a(\vec{p}),a^\dagger(\vec{p}')]=\;k(2\pi)^3\delta(\vec{p}-\vec{p}').$

The equal-time commutator is

$$
\begin{aligned}
&[\phi(\vec{x},t),\pi_\phi(\vec{x}',t)]\\
&= \frac{2E'}{\sqrt{3}}k(2\pi)^3\int\frac{d^3p}{(2\pi)^3 2E}\int\frac{d^3p'}{(2\pi)^3 2E'}\delta(\vec{p}-\vec{p}')e^{-\frac{i}{\sqrt{5}}(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{x}')+\frac{1}{\sqrt{3}}(E-E')t}\\
&= \frac{2E'}{\sqrt{3}}k\frac{(2\pi)^3}{(2\pi)^6 4EE'}\int d^3p e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot(\vec{x}-\vec{x}')+\frac{1}{\sqrt{3}}E(t-t')}\\
&= \frac{2E'}{\sqrt{3}}\frac{k}{(2\pi)^3}\frac{(2\pi)^3}{4EE'}\delta\left(\frac{1}{\sqrt{5}}(\vec{x}-\vec{x}')\right)\\
&= \frac{2E'}{\sqrt{3}}\frac{k}{4EE'}\sqrt{5^3}\delta(\vec{x}-\vec{x}')\\
&= \frac{\sqrt{5^3}}{2\sqrt{3}E}k\delta(\vec{x}-\vec{x}').
\end{aligned}
\tag{4.15}
$$

Then

$k = \frac{2\sqrt{3}}{\sqrt{5^3}}Ei$

and

$$
[a(\vec{p}),a^\dagger(\vec{p}')] = \sqrt{\frac{3}{5^3}}(2\pi)^3(2E)i\delta(\vec{p}-\vec{p}'). \tag{4.16}
$$

The number operator is defined to be

$$
\begin{aligned}
N &= k'\int\frac{d^3p}{(2\pi)^3 2E}a^\dagger(\vec{p})a(\vec{p})\\
Na^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)|0\rangle &= na^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)|0\rangle.
\end{aligned}
\tag{4.17}
$$

Since

$$
\begin{aligned}
&k'\int\frac{d^3p}{(2\pi)^3 2E}a^\dagger(\vec{p})a(\vec{p})a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)|0\rangle\\
&= k'\int\frac{d^3p}{(2\pi)^3 2E}a^\dagger(\vec{p})\{a^\dagger(\vec{p}_1)a(\vec{p}_1)+[a(\vec{p}_1),a^\dagger(\vec{p}_1]\}a^\dagger(\vec{p}_2)...a^\dagger(\vec{p}_n)|0\rangle\\
&= k'\frac{1}{(2\pi)^3 2E}n\left(\sqrt{\frac{3}{5^3}}(2\pi)^3 2Ei\right)a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)|0\rangle\\
&\quad + k'\int\frac{d^3p}{(2\pi)^3 2E}a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)a^\dagger(\vec{p})a(\vec{p})|0\rangle
\end{aligned}
\tag{4.18}
$$

equals $n|a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)|0\rangle$ if $k' = \sqrt{\frac{5^3}{3}}i$,

$$
N = \sqrt{\frac{5^3}{3}}i\int\frac{d^3p}{(2\pi)^3}a^\dagger(\vec{p})a(\vec{p}). \tag{4.19}
$$

The commutator function is defined to be

$$\Delta(\vec{x}-\vec{x}',t-t') = -i[\phi(\vec{x},t),\phi(\vec{x}',t')]$$
$$= -i\Big[\int \frac{d^3p}{(2\pi)^3 2E}[a(\vec{p})e^{\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}-\frac{1}{\sqrt{3}}Et} + a^\dagger(\vec{p})e^{-\frac{i}{\sqrt{3}}\vec{p}\cdot\vec{x}+\frac{1}{\sqrt{3}}Et}],$$
$$\int \frac{d^3p'}{(2\pi)^3 2E'}[a(\vec{p}')e^{\frac{i}{\sqrt{5}}\vec{p}'\cdot\vec{x}'-\frac{1}{\sqrt{3}}E't'} + a^\dagger(\vec{p}')e^{-\frac{i}{\sqrt{5}}\vec{p}'\cdot\vec{x}'+\frac{1}{\sqrt{3}}E't'}]\Big]$$
$$= \sqrt{\frac{3}{5^3}}\int \frac{d^3p}{(2\pi)^3 2E}[e^{\frac{i}{\sqrt{5}}\vec{p}\cdot(\vec{x}-\vec{x}')-\frac{1}{\sqrt{3}}E(t-t')} - e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot(\vec{x}-\vec{x}')+\frac{1}{\sqrt{3}}E(t-t')}]. \tag{4.20}$$

The commutator function of the Klein-Gordon scalar field in Minkowski space-time is

$$\Delta_{KG}(x) = -i\int \frac{d^3p}{(2\pi)^3 2E}(e^{-ip\cdot x} - e^{ip\cdot x}),$$

which has the representation

$$-\frac{1}{(2\pi)^4}\int_C \frac{d^4p}{p^2-m^2}e^{-ip\cdot x},$$

where C is a contour about the two poles at $\pm\sqrt{|\vec{p}|^2+m^2}$ in the complex p^0 plane.

The above integral (4.20) may recast in the form, through a change of variables $\vec{p} = \sqrt{5}\vec{\hat{p}}$ and $E = \sqrt{3}i\hat{E}$ with the four-vector $\hat{p} = (\hat{E},\vec{\hat{p}})$

$$\sqrt{\frac{3}{5^3}}\frac{\sqrt{5}^3}{\sqrt{3}i}\int \frac{d^3\hat{p}}{(2\pi)^3 2\hat{E}}[e^{i\hat{p}\cdot x} - e^{-i\hat{p}\cdot x}] = -i\int \frac{d^3\hat{p}}{(2\pi)^3 2\hat{E}}[e^{-i\hat{p}\cdot x} - e^{i\hat{p}\cdot x}] \tag{4.21}$$

Therefore,

$$\Delta(x) = -\frac{1}{(2\pi)^4}\int \frac{d^4\hat{p}}{\hat{p}^2-m^2}e^{-i\hat{p}\cdot x} \tag{4.22}$$

in (+ - --) signature. The configuration space propagator of the scalar field is given by

$$i\Delta_F(x-x') = \langle 0|T\phi(x)\phi(x')|0\rangle$$

Therefore, the momentum space propagator would be

$\frac{-i}{\hat{p}^2-m^2}$ in (+ - --) signature and $\frac{i}{\hat{p}^2+m^2}$ in (- + ++) signature.

Converting the momentum space variable to the $p = (E,\vec{p})$, the propagator equals

$$\frac{-i}{\left(\frac{E}{\sqrt{3}i}\right)^2 - \left(\frac{|\vec{p}|}{\sqrt{5}}\right)^2 - m^2} = \frac{i}{\frac{E^2}{3}+\frac{|\vec{p}|^2}{5}-m^2} = \frac{15i}{5E^2+3|\vec{p}|^2+15m^2}. \tag{4.23}$$

The momentum space rules for the quantum theory can be formulated with this momentum propagator, incoming external lines $e^{\frac{i}{\sqrt{5}}\vec{p}\cdot x-\frac{1}{\sqrt{3}}Et}$ and outgoing lines $e^{-\frac{i}{\sqrt{5}}\vec{p}\cdot\vec{x}+\frac{1}{\sqrt{3}}Et}$ and integration over internal momenta $\int\frac{d^4p}{(2\pi)^4}$.

5. CONCLUSION

The methods of quantum theory have been generalized for consistency with the correspondence with the reality of classical commutators. The addition of a matrix term with a real coefficient to the operators representing the momentum and the energy increases the lower bound in the uncertainty principle. Estimations of this lower bound from experimental evidence yield a value for the coefficient of the matrix term. The transformation of relativistic formula for the energy to differential operators for scalar fields that is a modification of the Klein-Gordon equation. Tracing of the matrix terms yields a scalar wave equation with different coefficients and mass. The quantization of the action for this field has been described. The commutators of φ and the conjugate momentum π_φ remain unchanged after evaluating the trace. Nevertheless, numerical factors must be included in the commutator of the annihilation and creation operators and the number operator. The commutator function was evaluated, and the momentum space propagator was calculated. The evaluation of scattering matrix elements then follows from rules for the diagrammatic expansion.

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